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DYNAMIC METHODS IN THE GENERAL THEORY OF CAUCHY TYPE FUNCTIONAL EQUATIONS

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1. Introduction. Definitions and Discussions

In the last 30-40 years, the study of functional equations has grown to be a large, independent branch of mathematics with its own methods and circle of problems and abounding in applications. If, at an early stage of development (XVIII - XIX centuries), functional equations played an auxiliary, perhaps even decorative role, describing in abstract form various fundamental functions from analysis, nowadays functional equations turn out to be a powerful tool when solving *analytical problems* in quite different fields of mathematics. Moreover, sometimes these equations arise as an adequate description of such problems. Then, having solved the corresponding functional equation, we thus solve the original problem. In this paper, we deal with precisely the latter situation.

One of the first functional equations was studied by Cauchy. He formulated the following problem: find all continuous functions $F(z)$ on $\mathbb{R}$ such that the equality

$$F(x+y) = F(x) + F(y) \tag{1}$$

holds for all points $(x, y) \in \mathbb{R}^2$. The solution of this problem is not complicated. From (1), it follows that

$$F\left(\sum_{j=1}^{k} z_j\right) = \sum_{j=1}^{k} F(z_j) \tag{2}$$

for arbitrary $k \geq 2$ and $z_j \in \mathbb{R}$. Setting $F(1) = \lambda$ and substituting $z_1 = \ldots = z_k = 1$, we get from (2)

$$F(k) = \lambda k.$$

If $z_1 = \ldots = z_k = 1/k$ in (2), then we arrive at the equality

$$F(1/k) = \lambda/k.$$

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From the last two equalities it immediately follows that for all integers m and n

$$F(m/n) = \lambda m/n,$$

and by continuity we find that

$$F(z) = \lambda z, \qquad z \in \mathbb{R}.$$

If we are interested in a function $F(z)$ defined not on the whole line but only on the interval $I = \{z : -1 \le z \le 1\}$, then it suffices that the equality (1) be valid at all points of the square

$$K = \{(x, y) : |x \pm y| \le 1\}.$$

The previous arguments then lead to the unique solution $F(z) = \lambda z,\ \ z \in I$.

The other Cauchy equation closely connected to (1) is that determining the exponential function, namely

$$F(x + y) = F(x)F(y), \ \ (x, y) \in \mathbb{R}^2. \tag{3}$$

Let F be a nonzero function. If $y = x$ in (3) then $F(2x) = (F(x))^2$ and hence $F(z) \ge 0$. If $F(y_0) = 0$ for some y_0 then $F(x + y_0) = 0$ for *all* x. Consequently $F(z) > 0$. But then the function

$$G(z) = \ln F(z), \ \ z \in \mathbb{R}$$

satisfies equation (1). Therefore,

$$G(z) = \lambda z \quad \text{ and } \quad F(z) = e^{\lambda z}.$$

Consider now a pair of continuous maps δ_1 and δ_2 in I [1] and let

$$\mathcal{D} = I \cup \mathcal{R}[\delta_1 + \delta_2],$$

where $\mathcal{R}[f]$ denotes the range of a map f.

Definition. Given a function $H : \ I \to \mathbb{R}$, the equation

$$F \circ (\delta_1 + \delta_2) - F \circ \delta_1 - F \circ \delta_2 = H \tag{4}$$

with F an unknown continuous function: $\mathcal{D} \to \mathbb{R}$ is called a *Cauchy type functional equation.*

Let Γ be a continuous nonsingular curve in the plane $\mathbb{R}^2$ with parametric representation

$$\Gamma = \{x = (x_1, x_2) : x_1 = \delta_1(t), \ x_2 = \delta_2(t); \ \ t \in I\},$$

where

$$\delta_1(-1) = \delta_2(-1) = 0. \tag{5}$$

If $H = 0$, then equation (4) under the name "the Cauchy equation on the curve Γ" was carefully investigated in the 70s - 80s in a series of works, starting with the pioneer paper of Zdun [**11**]; see [**1**] and [**2**] for references. The main goal of

[1] In what follows the words "a map A in I" mean (as it is usual in operator theory) "a map A of I into itself"

these works was to prove that (as in the case of equation (1)) the linear function $F(z) = \lambda z$ is the only solution of the homogeneous equation (4).

Note that condition (5) means from a geometrical point of view that the point $(-1, 0)$ in the (t, z)-plane is the end point of all three curves

$$z = \delta_1(t), \qquad z = \delta_2(t), \qquad z = \delta_1(t) + \delta_2(t). \tag{6}$$

Quite recently it was clarified (see [**4**]-[**9**]) that several problems in such diverse fields as integral geometry and boundary problems for partial differential equations can be reduced (sometimes in equivalent manner) to certain Cauchy type functional equations. What is characteristic is that in the latter case the corresponding curves (6) in the (t, z)-plane form a configuration not satisfying hypothesis (5). For convenience we fix this difference in configurations in the following definition.

Definition. Let

$$I = \{t : -1 \le t \le 1\} \qquad \text{and} \qquad \overset{\circ}{I} = \{t : -1 < t < 1\}.$$

We say that real-valued maps $\beta_1, \beta_2, \ldots, \beta_n$ of I form a *Z-configuration* if the functions $\beta_1, \ldots, \beta_n$ are all nondecreasing and

$$\beta_1(-1) = \ldots = \beta_n(-1) = 0.$$

We say that two maps β_1 and β_2 in I form a *$\mathcal{P}$-configuration* if both functions are nondecreasing,

$$\beta_1\beta_2(t) \ne 0 \quad \text{in} \quad \overset{\circ}{I},$$

and

$$\beta_1(-1) = \beta_2(1) = 0, \quad \beta_1(1) = 1, \quad \beta_2(-1) = -1. \tag{7}$$

Figures 1 and 2 represent typical examples of Z- and $\mathcal{P}$-configurations, respectively, $n = 2$. Dotted lines in both figures represent the graphs of functions $z = \beta_1(t) + \beta_2(t)$.

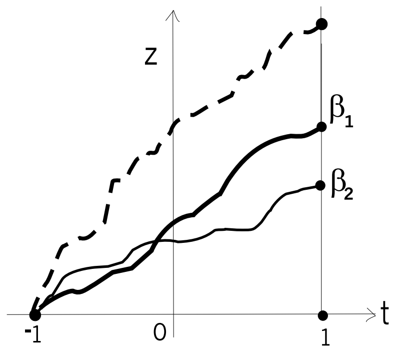

Figure 1

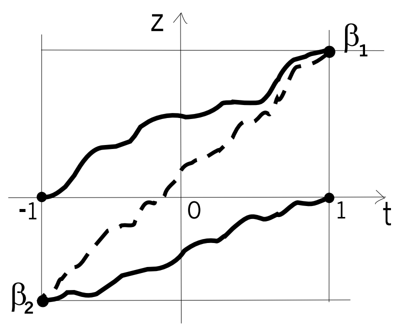

Figure 2

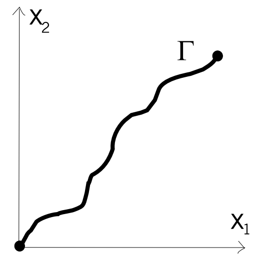

Figure 1′

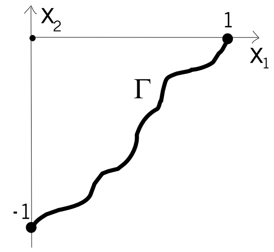

Figure 2′

Figures 1′ and 2′ represent the curves $\Gamma = \{(x_1, x_2) : x_1 = \beta_1(t), x_2 = \beta_2(t), t \in I\}$, corresponding to Z- and $\mathcal{P}$-configurations of pairs β_1, β_2, respectively.

It should be noted that when dealing with equation (4) a transition from a Z- to a $\mathcal{P}$-configuration leads to significant complications of the proofs even in the homogeneous situation. In particular, some new dynamic methods were worked out to overcome difficulties which arose. A new maximum principle for functional equations plays a significant role when studying homogeneous Cauchy equations. All these results and methods are the subject of this survey. It is worth mentioning that the new approach leads to completely new results also when curves (6) form a Z-configuration. It is interesting that the Cauchy type functional equation in the latter situation also turns out to be equivalent to a problem in integral geometry.

Our methods are admissible when dealing with sufficiently smooth functions (both given and unknown). It would be interesting to weaken restrictions of such type.

In conclusion, I thank both referees for their careful reading of the manuscript. Their questions and remarks allowed me to correct some formulations and to eliminate a number of misprints.

2. Solvability of the Cauchy Type Functional Equations

2.1. The case of a $\mathcal{P}$-configuration. In this subsection, we describe the results related to the unique solvability of the functional equation

$$F(\beta_1(t) + \beta_2(t)) - F(\beta_1(t)) - F(\beta_2(t)) = H(t), \quad t \in I \tag{8}$$

in the case when the functions $\beta_1(t)$ and $\beta_2(t)$ form a $\mathcal{P}$-configuration and satisfy a nondegeneracy condition

$$\beta_1'(t) + \beta_2'(t) > 0. \tag{9}$$

By virtue of hypotheses (7) and (9), the map $\beta = \beta_1 + \beta_2$ in I is a diffeomorphism preserving the boundary ∂I. Therefore, the maps

$$\delta_1 = \beta_1 \circ \beta^{-1} \qquad \text{and} \qquad \delta_2 = \beta_2 \circ \beta^{-1}$$

form a $\mathcal{P}$-configuration on I and, in addition, satisfy the condition

$$\delta_1(t) + \delta_2(t) = t, \qquad t \in I. \tag{10}$$

Thus, the change of variable $t \to \beta^{-1}(t)$ reduces equation (8) in equivalent manner to the form

$$\mathcal{B}F(t) := F(t) - F\big(\delta_1(t)\big) - F\big(\delta_2(t)\big) = H(t), \qquad t \in I. \tag{11}$$

It turns out that essential information related to the solvability of this equation may be derived by means of new dynamical methods, introduced in the author's papers [**4**],[**5**],[**7**]–[**9**]. The application of these methods becomes possible if we associate equation (11) with the semigroup Φ_δ of maps in I generated by δ_1 and δ_2. On the one hand, in terms of orbits of this semigroup, a necessary and sufficient condition for the existence of a unique solution to equation (11) is easily formulated. On the other hand, an essential part of the proof of both existence and uniqueness of a solution is based on the existence of very specific attractors of a noncommutative dynamical system generated by the semigroup Φ_δ.

We now turn to the exact formulations. We denote by Φ_δ the noncommutative semigroup of maps in I generated by δ_1 and δ_2. The elements of Φ_δ are maps in I of the form

$$\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1},$$

where $J = (j_1, \ldots, j_n)$ is an arbitrary n-tuple with all $j_k = 1$ or 2. The semigroup Φ_δ naturally generates a dynamical system. In what follows, we make use of the following geometric terminology related to Φ_δ.

(i) Given a map $\delta_J \in \Phi_\delta$, an ordered set $\mathcal{O} = (t_1, \ldots, t_{n+1})$ of points in I is called an *orbit* if

$$t_{k+1} = \delta_{j_k}(t_k) \quad \text{for} \quad 1 \le k \le n \le \infty. \tag{12}$$

Introduce the *guiding* sets

$$\mathcal{T}_1 = \{t \in I \mid \delta_2'(t) = 0\}, \quad \mathcal{T}_2 = \{t \in I \mid \delta_1'(t) = 0\}, \qquad \text{and} \qquad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2.$$

(ii) An orbit $\mathcal{O} = (t_1, \ldots, t_{n+1})$, $n = 1, 2, \ldots$, is called $\mathcal{T}$-*proper* if in (12)

$$\delta_{j_k} = \delta_1 \quad \text{when } t_k \in \mathcal{T}_1 \quad \text{and} \quad \delta_{j_k} = \delta_2 \quad \text{when } t_k \in \mathcal{T}_2.$$

(iii) If all the points of an orbit belong to the set $\mathcal{T}$, then the orbit is called $\mathcal{T}$-*guided*.

(iv) If the end points of an orbit $\mathcal{O}$ coincide, i.e. $t_{n+1} = t_1$, then the orbit $\mathcal{O} = (t_1, \ldots, t_{n+1})$ is called *periodic* or, in short, *cycle*.

Definition. We denote by $\mathfrak{N}_\delta^{\mathcal{T}}$ the set of all $\mathcal{T}$-proper $\mathcal{T}$-guided cycles in I.

Before formulating the main result related to the solvability of equation (11), we note that the kernel of the operator $\mathcal{B}$ contains all linear functions, as will be seen later on, the cokernel of $\mathcal{B}$ is also nonempty. This makes us look for some necessary conditions for solvability. To this end, substitute in (11) consecutively $t = -1$ and $t = 1$. As relations (7) remain true for the functions δ_1 and δ_2, we obtain

$$\mathcal{B}F(-1) = \mathcal{B}F(1) = -F(0). \tag{13}$$

for an arbitrary function $F \in C(I)$. It follows that the range of the operator $\mathcal{B}$ consists of functions H whose values at boundary points of I are equal. Moreover, the unknown function F has to be connected with a given function H by the equality

$$F(0) = -H(-1) = -H(1).$$

Taking into account that solutions $F(z) = \lambda z$ of the homogeneous equation (11) are uniquely determined by the value $F(1)$, we arrive at the following natural problem:

given numbers $\lambda, \mu \in \mathbb{R}$ and an arbitrary function $H \in C^2(I)$ satisfying the condition

$$H(1) = H(-1) = -\mu, \tag{14}$$

find a function $F \in C^2(I)$ such that

$$\mathcal{B}F = H \quad \text{on } I, \qquad F(0) = \mu, \quad F(1) = \lambda. \tag{15}$$

Denote by $\mathcal{T}_j'$ the sets of limit points of the sets $\mathcal{T}_j,\ j = 1, 2$. Now we are ready to formulate the first result.

Theorem 1. (*see* [**7**]). *Suppose that either at least one of the sets $\mathcal{T}_j,\ j = 1, 2$, is finite or*

$$\min_{\mathcal{T}_1'} t > \max_{\mathcal{T}_2'} t. \tag{16}$$

If

$$\mathfrak{N}_\delta^{\mathcal{T}} = \emptyset, \tag{17}$$

then, given arbitrary constants $\lambda,\ \mu$, and function $H \in C^2(I)$ satisfying condition (14)*, there is a unique solution $F \in C^2(I)$ of problem* (15)*. The inverse operator $H \mapsto F$ is continuous: $C^2(I) \to C^2(I)$.*

Remark. The assertion remains valid if we replace the boundary condition $F(1) = \lambda$ by the condition $F'(0) = \lambda$.

We now draw the reader's attention to the following interesting fact closely connected with the Cauchy equation (1).

In the plane $\mathbb{R}^2$ of variables (x, y) consider an arbitrary nondecreasing twice differentiable curve Γ which connects points $(0, -1)$ and $(1, 0)$ and admits a parametric representation of the form

$$x = \delta_1(t), \qquad y = -\delta_2(t), \qquad -1 \le t \le 1,$$

(see Fig. 2′). If these functions δ_1 and δ_2 satisfy conditions (16) and (17), then by Theorem 1 the homogeneous equation (11) has no solutions from $C^1(I)$ except for $F(z) = \lambda z,\ |z| \le 1$.

Thus, in order to determine a linear function F on the interval I the Cauchy equality (1) need not be valid for *all* points (x, y) of the square

$K = \{(x, y) : |x \pm y| \leq 1\}$. It suffices that the equality $F(x + y) = F(x) + F(y)$ be valid for all points of some above-mentioned *curve* Γ. For instance, a side

$$x = (t - 1)/2, \quad y = (t + 1)/2, \quad -1 \leq t \leq 1$$

of the square K can play the role of the curve Γ. Thus, the original Cauchy problem (1) turns out to be overdetermined (see also [**10**]).

In proving the uniqueness in Theorem 1, a maximum principle for functional equations of a rather general form plays a crucial role. This principle has been obtained for the first time in the author's papers [**7**],[**8**]. In the situation under consideration the corresponding assertion appears as follows.

Theorem 2. *If $G \in C(I)$ is a solution of equation*

$$G(t) - \delta_1'(t)G(\delta_1(t)) - \delta_2'(t)G(\delta_2(t)) = 0,$$

(*the differentiated homogeneous equation* (11)) *and the functions δ_1, δ_2 satisfy hypotheses* (16) *and* (17), *then G takes its maximum and minimum at the boundary ∂I.*

2.2. The case of a Z-configuration. In this subsection, we deal with the unique solvability of the functional equation

$$\mathcal{B}F(t) := F\left(\sum_{j=1}^{n} \beta_j(t)\right) - \sum_{j=1}^{n} F\big(\beta_j(t)\big) = H(t), \quad t \in I, \ \ n \geq 2, \tag{18}$$

where the functions $\beta_1, \ldots, \beta_n$ form a Z-configuration.

It is easily seen that the homogeneous equation $\mathcal{B}F = 0$ has the nontrivial solutions $F = \lambda z$, $\lambda \in \mathbb{R}$. Furthermore, substituting -1 for t in (18), we arrive at a necessary condition for the solvability of equation (18):

$$(1 - n)F(0) = H(-1), \quad n \geq 2.$$

On the other hand, the same procedure with the differentiated equation (18) leads to the equation (independent of F!)

$$H'(-1) = 0.$$

These observations enable us to formulate the following well-posed problem for equation (18).

Given numbers μ, $\lambda \in \mathbb{R}$ and a function $H \in C^2(I)$ satisfying conditions

$$H(-1) = \lambda, \qquad H'(-1) = 0, \tag{19}$$

find a function $F \in C^2(I)$ such that

$$\mathcal{B}F = H \quad \text{on } I, \qquad F(0) = \lambda/(1 - n), \qquad F'(0) = \mu. \tag{20}$$

Theorem 3. *Let $\beta_1, \dots, \beta_n$ be twice continuously differentiable functions on I satisfying the condition*

$$\sum_{\substack{j,k=1\\ j\neq k}}^{n} \beta_j'\beta_k' > 0 \quad \text{in a deleted neighborhood of the point } t=-1. \tag{21}$$

Let $\beta = \sum_{j=1}^n \beta_j$ and $\widehat{I} = [0, \beta(1)]$. Then if $H \in C^2(I)$ satisfies condition (19), *there exists a unique solution $F \in C^2(\widehat{I})$ of problem* (20).

The proof of the existence in this theorem is based (as in Theorem 1) on a dynamical approach and cannot be given in the framework of this short survey. But the proof of uniqueness is easier than in the case of $\mathcal{P}$-configuration. Indeed, let $H = 0$ and $\mu = \lambda = 0$ in (20). Then $\mathcal{B}F(0) = 0$, and consequently *the equations*

$$\mathcal{B}F = 0 \qquad \text{and} \qquad \frac{d}{dt}\mathcal{B}F = 0 \tag{22}$$

are equivalent. Introduce a new variable

$$z = \beta(t), \quad t \in I,$$

and let

$$\delta_j = \beta_j \circ \beta^{-1}$$

be new maps in $\widehat{I}$. Then $\sum \delta_j(z) = z$, and the second equation in (22) can be written in the form

$$G(z) - \sum_{j=1}^{n} \delta_j'(z) G\big(\delta_j(z)\big) = 0, \quad z \in \widehat{I}. \tag{23}$$

with $G = F'$. It is clear that $\delta_j'(z) \geq 0,\ 1 \leq j \leq n$, and

$$\sum_{j=1}^{n} \delta_j'(z) = 1. \tag{24}$$

Therefore, any constant solves equation (23). Let us show that there are no other solutions. Take an arbitrary solution $G(z)$ of (23) and let

$$\max_{\widehat{I}} G(z) = M.$$

Let $\mathcal{M} = \{z \in \widehat{I} : G(z) = M\}$. We show that $0 \in \mathcal{M}$. Indeed, let

$$T = \min\{z : z \in \mathcal{M}\}.$$

It is clear that $T \in \mathcal{M}$. If $T \neq 0$, substitute T for z in (23). Making use of (24), we obtain $G(\delta_{j'}(T)) = M$ for some indices j' (those for which $\delta_{j'}'(T) \neq 0$). If $\delta_{j'}(T) = T$, then $\delta_j(T) = 0$ for $j \neq j'$. It follows that $\delta_j(z) \equiv 0$ for $z \leq T$, and hence $\sum_{j\neq k} (\delta_j'\delta_k')(z) = 0$ in a neighborhood of $z = 0$. But this contradicts hypothesis (21). Therefore, $\delta_{j'}(T) < T$ in contradiction with the definition of T.

Repeating the same arguments with minimum instead of maximum results in the equality

$$\min_{\widehat{I}} G(z) = G(0).$$

Thus $\min G = \max G$, and therefore $G \equiv \text{const}$. As $F'(0) = 0$, it follows that $G = 0$. This proves the uniqueness in Theorem 3.

2.3. Multiplicative Cauchy type functional equations. In this brief subsection we are concerned with "nonhomogeneous" *multiplicative Cauchy functional equations*, i.e., functional equations of the form

$$F \circ \sum_{j=1}^{n} \beta_j = \left(\prod_{j=1}^{n} F \circ \beta_j \right) H, \tag{25}$$

where $H > 0$ is a given function and the functions $\beta_1, \ldots, \beta_n$ form one of the abovementioned configurations. When $H \equiv 1$, only the case of Z-configuration and $n = 2$ has been studied (see [**11**]).

Let the functions $\beta_1, \ldots, \beta_n$ on I form a Z-configuration and satisfy hypothesis (21). Repeating word for word the concluding arguments in Sec. 2.2, we conclude that with the exception of the zero solution, *only positive functions F may solve equation* (25). But then this equation is equivalent to the equation

$$(\ln F) \circ \sum_j \beta_j = \sum_j (\ln F) \circ \beta_j + \ln H,$$

which is nothing but equation (18). Consequently, applying Theorem 3 leads to the following result.

THEOREM 4. *Given real numbers μ and $\lambda > 0$, and an arbitrary positive function $H \in C^2(I)$ with $H(-1) = \lambda$, $H'(-1) = 0$, equation* (25) *has a unique solution $F \in C^2(\widehat{I})$ satisfying conditions*

$$F(0) = \lambda^{1/(1-n)} \qquad \textit{and} \qquad F'(0) = \mu\lambda^{1/(1-n)}.$$

Turn now to equation (25) with $n = 2$ and functions β_1, β_2 forming a $\mathcal{P}$-configuration. The main novelty here compared with a Z-configuration is that even the simplest equation

$$F = (F \circ \delta_1)(F \circ \delta_2) \qquad \text{in } I$$

with $\delta_1(t) + \delta_2(t) = t$ may have oscillating solutions. Nevertheless, the following assertion is valid.

THEOREM 5. (*see* [**7**]). *Given positive numbers $\lambda, \mu \in \mathbb{R}$ and an arbitrary positive function $H \in C^2(I)$ satisfying the condition*

$$H(1) = H(-1) = \mu,$$

there is a unique positive solution $F \in C^2(\widehat{I})$ of equation (25) *such that*

$$F(0) = 1/\mu \qquad \textit{and} \qquad F\big(\beta(1)\big) = e^{\lambda}.$$

3. Problems in Analysis Reducing to Cauchy Type Functional Equations

In this section, we discuss several problems in classical analysis which can be reduced (sometimes in an equivalent manner) to a Cauchy type functional equation. What is interesting is that none of these problems at first sight give even the merest hint of a connection with functional equations. But having solved a corresponding functional equation, we automatically solve the original problem. We trace this connection by considering two problems in integral geometry and in partial differential equations which were studied for the first time in the author's papers [**4**]-[**8**].

3.1. Some problems in integral geometry and Cauchy functional equations. A typical problem in integral geometry is to reconstruct a function in a domain D of $\mathbb{R}^n$ knowing its integrals over a family of subdomains in D. A peculiarity of the problem we deal with is that we consider bounded domains D with the boundary ∂D. The statement of this problem and the corresponding results turn out to be intimately connected with both local and global properties of ∂D. This connection is realized by means of a semigroup of maps of ∂D which we associate with the problem in question. But exactly the same situation arises in studying a Cauchy type functional equation (see Subsec. 2.1). It is no wonder: it will be proved below that *every Cauchy type functional equation is equivalent to* (*at least*) *two different problems in integral geometry.* These problems correspond to the two different configurations formed by the functions β_1 and β_2.

3.1.1. *The case of a $\mathcal{P}$-configuration*

3.1.1.1 Statement of the Problem

Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be smooth nonsingular transversal vector fields in a disk $B \subset \mathbb{R}^2$. Introduce a curvilinear triangle $D = OA_1A_2$ whose sides OA_1 and OA_2 are trajectories of the vector fields $\mathbf{l}_1$ and $\mathbf{l}_2$, respectively. The side $\Gamma = A_1A_2$ is assumed to be an arbitrary smooth curve without singularities which is transversal at its ends to $\mathbf{l}_1$ and $\mathbf{l}_2$. In addition, the closure $\overline{D}$ of a domain D is supposed to satisfy the following hypotheses.

1° For any point $p \in \overline{D}$, a trajectory of $\mathbf{l}_j$ passing through p meets OA_k, $k, j = 1, 2,\ k \neq j$, at a point $\pi_k p$.

2° The set $\overline{D}$ is $\mathbf{l}_j$-convex, $j = 1, 2$. This means that given points p and q on any trajectory γ_j of the field $\mathbf{l}_j$, all the points $r \in \gamma_j$ between p and q belong to $\overline{D}$.

Given an arbitrary point $q \in \Gamma$ let D_q be a curvilinear parallelogram qq_1Oq_2, where $q_j = \pi_j q,\ j = 1, 2$. The above conditions 1° and 2° guarantee the inclusion $\overline{D}_q \subset \overline{D}$ for all $q \in \Gamma$ (see Fig. 3).

In this subsection, we will deal with a solvability of an integral equation of the following form:

$$\int_{D_q} f d\sigma = h(q), \quad q \in \Gamma. \tag{26}$$

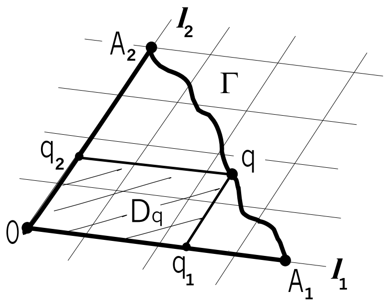

Figure 3

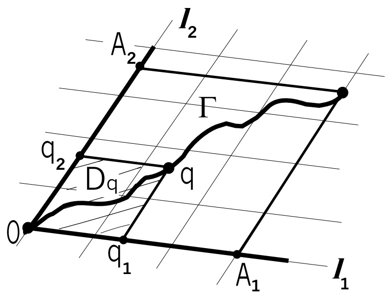

Figure 4

Here $d\sigma$ is a measure in B, $h(q) \in C(\Gamma)$ is a given function and $f \in C(\overline{D})$ is an unknown function.

The general problem formulated in [**3**] in connection with this equation is as follows: for which spaces of functions f and h is the map $f \mapsto h$ one-to-one, and which functions $h(q)$ can be represented by the integral (26)?

As to the second question, it follows from (26) that any such function h belongs to the space $\mathcal{H}(\Gamma) = (C^2 \cap C_0)(\Gamma)$ of twice continuously differentiable functions vanishing on the boundary $\partial\Gamma$. Therefore, the best possible solution of the problem consists in a description of spaces $\mathcal{F}(D) \in C(\overline{D})$ such that the map

$$\mathcal{A} : \mathcal{F}(D) \to \mathcal{H}(\Gamma)$$

is one-to-one. One possible class of such spaces is introduced below.

Definition. Given a smooth nonsingular vector field $\mathbf{l}$ in B, we denote by $C_{\langle \mathbf{l} \rangle}(D)$ the set of all functions in $C(\overline{D})$ which remain constant along any trajectory of the field $\mathbf{l}$.

In this paper, we consider the only case of vector fields

$$\mathbf{l} = r_1 \mathbf{l}_1 + r_2 \mathbf{l}_2, \quad r_1 r_2 > 0$$

with constant coefficients r_1 and r_2. In this situation, we obtain an exhaustive solution of the problem in question by formulating the necessary and sufficient condition for the curve Γ to ensure the above mentioned property of the operator

$$\mathcal{A} : C_{\langle \mathbf{l} \rangle}(D) \to \mathcal{H}(\Gamma).$$

Much more general vector fields $\mathbf{l}$ are considered in [**8**]. To formulate the above-mentioned condition, we introduce one more projection $\pi_{\mathbf{l}} : \overline{D} \to \Gamma$ along trajectories of the vector field $\mathbf{l}$. Let

$$\zeta_1 = \pi_{\mathbf{l}} \circ \pi_1 \qquad \text{and} \qquad \zeta_2 = \pi_{\mathbf{l}} \circ \pi_2$$

be two maps in Γ. Denote by Φ_ζ the noncommutative semigroup of maps in Γ, generated by ζ_1 and ζ_2. The analogy of Φ_ζ with semigroup Φ_δ considered in Subsec. 2.1 is obvious. As in the case of Φ_δ, we define an orbit in Γ as a sequence of points $(q_1, \dots, q_n, \dots)$ in Γ such that

$$q_{k+1} = \zeta_{j_k}(q_k), \quad k = 1, 2, \dots$$

and all ζ_{j_k} are equal ζ_1 or ζ_2. As above, the *guiding* sets [2]

$$\mathcal{T}_j = \{q \in \Gamma : \mathbf{l}_j(q) \in T_q(\Gamma)\}, \quad j = 1, 2,$$

and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ are introduced and condition (16) is supposed to be fulfilled. Repeating word for word what has been said in Subsec. 2.1, we define *periodic, $\mathcal{T}$-guided* and *$\mathcal{T}$ -proper orbits* corresponding to the semigroup Φ_ζ. Finally, we introduce the set $\mathfrak{N}_\zeta$ of all $\mathcal{T}$-proper periodic $\mathcal{T}$-guided orbits generated by Φ_ζ (see Fig.5 - Fig.7). Now everything is ready for the formulation of the main result of this subsection.

3.1.1.2 Main result and an outline of the proof

Theorem 6. *Assume that all above hypotheses concerning to domain D, curve Γ and vector fields $\mathbf{l}_1, \mathbf{l}_2$ and $\mathbf{l}$ are fulfilled as well as the conditions on the sets $\mathcal{T}_j'$, $j = 1, 2$, figuring in Theorem 1. Then given an arbitrary function $h \in \mathcal{H}(\Gamma)$, there is a unique solution $f \in C_{\langle \mathbf{l} \rangle}(D)$ of equation (26) if and only if $\mathfrak{N}_\zeta = \emptyset$. The inverse operator $h \mapsto f$ is continuous: $\mathcal{H}(\Gamma) \to C_{\langle \mathbf{l} \rangle}(D)$.*

Let us outline the proof of this theorem. First of all, choosing a special coordinate system (x_1, x_2) in the disk B, we reduce the integral equation (26) to the form

$$\int_0^{\alpha_1(z)} \int_0^{\alpha_2(z)} f(\omega(x)) dx_1 dx_2 = h(z), \qquad z \in I_z, \tag{27}$$

with f an unknown continuous function on the interval $I_t = \{t : -r_1 \le t \le r_2\}$. Here $I_z = \{z : -1 \le z \le 1\}$, $\alpha(z) = (\alpha_1(z), \alpha_2(z))$, and equalities $x_1 = \alpha_1(z)$, $x_2 = \alpha_2(z)$, $z \in I_z$, define a parametric representation of the curve Γ. It is clear that

$$\alpha_1'(z) \ge 0, \quad \alpha_2'(z) \le 0, \quad \text{and} \quad (\alpha_1' - \alpha_2')(z) > 0, \quad z \in I_z.$$

As to ω, this is a function

$$\omega(x) = r_2 x_1 - r_1 x_2$$

which does not change its values along trajectories of the vector field $\mathbf{l}$. Denote $\omega_1 = \omega(x_1, 0)$, $\omega_2 = \omega(0, x_2)$ and let

$$\sigma = \omega_\Gamma \circ \alpha : I_z \to I_t,$$

where ω_Γ is the restriction of ω to Γ. By the above, the function σ is invertible. Introducing a new unknown function

$$F(t) = -\int_0^t f(s)(t - s) ds / r_1 r_2, \quad t \in I_t,$$

[2] $T_q(\Gamma)$ denotes as usual the tangent space of Γ at a point q.

we arrive at the functional equation

$$F(\omega \circ \alpha) - F(\omega_1 \circ \alpha) - F(\omega_2 \circ \alpha) = h.$$

By setting

$$\delta_1 = \omega_1 \circ \alpha \circ \sigma^{-1}, \qquad \delta_2 = \omega_2 \circ \alpha \circ \sigma^{-1}$$

we rewrite this equation as

$$F - F \circ \delta_1 - F \circ \delta_2 = h \circ \sigma \qquad \text{on } I_t. \tag{28}$$

It is remarkable that the functions δ_1 and δ_2 which are connected one-to-one with the *geometric* problem (26) form a $\mathcal{P}$-configuration. Moreover, as $\delta_1(t) + \delta_2(t) = t$ for all $t \in I_t$, equation (28) is nothing but a Cauchy type functional equation on the corresponding curve. As we know from Sec. 2.1, the solvability of this equation depends on whether the corresponding set $\mathfrak{N}_\delta$ is empty or not. Therefore, to prove Theorem 6, it remains only to show that the sets $\mathfrak{N}_\delta^{\mathcal{T}}$ and $\mathfrak{N}_\zeta$ are empty or nonempty simultaneously. The corresponding proof is given in the author's paper [**8**]. This completes the proof of Theorem 6.

Remark. We wish to emphasize that by Theorem 6 *each Cauchy type functional equation* (4) *with the functions δ_1 and δ_2 forming a $\mathcal{P}$-configuration is equivalent to some problem in integral geometry of the described type.*

To illustrate this result, consider the domains D represented by Figures 5, 6 and 7.

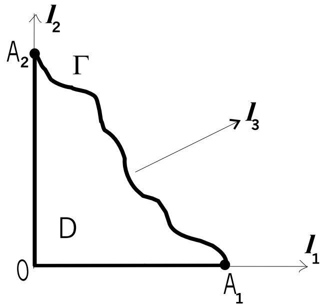

Figure 5

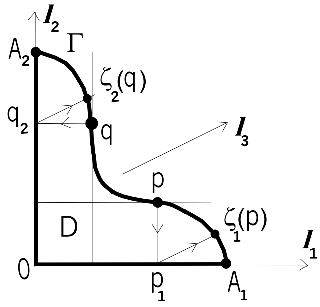

Figure 6

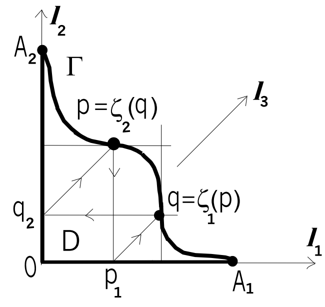

Figure 7

In these figures $p \in \mathcal{T}_1$, $q \in \mathcal{T}_2$ are the only points from $\mathcal{T}$. The curve Γ on Fig. 5 has no points in $\mathcal{T}$. It follows that $\mathfrak{N}_\zeta = \emptyset$, and by Theorem 3, problem (26) is uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. In Fig. 6, the orbits $(p, \zeta_1(p))$ and $(q, \zeta_2(q))$ are the only $\mathcal{T}$-proper orbits corresponding to the semigroup Φ_ζ with a beginning at points p and q, respectively. As neither orbits is $\mathcal{T}$-guided, we have $\mathfrak{N}_\zeta = \emptyset$ in this case, and problem (26) is also uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. In Fig. 7, as is easily seen, the orbit (p, q, p) is a $\mathcal{T}$-guided $\mathcal{T}$-proper cycle (as well as the orbit (q,p,q)). Therefore, by Theorem 6, the operator $\mathcal{A}: C_{\langle\mathbf{l}\rangle}(D) \to \mathcal{H}(\Gamma)$ is not one-to-one.

In the author's papers [**7**],[**8**], some necessary conditions for the right hand sides h are given to ensure the solvability of equation (26). It is interesting that the number of these conditions coincides with the number of elements in $\mathfrak{N}_\zeta$.

3.1.1. *The case of a $\mathcal{Z}$-configuration.* Let $\mathbf{l}, \mathbf{l}_1$ and $\mathbf{l}_2$ be a triple of smooth nonsingular vector fields in a disk $B \subset \mathbb{R}^2$ such that $\mathbf{l}_1$ and $\mathbf{l}_2$ are transversal and

$$\mathbf{l} = b_1\mathbf{l}_1 + b_2\mathbf{l}_2, \quad b_1b_2 \geq 0,$$

with b_1 and b_2 functions on B. Let $D = OA_1O'A_2$ be a (two-dimensional) curvilinear parallelogram in B whose sides OA_1, $O'A_2$ and OA_2, $O'A_1$ are trajectories of the vector fields $\mathbf{l}_1$ and $\mathbf{l}_2$, respectively, and the "diagonal" $\Gamma = OO'$ is a trajectory of the vector field $\mathbf{l}$. Given an arbitrary point $q \in \Gamma$, denote by D_q the curvilinear parallelogram qq_1Oq_2 with $q_j = \pi_j q,\ j = 1, 2$, the same projections of the point q as in Subsec. 3.1.1.1 (see Fig. 4). In this subsection, we briefly discuss the solvability of the integral equation

$$\int_{D_q} f d\sigma = h(q), \qquad q \in \Gamma, \tag{29}$$

which is none other than equation (26), if $d\sigma$, h and f in (29) have the same meaning as in (26). All the discussions around the statement of problem (26) are relevant to problem (29). The space of the right hand sides in (29) is contained in the space $\mathcal{H}_0(\Gamma)$ of all functions h which are twice differentiable in Γ and vanish at the point O along with their first derivative $\partial_{\mathbf{l}}h$. This leads to the following result.

THEOREM 7. *Let*

$$\mathbf{r} = r_1\mathbf{l}_1 + r_2\mathbf{l}_2, \qquad r_1r_2 < 0,$$

be a vector field with r_1 and r_2 constants. Then for each $h \in \mathcal{H}_0(\Gamma)$, there exists a unique solution $f \in C_{\langle\mathbf{r}\rangle}(D)$ of equation (29)*. The inverse operator $h \mapsto f$ is continuous: $\mathcal{H}_0(\Gamma) \to C_{\langle\mathbf{r}\rangle}(D)$.*

Repeating word for word what was said in Subsec. 3.1.1.2, we reduce equation (29) in an equivalent manner to a Cauchy type functional equation

$$F - F \circ \delta_1 - F \circ \delta_2 = H \quad \text{on} \ \ I_t$$

(see (28)). But in contrast to (28), this time the functions δ_1 and δ_2 form a Z-configuration. The desired result follows, therefore, directly from Theorem 3.

3.2. First boundary problem for hyperbolic differential equations and Cauchy type functional equations. To begin with we mention that in the framework of the classical theory of partial differential wquations, boundary problems for hyperbolic equations (all needed definitions are given below) are usually studied in domains closely connected with the equation under consideration. What was typical of these equations (and more generally, of any evolution equation) is that if a domain is bounded, then a part of the boundary is usually free of a priori information about an unknown solution. In the presence of characteristics, boundary conditions on the *whole* boundary of a *bounded* domain are usually treated as prohibited. Nevertheless, (see [**4**]-[**7**]), for a wide class of hyperbolic differential equations this taboo can be lifted. In other words, any equation of this kind defines (a wide class of) bounded domains $D \subset \mathbb{R}^2$ such that the problem of searching for a solution u of this equation in D with the prescribed values of u on the whole boundary ∂D is well-posed. As will be shown later, this new boundary problem turns out to be *equivalent* to a Cauchy type functional equation. Perhaps this explains why this problem has not been investigated previously.

3.2.1. *Statement of the problem.* For the sake of brevity, we restrict ourselves to a homogeneous differential operator with constant coefficients.

In the (x, y)-plane $\mathbb{R}^2$, we consider an arbitrary homogeneous x-strictly hyperbolic operator $P(\partial_x, \partial_y)$ of the 3rd order. The x-strict hyperbolicity means that the characteristic polynomial $P(\tau, \lambda)$ has, for any $\lambda \neq 0$, three distinct real roots in τ. It follows that the operator $P = P(\partial_x, \partial_y)$ can be uniquely represented in the form

$$P(\partial_x, \partial_y) = a(\partial_x - a_1\partial_y)(\partial_x - a_2\partial_y)(\partial_x - a_3\partial_y) \qquad (30)$$

with real constants a, a_1, a_2, a_3, where $a_j \neq a_k$ for $j \neq k$. The characteristics of the operator P are straight lines

$$y + a_1x = \text{const}, \quad y + a_2x = \text{const}, \quad y + a_3x = \text{const}.$$

Let $\mathbf{l}_1, \mathbf{l}_2$ and $\mathbf{l}_3$ be vector fields in $\mathbb{R}^2$ parallel to these lines, respectively. Denote by $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_6$ characteristic rays beginning at some point 0. Chose any triple of neighboring rays $\mathcal{R}_j$, say, $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$. Let $\mathcal{R}_3$ be the ray lying between $\mathcal{R}_1$ and $\mathcal{R}_2$. Consider a curvilinear triangle $D = OA_1A_2$ with sides $OA_1 \subset \mathcal{R}_1$, $OA_2 \subset \mathcal{R}_2$. The side $\Gamma = A_1A_2$ is assumed to be an arbitrary smooth curve without singularities which is transversal to OA_1 and OA_2 (cf. Subsec 3.1.1.1). We suppose the closure $\overline{D}$ to satisfy the hypotheses 1° and 2° of Subsec. 3.1.1.1. It follows in particular that Γ is transversal to the vector field $\mathbf{l}_3$.

The *first boundary problem* for the above operator $P(\partial_x, \partial_y)$ and domain D is as follows.

Given functions $F \in C(\overline{D})$ and $h \in C(\partial D)$, find a solution of the boundary problem

$$Pu = F \quad \textit{in } D, \qquad u = h \quad \textit{on } \partial D. \qquad (31)$$

We call a function u in $\overline{D}$ a *generalized solution* of problem (31) if $u \in C^2(D)$, $u = h$ on ∂D, and for all functions $\varphi \in C_0^\infty(D)$

$$\int\limits_{\mathbb{R}^2} u\,{}^tP\varphi dxdy = \int\limits_{\mathbb{R}^2} F\varphi dxdy,$$

where ${}^tP$ is the formally adjoint differential operator.

3.2.2. *The statement of the result and a sketch of the proof.* To formulate the main result related to the solvability of problem (31), let us consider the semigroup Φ_ζ of maps in Γ introduced in Subsec. 3.1.1.1 with $\mathbf{l} = \mathbf{l}_3$. The corresponding guiding sets $\mathcal{T}_j$ in the theory of partial differential equations are usually called *characteristic* sets. Similarly, we introduce the orbits corresponding to Φ_ζ, and accompanying notions of periodic, $\mathcal{T}$-guided orbits. Finally, we introduce the set $\mathfrak{N}_\zeta$ whose elements are all the $\mathcal{T}$-proper periodic orbits, consisting of only $\mathcal{T}$-guiding points in Γ.

Denote by $C^k(\partial D)$ the space of continuous functions on ∂D whose restrictions to all sides of the triangle D are k times continuously differentiable functions. The main result related to problem (31) is as follows.

Theorem 8. (*see* [**6**],[**7**]). *Assume that the characteristic sets $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy condition* (16). *Then for any functions $F \in C(\overline{D}\,)$ and $h \in C^2(\partial D)$ there exists a unique generalized solution $u(x, y)$ of the problem* (31) *if and only if the set $\mathfrak{N}_\zeta$ is empty. The inverse operator $(F, h) \mapsto u$ is continuous: $C(\overline{D}\,)\times C^2(\partial\,\Omega) \to C^2(\overline{D}\,)$. If $F \in C^k(D)$, $h \in C^{k+2}(\partial D)$, andk ≥ 1 is an integer, then $u \in C^{k+2}(D)$ is a classical solution of the problem in question.*

Proof. We restrict ourselves to the proof of the existence of a unique generalized solution to problem (31) with $F = 0$. Let us write the operator P in the form (30). It is obvious that there exists a linear transformation in $\mathbb{R}^2$ reducing the problem under consideration to the problem

$$(r_1\partial_x + r_2\partial_y)\partial_x\partial_y u = 0 \quad \text{in} \;\; D, \qquad u = h \quad \text{on} \;\; \partial D, \tag{32}$$

where $r_1r_2 > 0$. (For convenience, we preserve the previous notation for the domain and functions). Here D is a domain in $\mathbb{R}^2$ whose boundary ∂D consists of three parts $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{(x, y) : y = 0,\;\; 0 \leq x \leq 1\}, \qquad \Gamma_2 = \{(x, y) : x = 0,\;\; 0 \leq y \leq 1\},$$

$$\Gamma_3 = \{(x, y) : x = \alpha_1(t), y = \alpha_2(t);\;\; 0 \leq t \leq 1\},$$

and

$$\alpha_1(0) = 0,\;\; \alpha_1(1) = 1; \qquad \alpha_2(0) = 1,\;\; \alpha_2(1) = 0.$$

Note that the functions $\alpha_1(t)$ and $-\alpha_2(t)$ form a $\mathcal{P}$-configuration.

Let

$$h = h_1(x) \quad \text{on} \;\; \Gamma_1, \qquad h = h_2(y) \quad \text{on} \;\; \Gamma_2, \quad \text{and} \quad h = h_3(x, y) \quad \text{on} \;\; \Gamma_3.$$

The continuity of the function h leads to the natural compatibility conditions

$$h_1(0) = h_2(0), \qquad h_1(1) = h_3(1, 0), \qquad h_2(1) = h_3(0, 1). \tag{33}$$

Due to the assumptions related to the domain D, an arbitrary generalized solution u of the equation in (32), satisfying boundary condition only on $\Gamma_1 \cup \Gamma_2$, can be represented in the form

$$u(x,y) = \int_0^x \left(\int_0^y F(r_2 s - r_1 t) dt \right) ds + h_1(x) + h_2(y) - h_1(0), \quad 0 \le x, y \le 1. \tag{34}$$

The function F is an arbitrary continuous function on the interval $I = (-r_2, r_1)$. The requirement of satisfying the boundary condition $u = h_3$ on Γ_3 leads to the following integral equation for the unknown function $F \in C(I)$:

$$\int_0^{\alpha_1(t)} \left(\int_0^{\alpha_2(t)} F(r_2 x - r_1 y) dy \right) dx = H(t), \quad 0 \le t \le 1. \tag{35}$$

Here

$$H(t) = -h_1\big(\alpha_1(t)\big) - h_2\big(\alpha_2(t)\big) + h_3\big(\alpha_1(t), \alpha_2(t)\big) + h_1(0)$$

is a given function. What is important is that the function $H(t)$, generated by an arbitrary continuous and twice piecewise differentiable function h in (32), belongs to the space $\mathcal{H}(I) = C^2 \cap C_0(I)$ (see Subsec. 3.1.1.1). This follows from the compatibility conditions (33). Conversely, the function $u(x,y)$ which is defined by (34) with F a solution of equation (35) solves problem (32).

Thus, problem (32) turns out to be equivalent to equation (35), which is nothing but equation (27). The existence of a unique solution to problem (32) provided that $\mathfrak{N}_\zeta = \emptyset$ follows immediately from Theorem 1.

4. Functional Equations Determining Polynomials

This section is devoted to a class of functional equations whose solutions are only polynomials. This class contains the Cauchy equation (1) and for many reasons can be considered as a natural generalization of this equation.

Definition. Given a function F in $\mathbb{R}$ and an integer $n \ge 2$, we denote by $\mathcal{P}_n$ the operator

$$F \to (\mathcal{P}_n F)(x), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

of the form

$$\mathcal{P}_n F(x) = F\left(\sum_{j=1}^n x_j\right) - \sum_{k=1}^n F\left(\sum_{j \ne k} x_j\right) + \cdots + (-1)^n F(0).$$

With this notation, equation (1) has the form

$$\mathcal{P}_2 F(x) = F(0), \quad x \in \mathbb{R}^2;$$

and all continuous solutions F of this equation are polynomials of degree 1. The first assertion of this section treats the general case of the equation $\mathcal{P}_n F = 0$.

THEOREM 9. *If $n \geq 2$, then any continuous solution of the equation*

$$\mathcal{P}_n F = 0 \tag{37}$$

is a polynomial of degree $n-1$.

PROOF. We restrict ourselves to sufficiently smooth solutions F. Denote $F_k = F^{(k)}$, $k = 1, \ldots, n$, and let

$$x_{(k)} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), \quad \partial_k = \partial/\partial x_k, \quad \tau_k \Phi(x) = \Phi \big|_{x_k=0} .$$

Then, as can be easily verified, we have

$$\tau_k \partial_k \mathcal{P}_n F = (\mathcal{P}_{n-1} F_1)(x_{(k)}), \quad k = 1, \ldots, n. \tag{38}$$

Using consecutively this relation for $k = n, n-1, \ldots, 3$, we arrive at the equality

$$\mathcal{P}_2 \big(F_{n-2} \big)(x_1, x_2) = 0.$$

It follows by the above that

$$F_{n-2}(z) = a_0 + a_1 z.$$

Integrating this equality $n-2$ times leads to the desired result.

The following result considerably sharpens the previous theorem. It shows that, analogously to the original Cauchy equation (1), problem (37) is overdetermined (see Subsec. 2.1). In order to determine a polynomial of degree $(n-1)$, the equality $(\mathcal{P}_n F)(x) = 0$ need not hold for *all* points in $\mathbb{R}^n$. It suffices that it holds at points of some hypersurface Γ.

THEOREM 10. (cf. [**10**]) *Let $z = z(x_{(n)})$ be a smooth function with $z(0) = 0$ and*

$$\partial_j z(x_{(n)}) > 0, \quad \text{if} \quad x_{(n)} \geq 0, \quad j = 1, \ldots, n-1.$$

Then any solution $F \in C^{n-2}(\mathbb{R}_+)$ of the equation

$$\mathcal{P}_n F \big(x_{(n)}, z(x_{(n)}) \big) = 0$$

is a polynomial of degree $n-1$.

The proof of this theorem is reduced to the case $n = 2$ with the help of a variant of equality (38), and is completed making use Theorem 3.

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