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On Solvability of Functional Equations Relating to Dynamical Systems with Two Generators

B. P. Paneah

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Abstract. In this paper, some solvability problems for functional equations of the form

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = h(t), \qquad t \in I,$$

are studied. Here I is a finite closed interval in $\mathbb{R}$, F is an unknown continuous function, δ_1 and δ_2 are given continuous maps of I into itself, and $a_1(t)$, $a_2(t)$, and $h(t)$ are real-valued continuous functions on I. Such equations are of interest not only by themselves as an object of analysis, but they are also a necessary link in solving various problems in such diverse fields as integral and functional equations, measure theory, and boundary problems for hyperbolic differential equations. The major part of the proofs is based on the new results in the theory of dynamical systems generated by a noncommutative semigroup with *two* generators.

Key words: dynamical system, orbit, functional equation, boundary problem, hyperbolic differential equation.

1. Introduction

In this paper, we consider a wide class of functional equations of the form

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = h(t), \qquad t \in I, \tag{1}$$

on a finite closed interval $I \subset \mathbb{R}$. Here δ_1 and δ_2 are given continuous maps of I into itself, a_1, a_2, and h are given real-valued functions on I, and F is the unknown real function on I. In particular, this equation can be regarded as an analog of the well-known cohomological equation $F(t) - F(\delta(t)) = h(t)$ appearing in connection with many problems of the theory of dynamical systems and ergodic theory. An essential distinctive feature of Eq. (1) is that it involves *two* noncommuting maps, and the appearing dynamics is determined by a noncommutative semigroup with two generators. In this connection, it is worth mentioning Lemma 3 describing some regular properties of orbits of this semigroup. Precisely these properties form the technical basis of the majority of subsequent proofs. We also note that these equations are of interest not only as an object of analysis but also as an adequate technical tool for investigating some new problems in the theory of functional and integral equations as well as in the theory of boundary problems for higher-order (> 2) hyperbolic differential equations. This fact is partly reflected below in Secs. 5–7.

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2. Definitions and Notation

Throughout this paper, we use the notation

$$I = \{t \mid -1 \leqslant t \leqslant 1\} \quad \text{and} \quad \overset{\circ}{I} = \{t \mid -1 < t < 1\}.$$

In relation to the parameters determining Eq. (1), it is assumed that they are continuous (unless otherwise stipulated) and that

(i) both functions δ_1 and δ_2 do not decrease in I;

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(ii) the inequalities $\delta_2(t) < t < \delta_1(t)$, $t \in \overset{\circ}{I}$, hold;
(iii) the ranges of the maps δ_1 and δ_2 are the closed intervals $[0, 1]$ and $[-1, 0]$, respectively;
(iv) the coefficients $a_1(t)$ and $a_2(t)$ are nonnegative and satisfy the hypothesis

$$0 < a_1(t) + a_2(t) \leqslant 1, \qquad t \in I.$$

It follows from (ii) and (iii) that

$$\delta_1(-1) = \delta_2(1) = 0, \quad \delta_1(1) = 1, \quad \delta_2(-1) = -1. \tag{2}$$

Let us introduce the *guiding* sets

$$\mathscr{T}_1 = \{t \in I \mid a_2(t) = 0\}, \quad \mathscr{T}_2 = \{t \in I \mid a_1(t) = 0\}, \quad \text{and} \quad \mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2.$$

The points of the set $\mathscr{T}$ are said to be *$\mathscr{T}$-guiding*. It follows from (iv) that $\mathscr{T}_1 \cap \mathscr{T}_2 = \varnothing$.

The concluding hypothesis concerns a mutual property of the maps δ_j and the functions a_k, $1 \leqslant j, k \leqslant 2$, namely,

(v) $a_j(t) = 0$ on each interval of constancy of the function $\delta_j(t)$, $1 \leqslant j \leqslant 2$.

In particular, if the function $\delta_j(t)$ increases, then the function $a_j(t)$ satisfies the only hypothesis (iv). We note that hypotheses (i)–(v) are not artificial in the sense that all type (1) functional equations that have up to now arisen in various problems in geometry, in the theory of integral and functional equations, and also in the theory of boundary problems for hyperbolic partial differential equations satisfy these conditions (in particular, see Secs. 5–7).

The maps δ_1 and δ_2 generate a noncommutative semigroup Φ_δ. The elements of Φ_δ are all the maps of I into itself of the form $\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1}$, where $J = (j_1, \ldots, j_n)$ is a multi-index with j_k equal to 1 or 2, and $\circ$ denotes the composition of maps. It can be verified easily that this semigroup is not free in the general case. The semigroup Φ_δ naturally determines a dynamical system. In what follows, we use the geometric terminology below relating to Φ_δ and not coinciding completely with the traditional one.

1) Given a map δ_J of I into itself with a multi-index $J = (j_1, \ldots, j_n)$, an ordered set $\mathscr{O} = (t_1, \ldots, t_{n+1})$ of points in I is called an *orbit* (or, sometimes, an *orbit of the point* t_1) if

$$t_{k+1} = \delta_{j_k}(t_k) \tag{3}$$

for all $1 \leqslant k \leqslant n < \infty$. In the sequel, the term "orbit" is also used for describing infinite sequences $(t_1, t_2, \ldots)$ satisfying condition (3).

2) An orbit $\mathscr{O} = (t_1, \ldots, t_{n+1})$, $n = 1, 2, \ldots$, is said to be *$\mathscr{T}$-proper* if

$$\delta_{j_k} = \delta_1 \quad \text{for } t_k \in \mathscr{T}_1 \quad \text{and} \quad \delta_{j_k} = \delta_2 \quad \text{for } t_k \in \mathscr{T}_2$$

in (3).

3) If all points of an orbit $\mathscr{O}$ belong to a guiding set $\mathscr{T}$, then $\mathscr{O}$ is called a *$\mathscr{T}$-guided* orbit.

4) An orbit $\mathscr{O} = (t_1, \ldots, t_{n+1})$ is called a *periodic orbit* or a *cycle* if $t_1 = t_{n+1}$.

Definition. We denote by $\mathfrak{N}_\delta^{\mathscr{T}}$ the set of all $\mathscr{T}$-proper $\mathscr{T}$-guided periodic orbits in I.

3. Solvability of the Homogeneous Equation (1)

The main result of this section is Theorem 1 describing the conditions and the character of the solvability of Eq. (1) with $h = 0$. Its proof is based on the maximum principle for the class of functional equations in question, which is also of independent interest. In turn, the proof of the maximum principle is based on the study of the noncommutative dynamical system generated by the semigroup Φ_δ. This dynamic approach makes it possible to overcome analytical difficulties frequently appearing in the theory of functional equations and to obtain results of broad generality. The possibilities of this approach are certainly not exhausted by the problems solved in this paper. For example, a variation of a guiding set $\mathscr{T}$ along with a description of $\mathscr{T}$-proper attractors of the corresponding dynamical system may prove to be productive in some closely related problems.

Denote by $\mathscr{T}_j'$ and $\mathscr{T}'$ the sets of all limit points of the sets $\mathscr{T}_j$, $j = 1, 2$, and $\mathscr{T}$, respectively. In what follows, it is assumed that if both sets $\mathscr{T}_1$ and $\mathscr{T}_2$ are infinite, then the numbers

$$\tau_1 = \min\{t \mid t \in \mathscr{T}_1'\} \quad \text{and} \quad \tau_2 = \max\{t \mid t \in \mathscr{T}_2'\}$$

satisfy the inequality

$$\tau_1 > \tau_2. \tag{4}$$

Theorem 1. 1°. *Let*

$$a_1(t) + a_2(t) = 1 \quad \text{on } I \tag{5}$$

and let

$$a_1(-1)a_2(1) \neq 0. \tag{6}$$

Then all solutions of the homogeneous equation

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = 0 \tag{7}$$

are constant functions if

$$\mathfrak{N}_\delta^{\mathscr{T}} = \varnothing. \tag{8}$$

2°. *Let*

$$0 < a_1(t) + a_2(t) \leqslant 1 \quad \text{on } I, \tag{9}$$

and let the set $\mathscr{B} = \{t \mid a_1(t) + a_2(t) < 1\}$ contain some deleted neighborhood of the boundary ∂I. Then, under hypothesis* (8), *Eq.* (7) *has no nontrivial solution.*

Theorem 2 (Maximum principle). 1°. *If hypotheses* (5), (6), *and* (8) *are fulfilled, then any solution of Eq.* (7) *takes its maximal and minimal values at boundary points of the interval I.*

2°. *Assume that the set $\mathscr{B}$ is not empty. Then any solution of problem* (7) *takes its positive maximum and negative minimum at points of the set $I \setminus \mathscr{B}$. If hypothesis* (8) *is fulfilled, then these extremal values are taken on the boundary ∂I.*

We will obtain this result as a consequence of some general assertion concerning attractors of the dynamical system generated by the semigroup Φ_δ. This assertion is of interest by itself and it is by no means related to Eq. (1). However, for brevity we preserve *all* the preceding symbols, definitions, and conditions involving them with the only exception, namely, in Lemma 3 below, $\mathscr{T}_1$ and $\mathscr{T}_2$ denote arbitrary closed sets in I that have no common points and contain all intervals of constancy of the functions δ_2 and δ_1, respectively. In particular, in this case, the set $\mathfrak{N}_\delta^{\mathscr{T}}$ consists of all $\mathscr{T}$-proper periodic orbits contained in the set $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$.

Lemma 3. *Under hypothesis* (8), *for any point $t_1 \in \overset{\circ}{I}$, there is a $\mathscr{T}$-proper orbit $\mathscr{O} = (t_1, t_2, \dots)$ converging to one of the boundary points of I.*

Proof. If one of the sets $\mathscr{T}_j$ (say, $\mathscr{T}_2$) is empty, then the proof is obvious, namely, the orbit $\mathscr{O} = (t_1, \delta_1(t_1), \delta_1^2(t_1), \dots)$ is $\mathscr{T}$-proper and converges to the point $t = 1$. The convergence follows from the fact that the sequence $\delta_1(t_1), \delta_1^2(t_1), \dots)$ increases according to the right inequality in (ii) and is bounded. Let $\lim \delta_1^k(t_1) = \xi$ as $k \to \infty$. Then $\lim \delta_1^{k+1}(t_1) = \delta_1(\xi)$, and hence $\delta_1(\xi) = \xi$. By (ii), this results in relation $\xi = 1$.

In the subsequent part of the proof, the following fact plays a significant role.

Proposition 4. *If a periodic orbit $\mathscr{C}$ is a part of a $\mathscr{T}$-proper orbit $\mathscr{O} = (t_1, t_2, \dots)$, $t_1 \in \overset{\circ}{I}$, then $t_1 \in \mathscr{C}$ and $t_1 \neq 0$.*

In other words, $\mathscr{T}$-proper orbits have no points of self-intersection.

Proof of the proposition 4. Assume that $t_1 \notin \mathscr{C}$, and let t_q, $q \geqslant 2$, be the first point of the orbit $\mathscr{O}$ belonging to $\mathscr{C}$. Then $t_q \neq 0$ and $\mathscr{C} = (t_q, t_{q+1}, \dots, t_{q+m})$, where $t_q = t_{q+m}$. It is obvious that $t_{q-1} \neq t_{q+m-1}$ since $t_{q-1} \notin \mathscr{C}$, but $t_{q+m-1} \in \mathscr{C}$. Let $t_q = \delta_{j_1}(t_{q-1})$ and let $t_{q+m} = \delta_{j_2}(t_{q+m-1})$. Since $t_q = t_{q+m} \neq 0$, hypothesis (iii) implies that $j_1 = j_2$. Denote a common value of these indices by j. By virtue of condition (i), $\delta_j(t) = \text{const}$ for all t, $t_{q-1} \leqslant t \leqslant t_{q+m-1}$. By the definition of the

*If U is a neighborhood of a point x, then $U \setminus \{x\}$ is the corresponding *deleted* neighborhood of this point.

sets $\mathscr{T}_1$ and $\mathscr{T}_2$, this means that $t_{q-1} \in \mathscr{T}_{j'}$ for $j' \neq j$,, whence it follows that the orbit (t_{q-1}, t_q) can not be $\mathscr{T}$-proper. This proves the proposition as the point $t_1 = 0$ does not belong to any cycle in virtue of hypothesis (iii).

Continuing the proof of Lemma 3, we note that, as follows from inequality (4), for a sufficiently small $\varepsilon > 0$, there are some deleted neighborhoods U_1^ε and U_2^ε of the points $t = -1$ and $t = 1$, respectively, such that $U_1^\varepsilon \cap \mathscr{T}_1 = \varnothing$ and $U_2^\varepsilon \cap \mathscr{T}_2 = \varnothing$. It is evident that if $t \in U_1^\varepsilon$ or $t \in U_2^\varepsilon$, then the orbit $\mathscr{O}_- = \{t, \delta_2(t), \delta_2^2(t), \dots\}$ or $\mathscr{O}_+ = \{t, \delta_1(t), \delta_1^2(t), \dots\}$, respectively, turns out to be $\mathscr{T}$-proper and, as was proved above, converges to the point $t = -1$ or $t = 1$, respectively.

It is easily seen that for the above-mentioned number ε, there is a number ν such that the relations $\delta_1^\nu(t) \in U_2^\varepsilon$ and $\delta_2^\nu(t) \in U_1^\varepsilon$ are valid for all points $t \in \overset{\circ}{I}$. Indeed, for example, in the case of δ_1, the existence of an integer $\nu > 0$ such that $\delta_1^\nu(-1) > 1 - \varepsilon$ is a consequence of hypothesis (ii). The required relation now follows immediately from the monotonicity condition (i).

Take an arbitrary point τ_0 between τ_1 and τ_2 (which is possible according to condition (4)). Let V_1 and V_2 be open disjoint sets in I forming an open covering of the closed set $\mathscr{T}' = \mathscr{T}_1' \cup \mathscr{T}_2'$ and satisfying the following conditions:

1) $V_1 \supset \mathscr{T}_1'$, $V_2 \supset \mathscr{T}_2'$; $V_1 \cap \mathscr{T}_2 = \varnothing$, $V_2 \cap \mathscr{T}_1 = \varnothing$,

2) the set V_1 (V_2) is located on the right (on the left) of the point τ_0.

Such a covering undoubtedly exists. Let $V = V_1 \cup V_2$. It should be mentioned that the set $W = \mathscr{T} \backslash V$ is finite. Denote by W the number of points in W. Let us agree to say that an arbitrary $\mathscr{T}$-proper orbit $\mathscr{O} = (t_1, t_2, \dots)$ *is directed* if

$$\delta_{j_k} = \begin{cases} \delta_1 & \text{for } t_k \notin \mathscr{T}_2 \text{ and } t_k \geqslant \tau_0, \\ \delta_2 & \text{for } t_k \notin \mathscr{T}_1 \text{ and } t_k < \tau_0 \end{cases}$$

in (3). It is clear that *any directed orbit is uniquely determined by its starting point.* Furthermore, a point t_n will be called a *turning point* of an orbit $\mathscr{O} = (t_1, t_2, \dots)$ if $t_n = \delta_j(t_{n-1})$ and $t_{n+1} = \delta_{j'}(t_n)$, where $j' \neq j$. It can be immediately verified that, in every directed orbit $\mathscr{O} = (t_1, t_2, \dots)$, each turning point $t_N \in V$ follows a point $t_{N-1} \in W$, and the same is true for any turning point $t_N \in I \setminus \mathscr{T}$. Since $I = V \cup (I \setminus \mathscr{T}) \cup W$, what has been said means that the number of turning points in an orbit $\mathscr{O}$ does not exceed the number W if *this orbit contains no periodic suborbits.* Therefore it becomes clear that the terminal point in the *directed* orbit $\mathscr{O}_1 = (t_1, \dots, t_N)$ with $N = \mathsf{W}\nu + \mathsf{W} + \nu$ lies outside the interval $(-1 + \varepsilon, 1 - \varepsilon)$. Consequently, one of the sewed orbits $\mathscr{O}_1 = (t_1, \dots, t_N, \delta_j(t_N), \delta_j^2(t_N), \dots)$ with $j = 1$ or $j = 2$ possesses all the properties postulated in Lemma 3.

We now turn to the situation where *a directed orbit $\mathscr{O}$ of a point $t_1 \in \overset{\circ}{I}$ includes some cycle $\mathscr{C}$.* By Proposition 4, this cycle contains the point t_1, and consequently it has the form $\mathscr{C} = (t_1, \dots, t_m)$, where $t_m = t_1$. Since $\mathfrak{N}_\delta^{\mathscr{T}} = \varnothing$, the orbit $\mathscr{C}$ is not $\mathscr{T}$-guided. Therefore, one of its points t_q, $1 \leqslant q \leqslant m - 1$, does not belong to the set $\mathscr{T}$. Assume that $t_{q+1} = \delta_j(t_q)$. We introduce a new point $\hat{t}_{q+1} = \delta_{j'}(t_q)$ with $j' \neq j$ and define the directed orbit $\mathscr{O}_1 = (\hat{t}_{q+1}, \hat{t}_{q+2}, \dots)$. Since this orbit is $\mathscr{T}$-proper, it has no common points with the periodic orbit $\mathscr{C}$. Indeed, if it is not true, $\hat{t}_p$ is the first point of this kind, and we have $\hat{t}_p = t_r$, $1 \leqslant r \leqslant m - 1$, then, as was shown in the proof of Proposition 4, $\hat{t}_{p-1} = t_{r-1}$. However, this is impossible since $t_{r-1} \in \mathscr{C}$, whereas $\hat{t}_{p-1} \notin \mathscr{C}$. By virtue of the same Proposition 4, the $\mathscr{T}$-proper orbit $(t_q, \hat{t}_{q+1}, \hat{t}_{q+2}, \dots)$ contains no periodic suborbits. As above, this enables us to conclude that the sewed orbit

$$\mathscr{O}_j = (t_1, t_2, \dots, t_q, \hat{t}_{q+1}, \dots, \hat{t}_{q+N}, \delta_j(\hat{t}_{q+N}), \delta_j^2(\hat{t}_{q+N}), \dots),$$

where $N = \mathsf{W}\nu + \mathsf{W} + \nu$ and $j = 1$ $(j = 2)$ if $\hat{t}_{q+N} > \tau_0$ $(\hat{t}_{q+N} \leqslant \tau_0)$, satisfies all the conditions stated in Lemma 3. This completes the proof of the lemma.

Proof of Theorem 2. 1°. Let $M = \max_I F$ and let $\mathscr{M} = \{t \in I \mid F(t) = M\}$. If $F(\hat{t}) = M$, then $F(\delta_1(\hat{t})) = M - \varepsilon_1$ and $F(\delta_2(\hat{t})) = M - \varepsilon_2$ for some nonnegative numbers ε_1 and ε_2. Substituting $\hat{t}$ for t into Eq. (7), we obtain $a_1(\hat{t})\varepsilon_1 + a_2(\hat{t})\varepsilon_2 = 0$. It follows that if $\hat{t} \in \mathscr{M} \setminus \mathscr{T}$, then

$\varepsilon_1 = \varepsilon_2 = 0$, and hence $\delta_1(\hat{t}) \in \mathscr{M}$, $\delta_2(\hat{t}) \in \mathscr{M}$. And if $\hat{t} \in \mathscr{M} \cap \mathscr{T}_1$, then $\varepsilon_1 = 0$, and consequently $\delta_1(\hat{t}) \in \mathscr{M}$. In just the same way, if $\hat{t} \in \mathscr{M} \cap \mathscr{T}_2$, then $\delta_2(\hat{t}) \in \mathscr{M}$. Comparing these observations with the definition of a $\mathscr{T}$-proper orbit (see Sec. 2), we conclude that, along with every point $t_1 \in \mathscr{M}$, the next point t_2 of *any* $\mathscr{T}$-proper orbit (t_1, t_2) also belongs to $\mathscr{M}$. Since δ_1 and δ_2 are maps of I into itself, this argument can be applied to the point t_2. As a result, we obtain one or two points t_3 of the form $t_3 = \delta_{j_2}(t_2) = \delta_{j_2} \circ \delta_{j_1}(t_1)$ lying in the set $\mathscr{M}$. In this case, the orbit $\mathscr{O} = (t_1, t_2, t_3)$ turns out to be $\mathscr{T}$-proper. Arguing in the same way, we conclude that, *along with every point $t_1 \in \mathscr{M}$, its every $\mathscr{T}$-proper orbit $\mathscr{O} = (t_1, t_2, \dots)$ lies in the set $\mathscr{M}$.* By Lemma 3, at least one of these orbits converges either to the point $t = 1$ or to the point $t = -1$. By virtue of the continuity of the function F, one of the numbers $F(1)$, $F(-1)$ is equal to M. Note that what has been said about the spread of the maximal value of a solution along $\mathscr{T}$-proper orbits remains true for the minimal value m of the same solution. As a result, one of the numbers $F(1)$ and $F(-1)$ is equal to m, and this completes the proof of part 1° of Theorem 2.

2°. Let F be an arbitrary solution of Eq. (7) and let $M = \max F = F(\hat{t}) > 0$. If $\hat{t} \in \mathscr{B}$, then, as it follows from (7), $M \leqslant a_1(\hat{t})M + a_2(\hat{t})M < M$, which is impossible. Thus, any solution F takes the positive maximal value only at points of the set $I \setminus \mathscr{B}$. In just the same way, we establish that F takes its negative minimal value on the set $I \setminus \mathscr{B}$ only. To prove the concluding assertion of the Theorem 2, it remains to repeat literally the proof of part 1°. This completes the proof of the theorem.

Proof of Theorem 1. Note that any solution of Eq. (7) satisfies the "boundary" conditions

$$F(-1) = F(1) = F(0). \tag{10}$$

To show this it suffices to substitute consecutively the numbers -1 and 1 for t in Eq. (7) and use relations (2), (5), and also condition (6). To prove the assertion 1°, we note that, by the maximum principle, each of the numbers $M = \max F$ and $m = \min F$ coincides with one of the numbers $F(1)$, $F(-1)$. Combining this with (10), we arrive at the relation $m = M$, whence $F = \text{const}$.

In the case 2°, by part 2° of Theorem 2, any solution F of Eq. (7) can take neither positive maximal nor negative minimal values. But this is possible only if $F \equiv 0$. This completes the proof of Theorem 1.

The following assertion can be interpreted as some oscillation property of solutions to Eq. (7).

Theorem 5. *If the set $\mathscr{B}$ is nonempty, then, under hypotheses* (6) *and* (8)*, any solution of Eq.* (7) *does not change its sign on I. In addition, a nonnegative solution attains its minimum only at points of the set $\mathscr{B}$, and the same is true for the maximum of a nonpositive solution.*

Proof. Assume that a solution F of problem (7) takes both positive and negative values. Then

$$M = \max F > 0 \quad \text{and} \quad m = \min F < 0.$$

By Theorem 2, the function F takes the same values on the boundary ∂I, and, in addition, $\partial I \subset I \setminus \mathscr{B}$. However, the latter is impossible in view of (10).

Assume now that $F \geqslant 0$ on I, and let $F(\breve{t}) = m$. If $\breve{t} \in I \setminus \mathscr{B}$, then, repeating the proof of Theorem 2, we conclude that one of the numbers $F(1)$ or $F(-1)$ is equal to m. By virtue of the same theorem, the number M also coincides either with $F(-1)$ or with $F(1)$. But then, in view of (10), we have $M = m$, and hence, $F \equiv \text{const}$ which is impossible if $\mathscr{B} \neq \varnothing$. Thus, $\breve{t} \in \mathscr{B}$, and this completes the proof of the theorem.

4. Solvability of the Nonhomogeneous Equation (1)

In this section we consider the problems of the existence of solutions to the nonhomogeneous equation (1) under hypothesis (9) and in conditions of its violation.

Theorem 6. *Let the set $\mathscr{B}$ contain a nonempty deleted neighborhood of the boundary ∂I. Then under hypothesis* (8)*, for an arbitrary function $h \in C(I)$, Eq.* (1) *has a unique solution $F \in C(I)$.*

Proof. We introduce a linear operator L of the form

$$L\colon F(t) \mapsto a_1(t)F(\delta_1(t)) + a_2(t)F(\delta_2(t))$$

in the space $C(I)$, and let E be the identity map. As is well known, if the norm $\|L^m\|$ of the operator L^m is less than 1 for some positive integer m, then the operator $E - L$ is invertible. Therefore, to prove Theorem 6, it suffices to show that such a number m exists in the case of the operator L in question. We first of all note that, for an arbitrary positive integer N, the function $L^N F$ can be represented in the form

$$L^N F(t) = \sum_{j_1,\dots,j_N=1}^{2} a_{j_1}(t)a_{j_2}(\delta_{j_1}(t))\cdots a_{j_N}(\delta_{j_{N-1}}\circ\cdots\circ\delta_{j_1}(t))F(\delta_J(t)), \tag{11}$$

where $J = (j_1,\dots,j_N)$. Indeed, it is obvious if $N = 1$. To apply the induction on N, we assume relation (11) to be fulfilled for some N and prove it for $N+1$. From the definition of the operator L and from (11), it follows that

$$\begin{aligned}(L^{N+1}F)(t) = (L\circ L^N)F(t) &= \sum_{j_0=1}^{2} a_{j_0}(t)(L^N F)(\delta_{j_0}(t))\\ &= \sum_{j_0,j_1,\dots,j_N=1}^{2} a_{j_0}(t)a_{j_1}(\delta_{j_0}(t))\cdots a_{j_N}(\delta_{j_{N-1}}\circ\cdots\circ\delta_{j_0}(t))F(\delta_J(t)),\end{aligned}$$

where $J = (j_0, j_1,\dots,j_N)$. It remains to note that substituting j_{k+1} for j_k, $k = 0,\dots,N$, in this relation yields relation (11) with $N+1$ instead of N.

As follows from (11), for an arbitrary function F with $\|F\| = 1$, the inequality

$$|L^N F(t)| \leqslant \sum_{j_1,\dots,j_N=1}^{2} a_{j_1}(t)a_{j_2}(\delta_{j_1}(t))\cdots a_{j_N}(\delta_{j_{N-1}}\circ\cdots\circ\delta_{j_1}(t)) \tag{12}$$

holds at each point $t \in I$. Let us prove that, for any fixed $t \in I$, there is a positive integer N and a constant $\gamma < 1$ such that the inequality

$$|L^N F(t)| < \gamma \tag{13}$$

is fulfilled for the same functions F. If $t \in \mathscr{B}$, then the assertion is true for $N = 1$ according to the definition of the set $\mathscr{B}$. If $t \notin \mathscr{B}$, then we consider an arbitrary $\mathscr{T}$-proper orbit $\mathscr{O} = (t_1, t_2, \dots)$ with $t_1 = t$ converging to one of the boundary points of I. The existence of such an orbit is proved in Lemma 3. By the hypotheses of the theorem, for some sufficiently large natural N, the point $t_N = \delta_{j_{N-1}}\circ\ldots\circ\delta_{j_1}(t)$ lies inside the set $\mathscr{B}$, and hence

$$a_1(t_N) + a_2(t_N) < 1 \quad\text{and}\quad a_{j_1}(t)a_{j_2}(\delta_{j_1}(t))\cdots a_{j_{N-1}}(\delta_{j_{N-2}}\circ\cdots\circ\delta_{j_1}(t)) \neq 0. \tag{14}$$

The second relation in (14) follows from the definition of a $\mathscr{T}$-proper orbit. Let us show that inequality (13) holds for the same N. Introduce the notation $a_{j_k}(\delta_{j_{k-1}}\circ\cdots\circ\delta_{j_1}(t)) = a_{j_1\dots j_k}$, $k = 1,\dots,N$, where $\delta_{j_0}(t) = t_1$. It is convenient to consider the right-hand side of inequality (12) as an N-linear form

$$Q_N = \sum_{j_1,\dots,j_N=1}^{2} a_{j_1}a_{j_1j_2}\cdots a_{j_1\dots j_N}$$

of nonnegative variables $a_{j_1\dots j_k}$, $k = 1,\dots,N$, satisfying the condition $\sum_{j_k=1}^{2} a_{j_1\dots j_k} \leqslant 1$ for all k. It is evident that $0 < Q_N \leqslant 1$ for all $N > 0$. It can be verified easily that if, for at least one multi-index $J = (j_1,\dots,j_{N-1})$, the relations

$$a_{j_1}a_{j_1j_2}\cdots a_{j_1\dots j_{N-1}} \neq 0 \quad\text{and}\quad \sum_{j_N=1}^{2} a_{j_1\dots j_N} < 1 \tag{15}$$

are valid, then $Q_N < 1$. The assertion is obvious if $N = 2$. We will show that it is true for $N+1$ if it is valid for N. To this end, we write down the form Q_{N+1} as $Q_{N+1} = a_1 Q_N^{(1)} + a_2 Q_N^{(2)}$, where $Q_N^{(1)}$ and $Q_N^{(2)}$ are the corresponding N-linear forms of the variables $a_{1j_2\dots j_k}$ and $a_{2j_2\dots j_k}$, $k = 2,\dots,N+1$. It is easily verified that if conditions (15) hold for the form Q_{N+1}, then, for at least one index $j = 1$ or $j = 2$, the form $Q_N^{(j)}$ satisfies the same conditions and $a_j \neq 0$. Since $Q_N^{(j)} < 1$ by the induction hypothesis, the inequality $Q_{N+1} < 1$ becomes obvious because $a_1 + a_2 \leqslant 1$ and $Q_N^{(j')} \leqslant 1$ for $j' \neq j$. To complete the proof of inequality (13), it remains to note that condition (14) is nothing other than inequalities (15).

Furthermore, note that, by virtue of the continuity of all the functions in question, inequality (13) holds at all points of some neighborhood U of the point under consideration for the same number N, probably with a larger constant $\gamma < 1$. The collection of these neighborhoods forms an open covering of the closed set $I \setminus \mathscr{B}$. Let $\{U_j\}_{j=1}^k$ be a finite subsystem of these neighborhoods and let N_j and γ_j be the corresponding constants. Setting $m = \max N_j$ and $\gamma = \max \gamma_j$, we arrive at the desired inequality $\|L^m\| < 1$, and this completes the proof of Theorem 6.

A wide class of Eqs. (1) with the same solvability properties as those postulated in Theorem 6 is described in the following proposition.

Corollary 7. *Let $(0, \varepsilon_1, \varepsilon_2, \dots)$ be an arbitrary $\mathscr{T}$-proper orbit of the point $\varepsilon_0 = 0$. The result of Theorem 6 remains true if an arbitrary point ε_k, $k \geqslant 1$, is substituted for ∂I in conditions of this theorem.*

A simple proof of this fact is based on Lemma 3 and Theorem 6, and we omit it.

To conclude this section, we state another result relating to the solvability of Eq. (1) and, in some sense, complementing Theorem 5.

Theorem 8. *Let $a_1(t) + a_2(t) = 1$ everywhere on I and let the conditions (6) and (8) be fulfilled. Then Eq. (1) has no solution if the right-hand side h does not change sign on I and is nonzero in an arbitrarily small deleted neighborhood of the boundary ∂I.*

Proof. Let F be a solution to equation (1) for a given function h, and let $F(\hat{t}) = M$ and $F(\check{t}) = m$ be the maximal and minimal values of F, respectively. Consecutive substitution of $\hat{t}$ and $\check{t}$ for t in (1) results in the inequalities $h(\hat{t}) \geqslant 0$ and $h(\check{t}) \leqslant 0$ as in the proof of Theorem 2. Thus, the maximal and minimal values of the function h can not be of the same sign. Assume that $h \geqslant 0$ on I and $h > 0$ in a deleted neighborhood U of the boundary ∂I. Then $\min h = 0$ and, in particular, $h(\check{t}) = 0$. As follows from the proof of Theorem 2, the minimal value m of a solution F spreads along $\mathscr{T}$-proper orbits of the point $\check{t}$. Consequently, the function h vanishes on any such orbit. By virtue of Lemma 3, $h = 0$ at points lying arbitrarily close to the boundary ∂I. But this contradicts the choice of the function h, which completes the proof of Theorem 8.

Remark. Comparing results of Theorem 5 and Theorem 8, we see that the nonhomogeneous equation (1) may be solvable only if the right hand-side h oscillates, whereas the homogeneous equation (1) has no oscillating solutions (i.e., those whose values have different signs).

5. On the Solvability Theory of the Cauchy Type Functional Equations

One of the first functional equations was considered by Cauchy. He proved that if a continuous (at one point) function $F(z)$, $|z| \leqslant 1$, satisfies the condition

$$F(x+y) = F(x) + F(y) \tag{16}$$

at all points (x, y) of the square $K = \{(x,y) \mid |x \pm y| \leqslant 1\}$, then $F(z) = \lambda z$.

In this section, as one of the applications of the results in Secs. 3–4, we will study the solvability problem for the functional equation

$$F((\beta_1(t) + \beta_2(t)) - F(\beta_1(t)) - F(\beta_2(t)) = h(t). \tag{17}$$

Here, as in Sec. 2, β_1 and β_2 are given continuous maps of I into itself and h and F are given and unknown real functions on I, respectively. Note that these equations are of interest not only by themselves, but they are also a necessary technical tool for solving some integral equations (see Sec. 6) and also for studying some boundary problems for hyperbolic partial differential equations of higher (> 2) order in a bounded domain (see Sec. 7). The nonhomogeneous equation (17) has never been considered earlier. Let Γ be a nonsingular curve in the plane $\mathbb{R}^2$ and let

$$x = \beta_1(t), \quad y = \beta_2(t), \qquad t \in I,$$

be its parametric representation. Then, by analogy with Eq. (16), it is natural to call Eq. (17) a *Cauchy type functional equation* (*on a curve* Γ). Such an approach to Eq. (17) will make it possible to treat the main result of this section as an unexpected sharpening of the above-mentioned Cauchy theorem relating to Eq. (16) (see Corollary 10).

We now turn to the precise statement of the problem. Consider a linear operator B of the form

$$B\colon F(t) \mapsto F((\beta_1 + \beta_2)(t)) - F(\beta_1(t)) - F(\beta_2(t))$$

in the space $C(I)$, where β_1 and β_2 are the above C^2-functions on I satisfying conditions (i)–(iii) (see Sec. 2) and also the inequalities*

$$(\beta_1' + \beta_2')(t) > 0, \quad t \in I, \qquad \beta_1'\beta_2' > 0 \quad \text{on } \partial I. \tag{18}$$

It follows that the map $\beta = \beta_1 + \beta_2$ is a diffeomorphism on I preserving the boundary ∂I. As a consequence, the maps

$$\delta_1 = \beta_1 \circ \beta^{-1} \quad \text{and} \quad \delta_2 = \beta_2 \circ \beta^{-1}$$

of I into itself have all the properties of the maps also denoted by δ_1 and δ_2 that were introduced in Sec. 2 and, in addition, satisfy the relation

$$\delta_1' + \delta_2' = 1 \quad \text{on } I. \tag{19}$$

The sets $\mathscr{T}_j$ of all points $t \in I$ such that $\delta_j'(t) = 1$, $j = 1, 2$, are said to be *guiding*, and we write $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. It is clear that the functions δ_1' and δ_2' have all the properties of the functions a_1 and a_2 (see Sec. 2) including property (v). As in Sec. 2, for the two maps δ_1 and δ_2 of I into itself and for the subsets $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ in I, it is possible to introduce the semigroup Φ_δ and all the corresponding geometric notions related to Φ_δ and $\mathscr{T}$. In particular, $\mathfrak{N}_\delta^{\mathscr{T}}$ denotes the set of all periodic $\mathscr{T}$-proper orbits entirely lying in $\mathscr{T}$.

It is obvious that the kernel of the operator B contains all linear functions. On the other hand, the cokernel of B is also nonempty (see (20)). Therefore, to obtain the best possible results related to the solvability of Eq. (17), it is desirable to establish some necessary conditions for this solvability beforehand. Let us substitute consecutively the values $t = -1$ and $t = 1$ in (17). Using relations (2) (which hold for the functions β_j as well), we conclude that

$$BF(-1) = BF(1) = -F(0) \tag{20}$$

for any function $F \in C(I)$. This means that the range of the operator B consists only of the functions h taking equal values at the boundary points of I. Moreover, an unknown function F must be related to the given function h by the condition

$$F(0) = -h(-1) = -h(1).$$

Taking into account that the solution $F(z) = \lambda z$ of Eq. (17) with $h = 0$ is uniquely determined by setting the value of $F(1)$, we arrive at the following natural problem: *given some numbers* λ, $\mu \in \mathbb{R}$ *and an arbitrary function* $h \in C^2(I)$ *satisfying the condition*

$$h(1) = h(-1) = -\mu, \tag{21}$$

to find a function $F \in C^2(I)$ *such that*

$$BF = h \text{ on } I, \quad F(0) = \mu, \quad F(1) = \lambda. \tag{22}$$

The main result of this section is the following theorem.

*Under some different assumptions about the functions β_1 and β_2, Eq. (17) was also studied in [4].

Theorem 9. *Suppose that condition* (4) *holds and* $\mathfrak{N}_\delta^{\mathscr{T}} = \varnothing$. *Then, for all constants* λ *and* μ *and for an arbitrary function* $h \in C^2(I)$ *satisfying condition* (21), *there exists a unique solution* $F \in C^2(I)$ *of problem* (22). *The inverse operator* $h \mapsto F$ *from* $C^2(I)$ *to* $C^2(I)$ *is continuous.*

Remark. The assertion of the theorem remains true if the condition $F(1) = \lambda$ in problem (22) is replaced by the condition $F'(0) = \lambda$. This leads to some insignificant changes in the proof.

Proof. Replacing F by $F + t(\lambda - \mu) + \mu$ in problem (22), we arrive at the problem

$$BF = h_\mu \ \text{ on } I, \quad F(0) = 0, \quad F(1) = 0, \tag{23}$$

where $h_\mu = h + \mu$. The change of variable $t \to \beta^{-1}(t)$ in the equation entering (23) reduces problem (23) to the form

$$\widehat{B}F(t) := F(t) - F(\delta_1(t)) - F(\delta_2(t)) = \hat{h}_\mu(t), \qquad F(0) = 0, \quad F(1) = 0. \tag{24}$$

This form is precisely the starting point in studying problem (22). As follows from relations (20) and (21), both functions $\widehat{B}F$ and $\hat{h}_\mu$ vanish at the points $t = -1$ and $t = 1$. Therefore, differentiating the equation $\widehat{B}F(t) = \hat{h}_\mu(t)$ twice, we obtain two new problems which are *equivalent* to problem (24). These new problems are

$$B_1F'(t) := F'(t) - \delta_1'(t)F'(\delta_1(t)) - \delta_2'(t)F'(\delta_2(t)) = \hat{h}_\mu'(t), \tag{25}$$

$$B_2F''(t) := F''(t) - \delta_1'^2(t)F''(\delta_1(t)) - \delta_2'^2(t)F''(\delta_2(t)) - KF''(t) = \hat{h}_\mu''(t), \tag{26}$$

where K denotes a linear operator in $C(I)$,

$$K\colon H \mapsto \delta_1''(t)\int_\xi^{\delta_1(t)} H(s)\,ds + \delta_2''(t)\int_\xi^{\delta_2(t)} H(s)\,ds.$$

Here ξ is an arbitrary point of the interval $(0,1)$ such that $F'(\xi) = 0$. The existence of such point ξ is guaranteed by the boundary conditions. For brevity, we omit the conditions $F(0) = 0$ and $F(1) = 0$ in (25) and (26).

Since all the three problems in question are equivalent, the theorem will be proved if we establish the injectivity of the operator B_1 and also show that the index $\operatorname{ind} B_2$ of the operator B_2 in the space $C(I)$ is zero.

To prove the injectivity of the operator B_1 in the space of functions F vanishing at the points $t = 0$ and $t = 1$, we set $\hat{h}_\mu' = 0$ in (25). By virtue of relation (19), the new equation is none other than the homogeneous equation (1) whose coefficients a_j coincide with the coefficients δ_j', $j = 1, 2$. Therefore, the equation in question satisfies all the hypotheses of Theorem 1. By virtue of part 1° of this theorem, we conclude that the derivative F' is a constant, and this constant is zero since the function F vanishes at two points.

Passing to the operator B_2, we represent it in the form $B_2 = E - L - K$, where L denotes a linear operator of the form

$$L\colon H \mapsto \delta_1'^2 H \circ \delta_1 + \delta_2'^2 H \circ \delta_2$$

in the space $C(I)$. As follows from (19), the inequality $\delta_1'^2 + \delta_2'^2 < 1$ holds everywhere on I except at points of the set $\mathscr{T}$. Therefore, the equation $G - LG = h$ is none other than Eq. (1) whose coefficients a_j coincide with $\delta_j'^2$, $j = 1, 2$, and hence satisfy condition (9). In this case, the set $I \setminus \mathscr{T}$ plays the role of the set $\mathscr{B}$, and, in view of (18), the condition $\partial I \subset \mathscr{B}$ also holds for the a_j's in question. Thus, Theorem 6 can be applied to the equation under consideration, and we establish that the operator $E - L$ is invertible (although $\|L\| = 1$). For this reason, the operator B_2 turns out to be a sum of an invertible operator and a compact operator in the space $C(I)$. By the Riesz–Schauder theorem, it follows that $\operatorname{ind} B_2 = 0$. The unique solvability of problem (24) is now a consequence of the relation $\dim\ker B_2 = \dim\ker B_1 = 0$. It remains to prove the continuity of the inverse operator in problem (22). Clearly, the operator B from $C^2(I)$ to $C^2(I)$ is continuous. By the Banach closed graph theorem, it follows that the operator B^{-1} is continuous in the same space. This completes the proof of Theorem 9.

Corollary 10. *In the plane $\mathbb{R}^2$ with coordinates x and y, consider an arbitrary nondecreasing twice differentiable curve Γ joining the points $(0,-1)$ and $(1,0)$ and admitting a parametric representation of the form $x=\beta_1(t)$, $y=\beta_2(t)$, $-1\leqslant t\leqslant 1$, in which the functions β_1 and β_2 satisfy the hypotheses of Theorem 9. Then the Cauchy type functional equation on the curve Γ has no solutions distinct from $F(t)=\lambda t$.*

Thus, to determine the function $F(z)$ on the interval $\{z\mid -1\leqslant z\leqslant 1\}$, there is no need in the fulfillment of the Cauchy relation (16) at *all* points (x,y) of the square $K=\{(x,y)\mid |x\pm y|\leqslant 1\}$. It is sufficient (as it follows from Corollary 10) that the relation $F(x+y)=F(x)+F(y)$ hold at all points of a curve Γ of the above type. For example, the side

$$\Gamma=\{(x,y)\mid x=(t-1)/2,\ y=(t+1)/2\},\qquad -1\leqslant t\leqslant 1,$$

of the square K can play the role of this curve.*

6. On an Integral Equation Relating to Some Geometric Problem

The problem of reconstructing a function in a given domain D from the values of its integrals over a family $\{D_q\}$ of subdomains in D is always of interest not only as an object of pure analysis, but also in connection with various applications in practical disciplines. The most remarkable example of such a connection is the Radon problem and tomography. In this section, using the above results, we show that one of such problems can be solved in the case of a bounded domain D with a piecewise smooth boundary.

We begin with the statement of the general problem. Let ℓ_1 and ℓ_2 be some nonsingular smooth transversal vector fields in a disk $S\subset\mathbb{R}^2$. Consider a curvilinear triangle $D=OA_1A_2$ whose sides OA_1 and OA_2 coincide with trajectories of the vector fields ℓ_1 and ℓ_2, respectively, and the side $\Gamma=A_1A_2$ is an arbitrary nonsingular smooth curve transversal to both the fields ℓ_1 and ℓ_2 at the points A_1 and A_2. In addition, we assume the closure $\overline{D}$ of the domain D to satisfy the following hypotheses:

1) For any point $p\in\overline{D}$, a trajectory of ℓ_j passing through p meets OA_k at a point $\pi_k p$, $j\neq k$, $1\leqslant j,k\leqslant 2$.

2) The set $\overline{D}$ is ℓ_j-convex, $j=1,2$. This means that if given points p and q in $\overline{D}$ lie on some trajectory γ_j of the field ℓ_j, then all the points $r\in\gamma_j$, $j=1,2$, between p and q belong to $\overline{D}$.

Given an arbitrary point $q\in\Gamma$, let D_q be a curvilinear parallelogram qq_1Oq_2, where $q_j=\pi_j q$, $j=1,2$. Hypotheses 1) and 2) guarantee the inclusion $\overline{D}_q\subset\overline{D}$ for all $q\in\Gamma$. In the case under consideration, the above-mentioned geometric problem of reconstructing a function in $\overline{D}$ takes the form of the following integral equation:

$$Af:=\int_{D_q} f\,d\sigma=h(q),\qquad q\in\Gamma. \tag{27}$$

Here σ denotes a measure on S, $h(q)\in C(\Gamma)$ is a given function, and $f\in C(\overline{D})$ is an unknown function. The general problem stated in [1], reads as follows in application to the above equation: for what spaces of functions f and h is the map $A\colon f\mapsto h$ one-to-one, and what functions $h(q)$ can be represented by the integral (27)? As to the second question, it readily follows from (27) that any such function h belongs to the space $\mathscr{H}(\Gamma)=(C^2\cap C_0)(\Gamma)$ of all twice differentiable functions vanishing at the boundary points of Γ. Therefore, the best possible answer to the first question consists in describing a subspace $\mathscr{L}(D)$ of the space $C(\overline{D})$ for which the map $A\colon\mathscr{L}(D)\to\mathscr{H}(\Gamma)$ is one-to-one. Certainly, such subspaces can be determined in many ways. Here we will describe a class of such subspaces $\mathscr{L}(D)$ which naturally appear in studying the above-mentioned boundary problems for hyperbolic partial differential equations of the third order in a bounded domain (see Sec. 7).

*When this paper was already prepared for publication, the author learned that the result of Corollary 10 was obtained earlier in [5] by a quite different method and under absolutely different assumptions about the functions β_j.

Definition. Given a smooth nonsingular vector field ℓ in S, we denote by $C_{\langle\ell\rangle}(D)$ the subset of all functions in $C(\overline{D})$ which remain constant along each trajectory of the field ℓ.

For the sake of brevity, only the case of a vector field $\ell = r_1\ell_1 + r_2\ell_2$ with positive constants r_1 and r_2 will be studied in this paper*. However, in this situation, an exhaustive solution of the problem under consideration will be obtained, namely, we state a necessary and sufficient condition on Γ ensuring the single-valuedness of the operator $A\colon C_{\langle\ell\rangle}(D) \to \mathscr{H}(\Gamma)$.

To state this condition, in addition to the projections π_j, $j = 1, 2$, along the vector fields l_k, $j \neq k$, we consider the projection $\pi_\ell\colon \overline{D} \to \Gamma$ along the trajectories of the field ℓ. In other words, given an arbitrary point $p \in \overline{D}$, its projection $\pi_\ell p$ coincides with the intersection of the curve Γ and the trajectory of the field ℓ passing through p. Introduce two maps in Γ,

$$\zeta_1 = \pi_\ell \circ \pi_1 \quad \text{and} \quad \zeta_2 = \pi_\ell \circ \pi_2,$$

and denote by Φ_ζ the noncommutative semigroup of maps on Γ generated by ζ_1 and ζ_2, which is analogous to the semigroup Φ_δ considered in Sec. 2. As in Sec. 2, we define by means of the maps ζ_1 and ζ_2 an *orbit* in Γ as a sequence of points $(q_1, \dots, q_n, \dots)$ such that $q_{k+1} = \zeta_{j_k}(q_k)$, $k = 1, 2, \dots$. As above, we introduce the *guiding* sets

$$\mathscr{T}_j = \{q \in \Gamma \mid \ell_j(q) \in T_q(\Gamma)\}, \quad j = 1, 2, \quad \text{and} \quad \mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2,$$

where $T_q(\Gamma)$ denotes a tangent space to Γ at q. Repeating literally what was said in Sec. 2, we define periodic, $\mathscr{T}$-guided, and also $\mathscr{T}$-proper orbits corresponding to the semigroup Φ_ζ and denote by $\mathfrak{N}_\zeta^{\mathscr{T}}$ the set of $\mathscr{T}$-proper $\mathscr{T}$-guided cycles of the type in question. It is assumed by analogy with condition (4) that if both the sets $\mathscr{T}_1$ and $\mathscr{T}_2$ are infinite, then arbitrary points $\tau_1 \in \mathscr{T}_1'$ and $\tau_2 \in \mathscr{T}_2'$ are situated on Γ in the order A_1, τ_1, τ_2, A_2. The main result of this section is stated as the following assertion.

Theorem 11. *If all the above hypotheses related to the domain D, the curve Γ, and the vector fields ℓ, ℓ_1, and ℓ_2 are fulfilled, just as the conditions on the sets $\mathscr{T}_j$, $j = 1, 2$, figuring in Theorem 1, then, for an arbitrary function $h \in \mathscr{H}(\Gamma)$, Eq. (27) has a unique solution $f \in C_{\langle\ell\rangle}(D)$ if and only if $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$. The inverse operator $A^{-1}\colon h \mapsto f$ is a continuous operator from $\mathscr{H}(\Gamma)$ to $C_{\langle\ell\rangle}(D)$.*

Proof. Without loss of generality, we assume that the vector fields ℓ_1 and ℓ_2 are parallel to the coordinate axes x_1 and x_2, respectively, and consider as D a curvilinear triangle whose two sides coincide with the intervals $\{x_1 \mid 0 \leqslant x_1 \leqslant 1\}$ and $\{x_2 \mid 0 \leqslant x_2 \leqslant 1\}$ on the axes x_1 and x_2, respectively. As to the third side Γ, it is assumed to be a smooth nonsingular curve transversal to the coordinate axes. Let $x_1 = \alpha_1(z)$, $x_2 = \alpha_2(z)$, $z \in I$, be an arbitrary parametric representation of Γ and let $\alpha(z) = (\alpha_1(z),\, \alpha_2(z))$. It can be shown that, by virtue of the topological conditions 1) and 2) on the domain D, these functions $\alpha_1(z)$ and $\alpha_2(z)$ satisfy the conditions

$$\alpha_1'(z) \geqslant 0, \quad \alpha_2'(z) \leqslant 0 \quad \text{and} \quad |\alpha_1'(z)| + |\alpha_2'(z)| > 0, \quad z \in I, \tag{28}$$

provided that $\alpha(1) = (1, 0)$. We also introduce the function $\omega(x) = r_2 x_1 - r_1 x_2$ in $\overline{D}$ and denote by ω_Γ its restriction to Γ. The role of the function ω becomes clear if we note that it is constant on trajectories of the field ℓ. Therefore, any function $g \in C_{\langle\ell\rangle}(D)$ can be represented in the form $g = f \circ \omega$, where f is some continuous function on the closed interval $I_t = [-r_1, r_2]$. In the following we will also use the functions $\omega_1(x) = \omega(x_1, 0)$ and $\omega_2(x) = \omega(0, x_2)$. Note that the function $\sigma = \omega_\Gamma \circ \alpha\colon I \to I_t$ is invertible since $\sigma'(z) > 0$ according to (28). In the coordinate form, Eq. (27) looks as follows:

$$\int_0^{\alpha_1(z)} \int_0^{\alpha_2(z)} f(\omega(x))\, dx_2\, dx_1 = h(z), \qquad z \in I. \tag{29}$$

We introduce the new unknown function $F(t) = -\int_0^t f(s)(t-s)ds/r_1 r_2$. Substituting F for f in (29) we arrive (after some routine calculations) at the following functional equation for the

*In the case of variables r_1 and r_2, the problem in question is studied in [3].

function F:

$$F(\omega \circ \alpha) - F(\omega_1 \circ \alpha) - F(\omega_2 \circ \alpha) = h. \tag{30}$$

Since $F''(t) = -f(t)/r_1 r_2$ and $F(0) = F'(0) = 0$, the map $f \mapsto F$ is one-to-one. Therefore, to prove the theorem, it is sufficient to establish the unique solvability of equation (30). We introduce the new maps

$$\rho_1 = \omega_1 \circ \alpha \circ \sigma^{-1} \quad \text{and} \quad \rho_2 = -\omega_2 \circ \alpha \circ \sigma^{-1}$$

of I_t into itself and rewrite Eq. (30) in the form

$$F(t) - F(\rho_1(t)) - F(\rho_2(t)) = h(\sigma^{-1}(t)), \qquad t \in I_t. \tag{31}$$

It is remarkable that the functions ρ_1 and ρ_2 intrinsically connected with geometric problem (27) possess all the properties of β_1 and β_2 postulated in the previous section. In addition, since $\rho_1(t) + \rho_2(t) = t$ for all $t \in I_t$, Eq. (31) coincides with Eq. (17) and hence is a Cauchy type functional equation on the interval I_t. Denote by Φ_ρ the semigroup of maps in I_t generated by ρ_1 and ρ_2 and by O_ρ the corresponding set of orbits. Furthermore, we introduce the guiding sets

$$\mathscr{T}_{\rho_1} = \{t \in I_t \mid \rho_1'(t) = 1\}, \quad \mathscr{T}_{\rho_2} = \{t \in I_t \mid \rho_2'(t) = 1\}, \quad \text{and} \quad \mathscr{T}_\rho = \mathscr{T}_{\rho_1} \cup \mathscr{T}_{\rho_2}.$$

We can now define $\mathscr{T}_\rho$-proper, periodic, and $\mathscr{T}_\rho$-guided orbits in the standard way. Denote by $\mathfrak{N}_\rho^{\mathscr{T}_\rho}$ the subset in O_ρ consisting of all orbits that are simultaneously $\mathscr{T}_\rho$-proper, periodic, and $\mathscr{T}_\rho$-guided. By the constancy of the function ω on the trajectories of the vector field ℓ, the relation $\omega_j \circ \alpha = \omega_\Gamma \circ \zeta_j \circ \alpha$, $j = 1, 2$, holds. Substituting these relations into the formulas determining the functions ρ_1 and ρ_2, we conclude that $\rho_j = \omega_\Gamma \circ \zeta_j \circ \omega_\Gamma^{-1}$ for the same j. It follows immediately that $\rho_J = \omega_\Gamma \circ \zeta_J \circ \omega_\Gamma^{-1}$ for any multi-index J. Consequently, *if* $(q_1, \ldots, q_{n+1})$ *is a* $\mathscr{T}$*-proper* ($\mathscr{T}$*-guided*) *orbit in* Γ*, then its* ω_Γ*-image is a* $\mathscr{T}_\rho$*-proper* ($\mathscr{T}_\rho$*-guided*) *orbit in* I_t. This leads to the conclusion that the sets $\mathfrak{N}_\zeta^{\mathscr{T}}$ and $\mathfrak{N}_\rho^{\mathscr{T}_\rho}$ can be empty only simultaneously. Therefore, it follows from the hypotheses of Theorem 11 that $\mathfrak{N}_\rho^{\mathscr{T}_\rho} = \varnothing$. To complete the proof of solvability in Theorem 11, it remains to use the result in Theorem 9. To prove the continuity of the inverse operator A^{-1}, it suffices to use the definition of function F and the continuity of the operator $h(\sigma^{-1}(t)) \mapsto F(t)$ in $C^2(I)$ defined by Eq. (31).

It remains to prove the necessity of the hypothesis $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$. If $\mathfrak{N}_\zeta^{\mathscr{T}} \neq \varnothing$, then $\mathfrak{N}_\rho^{\mathscr{T}_\rho} \neq \varnothing$ and, consequently, there is a $\mathscr{T}_\rho$-proper periodic orbit $\mathscr{O} = (t_1, \ldots, t_{n+1})$ entirely lying in the critical set $\mathscr{T}_\rho$. We replace the function $h(\sigma^{-1}(t))$ in Eq. (31) by $H(t)$ and differentiate the resulting relation. Setting $F' = G$, we arrive at the relation

$$G(t) - \rho_1'(t)G(\rho_1(t)) - \rho_2'(t)G(\rho_2(t)) = H'(t). \tag{32}$$

We now note that, by definition, the points $t_1, \ldots, t_{n+1}$ satisfy the conditions

$$t_{k+1} = \rho_{j_k}(t_k), \quad \rho'_{j'_k}(t_k) = 0 \quad \text{for} \quad j'_k \neq j_k \text{ and } k = 1, \ldots, n$$

and $t_{n+1} = t_1$. Let us substitute t_1 for t in Eq. (32). Then one of the numbers $\rho_j'(t_1)$ is equal to zero, whereas $\rho'_{j'}(t_1)$, $j' \neq j$, is equal to unity. Furthermore, by the definition of $\mathscr{T}$-properness, $t_2 = \rho_{j'}(t_1)$. This results in the relation $G(t_1) - G(t_2) = H'(t_1)$. Continuing this procedure, we arrive at the chain of relations $G(t_2) - G(t_3) = H'(t_2), \ldots, G(t_n) - G(t_1) = H'(t_n)$, where the periodicity of the orbit $\mathscr{O}$ is used on the terminal step. Adding all these relations together, we obtain

$$\sum_{j=1}^{n} H'(t_j) = 0. \tag{33}$$

Thus, if there is a $\mathscr{T}$-proper $\mathscr{T}$-guided cycle $\mathscr{O} = (q_1, \ldots, q_n)$, this relation is a necessary condition for the solvability of Eq. (27). Hence, the necessity of the condition $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$ is proved, and this completes the proof of Theorem 11.

Remark. The above result makes it possible to interpret the set of orbits $\mathfrak{N}_\zeta^{\mathscr{T}}$ as an obstruction to constructing a solution of Eq. (27).

7. On the Solvability of Some Boundary Problems for Hyperbolic Partial Differential Equations with Data on the Entire Boundary

As another application of the results relating to Eq. (1), we consider, in the (x, y)-plane, an arbitrary homogeneous x-strictly hyperbolic differential operator $P(\partial_x, \partial_y)$ of the third order with constant coefficients. Such an operator can be uniquely represented in the form

$$P(\partial_x, \partial_y) = a(\partial_x - a_1\partial_y)(\partial_x - a_2\partial_y)(\partial_x - a_3\partial_y),$$

where all coefficients a, a_1, a_2, and a_3 are real numbers and $a_j \neq a_k$ if $j \neq k$. The characteristics of the operator P are the straight lines $y + a_1x = \text{const}$, $y + a_2x = \text{const}$, and $y + a_3x = \text{const}$. Denote by ℓ_1, ℓ_2, and ℓ_3 the related vector fields in $\mathbb{R}^2$ parallel to these lines. Let OA_1, OA_2, and OA_3 be an arbitrary triple of *neighboring* characteristic rays issuing from some point O (there are 6 rays of this kind). Assume that the ray OA_3 lies between OA_1 and OA_2.

Take a curvilinear triangle $D = OA_1A_2$ with the sides OA_1, OA_2, and $\Gamma = A_1A_2$, where Γ is a smooth nonsingular curve transversal to OA_1 and OA_2. The closure $\overline{D}$ is assumed to satisfy hypotheses 1) and 2) in Sec. 6. Consider the following boundary problem: *given some functions $F \in C(\overline{D})$ and $h \in C(\partial D)$, to find a function u such that*

$$P(\partial_x, \partial_y)u = F \quad \text{in } D, \qquad u = h \quad \text{on } \partial D. \tag{34}$$

To state the main result, we introduce the semigroup Φ_ζ of maps in Γ considered in Sec. 6 with $\ell = \ell_3$. In the new situation, the guiding sets $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ turn out to be none other than characteristic subsets in Γ. Introduce the set $\mathfrak{N}_\zeta^{\mathscr{T}}$ of all $\mathscr{T}$-proper periodic orbits of Φ_ζ consisting of characteristic points. Denote by $C^k(\partial D)$ the set of all continuous functions on ∂D whose restrictions to all sides of the triangle OA_1A_2 are k times continuously differentiable, $k > 2$.

Theorem 12 (cf. [2]). *Assume that the characteristic sets $\mathscr{T}_1$ and $\mathscr{T}_2$ satisfy the hypothesis stated before Theorem 11. Then, for any functions $F \in C^k(\overline{D})$ and $h \in C^{k+2}(\partial D)$, $k \geqslant 1$, there is a unique solution $u \in C^{k+2}(\overline{D})$ of problem (34) if and only if $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$. The inverse operator $(F, h) \mapsto u$ is bounded in the corresponding pair of the spaces.*

Proof. We restrict ourselves to the case $F = 0$. It is clear that a linear change of variables reduces (34) to the problem

$$(r_1\partial_x + r_2\partial_y)\partial_x\partial_y u = 0 \quad \text{in } D, \qquad u = h \quad \text{on } \partial D \tag{35}$$

with $r_1r_2 > 0$. The boundary ∂D now consists of three parts,

$$\Gamma_1 = \{(x, y) \mid y = 0,\ 0 \leqslant x \leqslant 1\}, \quad \Gamma_2 = \{(x, y) \mid x = 0,\ 0 \leqslant y \leqslant 1\}, \quad \text{and}$$
$$\Gamma = \{(x, y) \mid x = \alpha_1(t),\ y = \alpha_2(t),\ -1 \leqslant t \leqslant 1\},$$

where $\alpha_1(-1) = 0$, $\alpha_1(1) = 1$, $\alpha_2(-1) = -1$, and $\alpha_2(1) = 0$. Let $h = h_1(x)$ on Γ_1, $h = h_2(y)$ on Γ_2, and $h = h_3(x, y)$ on Γ. By the continuity of the function h on Γ, the following compatibility conditions are fulfilled:

$$h_1(0) = h_2(0), \quad h_1(1) = h_3(1, 0), \quad h_2(1) = h_3(0, 1), \tag{36}$$

Using the postulated properties of the domain D, it is easy to verify that the function

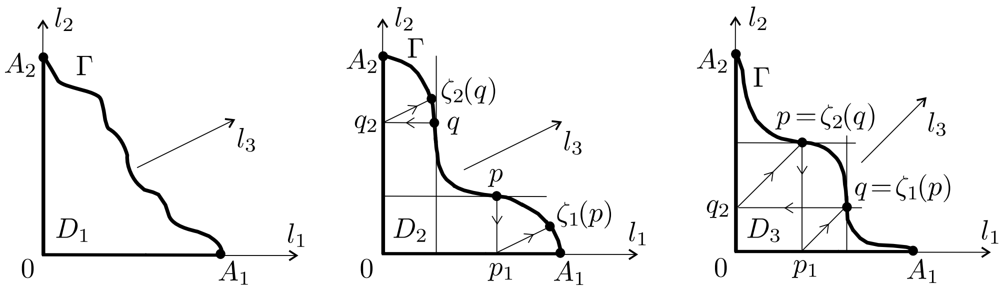
$$u(x, y) = \int_0^x \left(\int_0^y F(r_2s - r_1t)\, dt \right) ds + h_1(x) + h_2(y) - h_1(0), \quad 0 \leqslant x, y \leqslant 1, \tag{37}$$

satisfies the equation in (35) and the condition $u = h$ on the part $\Gamma_1 \cup \Gamma_2$ of the boundary for *any* function $F \in C^1(-r_1, r_2)$. Since it is necessary that the condition $u = h_3$ hold on Γ, we now arrive at the integral equation

$$\int_0^{\alpha_1(t)} \left(\int_0^{\alpha_2(t)} F(r_2x - r_1y)\, dy \right) dx = H(t), \qquad -1 \leqslant t \leqslant 1. \tag{38}$$

Here, as in Sec. 6, the relations $x = \alpha_1(t)$ and $y = \alpha_2(t)$, $t \in I$, describe the curve Γ in parametric form, and $H(t) = -h_1(\alpha_1(t)) - h_2(\alpha_2(t)) + h_3(\alpha_1(t), \alpha_2(t)) + h_1(0)$. It is important to note that the function H generated by *an arbitrary* k times piecewise differentiable functions h belongs to the space $\mathscr{H}(I) = (C^2 \cap C_0)(I)$. This follows from (36). The converse is also true, namely, the function u defined by formula (37) is a solution of problem (35). Thus, this problem turns out to be equivalent to the integral equation (38), which is none other than Eq. (29). The unique solvability of problem (35) for an arbitrary function h of the required form under the hypothesis $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$ follows directly from Theorem 11.

To illustrate the above result, we consider problem (35) in the domains D_1, D_2, and D_3 (see below).



In these figures, only the points p and q are characteristic. It is easy to verify that $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$ for the domains D_1 and D_2, and hence, problem (35) is well posed in D_1 and in D_2. On the other hand, in the case of D_3, the set $\mathfrak{N}_\zeta^{\mathscr{T}}$ contains the (unique) characteristic $\mathscr{T}$- proper periodic orbit (p, q, p). That is why Theorem 12 is inapplicable here.

In conclusion, we show that, at least in the case of a finite characteristic set $\mathscr{T}$, the condition $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$ of well-posedness of problem (34) is *typical*. This means that, in the space of C^1-curves Γ of the form

$$\Gamma = \{(x, y) \mid x = \delta_1(t),\ y = \delta_2(t),\ t \in I\},$$

where the functions δ_1 and δ_2 are described in Sec. 2, the curves Γ satisfying the above condition form a set of the second category in some complete metric space.

If one of the sets $\mathscr{T}_1$ and $\mathscr{T}_2$ is empty, then the generic character of the condition $\mathfrak{N}_\zeta^{\mathscr{T}} = \varnothing$ is obvious as well as the same character of the noncharacteristicness condition. Let us examine the simplest case of one-point sets $\mathscr{T}_1 = \{p\}$ and $\mathscr{T}_2 = \{q\}$ (see the domains D_2 and D_3 above). By definition, in this situation, the set $\mathfrak{N}_\zeta^{\mathscr{T}}$ contains at most one orbit $\mathscr{O} = (p, q, p)$ (the orbit (q, p, q) is naturally identified with $\mathscr{O}$). Therefore, the problem under consideration can now be restated in the following way.* Let $C^1 = C^1(I) \times C^1(I)$ and let

$$B = \{(\delta_1, \delta_2) \in C^1 \mid \delta_1' \geqslant 0,\ \delta_2' \geqslant 0,\ \delta_1'(s) = \delta_2'(t) = 0\},$$

where the sets of the corresponding values of s and t may change in the passage from one pair (δ_1, δ_2) to another. It is clear that the set B is closed in C^1, and hence B is a complete metric space. Introduce the set $W \subset B$ consisting of the functions (δ_1, δ_2) such that, for each of them, there is a unique point $(a, b) \in I \times I$ with $\delta_1'(a) = \delta_2'(b) = 0$. Finally, consider the set $V \subset W$ of those pairs (δ_1, δ_2) for which $\delta_2(a) = b$ and $\delta_1(b) = a$. It can be verified easily that W is dense in B and that V is a set of the first category in B. We now show that $W \setminus V$ is a set of the second category in B (and this solves the problem in question). It suffices to show that W is a G_δ-type set. This follows from the representation $B \setminus W = \bigcup_{n=1}^{\infty} A_n$, where each set

$$A_n = \{(\delta_1, \delta_2) \in B \mid \operatorname{diam}\{t \in I \mid \delta_1'(t) = 0\} \geqslant \tfrac{1}{n} \text{ or } \operatorname{diam}\{t \in I \mid \delta_2'(t) = 0\} \geqslant \tfrac{1}{n}\}$$

is obviously closed because $W = \bigcap_{n=1}^{\infty} (B \setminus A_n)$.

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Department of Mathematics, Technion
Haifa, Israel
e-mail: peter@techunix.technion.ac.il

Translated by B. P. Paneah