DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **9**, Number **6**, November **2003**

Website: http://AIMsciences.org

pp. **1411–1422**

NONCOMMUTATIVE DYNAMICAL SYSTEMS WITH TWO GENERATORS AND THEIR APPLICATIONS IN ANALYSIS

Boris Paneah

Department of Mathematics
Technion
Haifa, 32000, Israel

(Communicated by Yuri Kifer)

Abstract. In this paper, some new dynamical systems which are determined by a semigroup Φ of maps in a closed interval I are studied.The main peculiarity of these systems is that Φ is generated by *two* noncommuting maps. Introducing certain closed subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ in I makes it possible to determine some specific orbits corresponding to Φ and some specific attractors in I. These orbits play a crucial role in solving a wide variety problems in such diverse fields of analysis as functional and functional-integral equations, integral geometry, boundary problems for hyperbolic partial differential equations of higher (> 2) order. In the first part of this work we describe some conditions which ensure the existence of attractors in question of a special structure. In the second part several new problems in the above-mentioned fields of analysis are formulated, and we trace how the above dynamic approach works in solving this problems.

1. **Introduction.** The noncommutative dynamical systems we deal with in this paper are defined by a semigroup Φ_δ of maps of a closed interval I into itself. Each such semigroup is generated by two continuous maps δ_1 and δ_2 of I into itself. The set $\mathcal{O}$ of orbits corresponding to Φ_δ is considerably larger than in the case of one map δ since a transition from any point t_k of an orbit to the next point $t_{k+1} = \delta_j(t_k)$, $j = 1, 2$ may be realized in *two* different ways. Since a number of different n-pointed orbits in $\mathcal{O}$ grows exponentially as $n \to \infty$ it does not seem reasonable to expect meaningful general results related to such amorphous formation as the set $\mathcal{O}$. The situation changes abruptly when, in addition to maps δ_1 and δ_2, certain closed subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ in I without common points are given, and the only orbits we consider are those for which

$$t_{k+1} = \delta_{j_k}(t_k), \quad \text{where } j_k = 1 \quad \text{if } t_k \in \mathcal{T}_1 \quad \text{and } j_k = 2 \quad \text{if } t_k \in \mathcal{T}_2.$$

We call such orbits $\mathcal{T}$ - *proper.* One can say that the set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ plays a guiding role when forming orbits in question. Any above pair $(\Phi_\delta, \mathcal{T})$ makes it possible to determine some specific $(\mathcal{T}, \delta)$ - attractors in I which play a crucial role in solving a wide variety of problems in quite different fields of analysis. Each of these problems we associate with a pair $(\Phi_\delta, \mathcal{T})$ such that solvability properties of these problems and even some qualitative peculiarities of their solutions turn out to be closely related to the structure of the corresponding $(\mathcal{T}, \delta)$ - attractors. This observation leads naturally to the *problem of describing* $(\mathcal{T}, \delta)$ *- attractors for*

1991 *Mathematics Subject Classification.* 37N99, 39B22, 35L35.

Key words and phrases. Dynamical system, orbit, attractor, functional equation, integral geometry.

a given pair $(\Phi_\delta, \mathcal{T})$. A particular case of this problem is of the special interest, namely, given the above pair, *whether the boundary of the interval* I *contains some* $(\mathcal{T}, \delta)$ *- attractor.*

In the first part of this work, under some hypotheses about maps $\delta_1,\ \delta_2$ and sets $\mathcal{T}_1, \mathcal{T}_2$, we give an exhaustive answer to the last question (Theorem 1) in terms of some subset in $\mathcal{O}$. The condition which ensures a desirable property of a $(\mathcal{T}, \delta)$ - attractor is transparent from a geometrical point of view and easily verified (see Fig. 1 - 3).

The second part of the work is devoted to an application of this purely dynamic result to two problems in such diverse fields of analysis as integral geometry and functional equations. Note that both problems have never been investigated earlier. In discussing these problems we will trace how all the introduced notions appear in solving analytical problems, and how the above dynamic result works. This dynamic approach has also applications to boundary problems for higher (> 2) order hyperbolic differential equations in bounded domains, but we will not deal with this question here. For the first time such problems were studied (on the base of a prototype of Theorem 1) in the author's papers [3], [5].

2. **Some notation and definitions. Statement of the problem.** In the following two sections we denote by I and $\overset{\circ}{I}$ the closed and the open intervals

$$I = \{t | -1 \le t \le 1\} \qquad \text{and} \qquad \overset{\circ}{I} = \{t | -1 < t < 1\},$$

respectively. Let δ_1 and δ_2 be continuous maps of I into itself, and let $\mathcal{T}_1, \mathcal{T}_2$ be closed subsets in I satisfying the following conditions:

(i) both functions δ_1 and δ_2 do not decrease on I;

(ii) the relation

$$\delta_j(t) \ne t, \qquad j = 1, 2,$$

holds at all points $t \in \overset{\circ}{I}$.

(iii) The ranges $\mathcal{R}[\delta_1]$ and $\mathcal{R}[\delta_2]$ of maps δ_1, δ_2 are the closed intervals $(0, 1)$ and $(-1, 0)$, respectively.

(iv) $\mathcal{T}_1 \cap \mathcal{T}_2 = \varnothing$.

The concluding condition relates to mutual properties of the maps $\delta_j,\ j = 1, 2$, and the sets $\mathcal{T}_j$.

(v) The set $\mathcal{T}_j$ contains each interval of constancy of the function $\delta_{j'},\ j, j' = 1, 2;\ j \ne j'$.

We denote $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and call all the three sets $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}$ *guiding* (as well as all points $t \in \mathcal{T}$).

It is immediately verified that under conditions (i) - (iii) the relations

$$\delta_2(t) < t < \delta_1(t), \quad t \in \overset{\circ}{I}, \tag{1}$$

and

$$\delta_1(-1) = \delta_2(1) = 0, \quad \delta_1(1) = 1, \quad \delta_2(-1) = -1 \tag{2}$$

are valid (if δ_2 denotes a smaller function). The maps δ_1 and δ_2 generate a non-commutative semigroup Φ_δ. The elements of Φ_δ are all the maps of I into itself of the form

$$\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1},$$

where $J = (j_1, \ldots, j_n),\ n = 1, 2, \ldots$, is an arbitrary multi-index with j_k equal to 1 or 2 and $\circ$ denotes the composition of maps. The semigroup Φ_δ naturally determines a noncommutative dynamic system on I. In what follows we use the following geometric terminology relating to Φ_δ and not coinciding completely with the traditional one.

1) Given a multi-index J and a point $t_1 \in I$, an ordered set $\mathcal{O} = (t_1, t_2, \dots, t_{n+1})$ of points in I is called an *orbit* (of the point t_1) if

$$t_{k+1} = \delta_{j_k}(t_k), \quad k = 1, \dots, n. \tag{3}$$

In the sequel it will be convenient to use the term "orbit" also in describing infinite sequences $(t_1, t_2, \dots)$ satisfying condition (3) (in this situation one can determine a multi-index J but not the corresponding map δ_J).

2) An orbit $\mathcal{O} = (t_1, \dots, t_{n+1})$ is called $\mathcal{T}$ - *proper* if

$$\delta_{j_k} = \delta_1 \quad \text{when} \quad t_k \in \mathcal{T}_1 \quad \text{and} \quad \delta_{j_k} = \delta_2 \quad \text{when} \quad t_k \in \mathcal{T}_2.$$

in (3).

3) If all points of an orbit $\mathcal{O}$ belong to a guiding set $\mathcal{T}$, then $\mathcal{O}$ is called $\mathcal{T}$ - *guided* orbit.

4)An orbit $\mathcal{O} = (t_1, \dots, t_{n+1})$ is said to be *periodic* orbit or, in short, *cycle* if $t_1 = t_{n+1}$.

Definition We denote by $\mathfrak{N}_\delta^{\mathcal{T}}$ the set of all $\mathcal{T}$ - proper $\mathcal{T}$ - guided cycles in I.

Definition A set $\mathcal{A} \subset I$ is called $(\mathcal{T}, \delta)$- *attractor* if for any point $t_1 \in I$ there is a converging $\mathcal{T}$ - proper orbit $(t_1, t_2, \dots)$ with a limit in $\mathcal{A}$, and no proper subset of $\mathcal{A}$ possesses this property.

As will be shown later (see Sec.3 - 5) a wide diverse of different problems in analysis at first sight looking stationary can be successfully solved within a framework of the dynamical systems in question. With any of these problems we associate a corresponding semigroup Φ_δ and a guiding set $\mathcal{T}$. This makes it possible to reduce the problem under consideration to the question, whether there is a $(\mathcal{T}, \delta)$ - attractor contained in the boundary ∂I of I. Theorem 1 below contains an exhaustive answer to this question if the set $\mathcal{T}$ satisfies some geometric conditions.

3. **The main dynamic result.** Denote by $\mathcal{T}_j'$ and $\mathcal{T}'$ the sets of all limit points of the sets $\mathcal{T}_j,\ j = 1, 2$, and $\mathcal{T}$, respectively. If both sets $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are infinite we set

$$\tau_1 = \min\{t | t \in \mathcal{T}_1'\}, \quad \tau_2 = \max\{t | t \in \mathcal{T}_2'\}.$$

Theorem 1. *Let Φ_δ and $\mathcal{T}$ be the above semigroup and closed set, respectively, satisfying conditions* (i) − (v). *Assume that $\mathcal{T}' \cap \partial I = \varnothing$, and if the sets $\mathcal{T}_1'$, $\mathcal{T}_2'$ are infinite, the inequality*

$$\tau_1 > \tau_2 \tag{4}$$

holds. Then there is a $(\mathcal{T}, \delta)$ - attractor contained in ∂I if and only if

$$\mathfrak{N}_\delta^{\mathcal{T}} = \varnothing. \tag{5}$$

Proof: The necessity of condition (5) follows from the definitions. To prove its sufficiency we note at first that if one of the sets $\mathcal{T}_j,\ j = 1, 2$, is empty, then for an arbitrary point t_1 the orbit

$$\mathcal{O}_{j'} = (t_1, \delta_{j'}(t_1), \dots, \delta_{j'}^n(t_1), \dots), \quad j' \neq j,$$

is $\mathcal{T}$ - proper and converges to the point $t = (-1)^j$. For, if $j = 2$, for example, then the orbit $\mathcal{O}_1$, being increasing (due to the right inequality in (1)) and bounded, has a limit $\xi = \lim\limits_{k\to\infty} \delta_1^k(t_1)$. By continuity of the function δ_1 we have $\delta_1(\xi) = \lim \delta_1^k(t_1)$, and hence $\delta_1(\xi) = \xi$. By virtue of (ii), we find that $\xi = 1$. In what follows the sets $\mathcal{T}_1$ and $\mathcal{T}_2$ are assumed to be nonempty.

The following assertion turns out to be essential in proving the theorem.

Lemma 2. *If a periodic orbit $\mathcal{C}$ is a part of a $\mathcal{T}$ - proper orbit $\mathcal{O} = (t_1, t_2, \ldots)$, then $t_1 \in \mathcal{C}$ and $t_1 \neq 0$.*

Proof: Assume that $t_1 \notin \mathcal{C}$ and let t_q, $q \geq 2$, be the first point in $\mathcal{O}$ belonging to $\mathcal{C}$. Then

$$\mathcal{C} = (t_q, \ldots, t_{q+n}), \quad t_q = t_{q+n}.$$

for some $n > 1$. It is clear that $t_{q-1} \neq t_{q+n-1}$, since $t_{q-1} \notin \mathcal{C}$ and $t_{q+n-1} \in \mathcal{C}$. Let

$$t_q = \delta_{j_1}(t_{q-1}) \qquad \text{and} \qquad t_{q+n} = \delta_{j_2}(t_{q+n-1}).$$

Since $t_q = t_{q+n} \neq 0$, it follows by condition (iii) that $j_1 = j_2$. Denote a common value of these indices by j. By virtue of condition (i),

$$\delta_j(t) = \text{const} \quad \text{for all } \ t, \quad t_{q-1} \leq t \leq t_{q+n-1}.$$

But then $t_{q-1} \in \mathcal{T}_{j'}$ for $j' \neq j$, due to condition (v), and the orbit (t_{q-1}, t_q) can not be $\mathcal{T}_\delta$ - proper. This proves Lemma 2, as the second assertion is obvious.

Continuing the proof of Theorem 1 we note that, for a sufficiently small $\varepsilon > 0$, there are some deleted neighborhoods $\mathcal{U}_1^\varepsilon$ and $\mathcal{U}_2^\varepsilon$ of the points $t = -1$ and $t = 1$, respectively, such that

$$\mathcal{U}_1^\varepsilon \cap \mathcal{T}_1 = \varnothing, \qquad \mathcal{U}_2^\varepsilon \cap \mathcal{T}_2 = \varnothing.$$

This is a direct consequence of condition (4). It is worth mentioning that if $t \in \mathcal{U}_1^\varepsilon$ or $t \in \mathcal{U}_2^\varepsilon$, then the orbit

$$\mathcal{O}_1 = \{t, \delta_2(t), \delta_2^2(t), \ldots\} \quad \text{or} \quad \mathcal{O}_2 = \{t, \delta_1(t), \delta_1^2(t), \ldots\},$$

respectively, turns out to be $\mathcal{T}$ - proper, and, as was proved above, converges to the point $t = -1$ or $t = 1$, respectively.

It is important to note that for the above-mentioned number ε there is a number s such that inclusions

$$\delta_1^s(t) \in \mathcal{U}_2^\varepsilon \quad \text{and} \quad \delta_2^s(t) \in \mathcal{U}_1^\varepsilon$$

are true for all points $t \in \overset{\circ}{I}$. For, in the case $\mathcal{U}_2^\varepsilon$, for example, the existence of an integer $s > 0$ such that

$$\delta_1^s(-1) > 1 - \varepsilon$$

is a consequence of inequality (1). The required inclusion follows immediately from the monotonicity condition (i).

Take an arbitrary point τ_0 between τ_1 and τ_2 (see (4)). It is easily seen that there exists an open covering $V = V_1 \cup V_2$ of the closed set $\mathcal{T}' = \mathcal{T}_1' \cup \mathcal{T}_2'$ such that

1° $\quad V_1 \cap V_2 = \varnothing$;
2° $\quad V_1 \supset \mathcal{T}_1', \quad V_2 \supset \mathcal{T}_2'$;
3° $\quad V_1 \cap \mathcal{T}_2 = \varnothing$; and $V_2 \cap \mathcal{T}_1 = \varnothing$;
4° $\quad$ the set V_1 (V_2) lies on the right (on the left) of the point τ_0.

It is important to note that the set $\mathcal{M} = \mathcal{T} \backslash V$ is finite. Let us agree to call an arbitrary $\mathcal{T}$ - proper orbit $\mathcal{O} = (t_1, t_2, \ldots)$ *directed* if

$$\delta_{j_k} = \begin{cases} \delta_1 \text{ when } t_k \notin \mathcal{T}_2 \text{ and } t_k \geq \tau_0 \\ \delta_2 \text{ when } t_k \notin \mathcal{T}_1 \text{ and } t_k < \tau_0 \end{cases}.$$

in (3). Unlike the definition (2) (see Sec.2) the number j_k here is uniquely defined by any point $t_k \in I\backslash\mathcal{T}$. Obviously, *any directed orbit is uniquely determined by its starting point.* Furthermore, a point t_n of an orbit $\mathcal{O} = (t_1, t_2, \ldots)$ will be called *a turning point of* $\mathcal{O}$ if

$$t_n = \delta_j(t_{n-1}) \text{ and } t_{n+1} = \delta_{j'}(t_n), \text{ where } j \neq j'.$$

It is easy to verify that, in every directed orbit $\mathcal{O}$, each turning point $t_n \in V$ follows a point $t_{n-1} \in \mathcal{M}$, and the same is true for any turning point $t_n \subset I\backslash\mathcal{T}$. Since

$$I = V \cup (I\backslash\mathcal{T}) \cup \mathcal{M},$$

what has been said means that the number of turning points in an orbit $\mathcal{O}$ does not exceed the number M of points in the set $\mathcal{M}$, if *this orbit does not contain cyclic suborbits.* But then it becomes clear that the terminal point in the directed orbit

$$\mathcal{O}_1 = (t_1, t_2, \ldots, t_N)$$

with $N = Ms + M + s$ lies in a neighborhood $\mathcal{U}^\varepsilon = \mathcal{U}_1^\varepsilon \cup \mathcal{U}_2^\varepsilon$ of the boundary ∂I. Therefore, one of the sewed orbits

$$\mathcal{O} = (t_1, \ldots, t_N, \delta_j(t_N), \delta_j^2(t_N), \ldots)$$

with $j = 1$ or $j = 2$ solves the problem, i.e. it possesses all the properties postulated in Theorem 1.

Turn now to the situation where *a directed orbit* $\mathcal{O}$ *of a point* t_1 *includes some cycle* $\mathcal{C}$. By Lemma 2, this cycle contains the point t_1, and consequently it has the form $\mathcal{C} = (t_1, \ldots, t_m)$ with $t_m = t_1$. But since $\mathfrak{N}_\delta^{\mathcal{T}} = \varnothing$, the orbit $\mathcal{C}$ is not $\mathcal{T}$ - guided. Therefore, one of its points t_q, $1 \leq q \leq m-1$, does not belong to the set $\mathcal{T}$. Assume that $t_{q+1} = \delta_j(t_q)$. We introduce a new point $\widehat{t}_{q+1} = \delta_{j'}(t_q)$ with $j' \neq j$ and define the directed orbit

$$\mathcal{O}_1 = (\widehat{t}_{q+1}, \widehat{t}_{q+2}, \ldots).$$

Being $\mathcal{T}$ - proper this orbit has no common points with the cycle $\mathcal{C}$. Indeed, if $\widehat{t}_p$ is a first point of this kind and $\widehat{t}_p = t_r$, $1 \leq r \leq m-1$, then $\widehat{t}_{p-1} = t_{r-1}$ (as it was shown in the proof of Lemma 2). However, this is impossible, since $t_{r-1} \in \mathcal{C}$ whereas $\widehat{t}_{r-1} \notin \mathcal{C}$. By virtue of the same lemma, the $\mathcal{T}_\delta$ - proper orbit $(t_q, \widehat{t}_{q+1}, \widehat{t}_{q+2}, \ldots)$ does not contain any periodic suborbits. As above, this enables us to conclude that the sewed orbit

$$\mathcal{O}_j = (t_1, t_2, \ldots, t_q, \widehat{t}_{q+1}, \ldots, \widehat{t}_{q+N}, \delta_j(\widehat{t}_{q+N}), \delta_j^2(\widehat{t}_{q+N}), \ldots),$$

with $N = Ms + M + s$ and $j = 1$ or 2 if $\widehat{t}_{q+N} > \tau_0$ or $\widehat{t}_{q+N} \leq \tau_0$, respectively, satisfies all the conditions stated in Theorem 1. This completes the proof of the theorem.

4. **On solvability of a class of functional equations.**

In this section we deal with a functional equation of the following type

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = H(t), \tag{6}$$

where a_1, a_2 and H are given real-valued continuous functions on I, δ_1 and δ_2 are given continuous maps of I into itself, and F is the unknown function: $I \to \mathbb{R}$. For the first time such equations were investigated in the author's papers [3],[4],[5], although some particular cases (always with $H = 0, a_1 = a_2 = 1$) had been investigated earlier, in the 70s - 80s (see [2],[7]).

Being interesting by themselves these equations appear as a necessary technical tool for solving some problems from other fields of analysis. One of these problems, namely, that from integral geometry will be considered later, in Sec. 5. It is worth mentioning that equation (6) (at first sight by no means related to dynamical systems) is a suitable model to trace how the above concepts "$\mathcal{T}$- guided" and "$\mathcal{T}$- proper" orbit arise in solving this really analytic problem. We will see that in spite of a deep difference between proofs of the existence and the uniqueness of a solution to equation (6) both of them are based on the existence of a $(\mathcal{T}, \delta)$ - attractor with properties described in Theorem 1. In proving uniqueness such an attractor ensures the validity of a maximum principle, but in proving existence the same attractor makes it possible to obtain some necessary a priori estimates for the norm of the linear operator $F \to a_1 F \circ \delta_1 + a_2 F \circ \delta_2$.

Let us now pass to exact statements. With regard to the parameters determining equation (6), we assume that all the conditions (i) - (iii) are fulfilled. In particular, relations (1) and (2) are valid. As to the coefficients $a_1(t)$ and $a_2(t)$, they are assumed to be nonnegative and to satisfy the inequalities

(iv)′ $$0 < a_1(t) + a_2(t) \leq 1, \quad t \in I.$$

Introduce the guiding sets

$$\mathcal{T}_1 = \{t \in I \mid a_2(t) = 0\}, \qquad \mathcal{T}_2 = \{t \in I \mid a_1(t) = 0\},$$

and let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. In view of (iv)′, $\mathcal{T}_1 \cap \mathcal{T}_2 = \varnothing$.

The concluding hypothesis relates to some mutual properties of the functions a_j and the maps δ_k, $j, k = 1, 2$, namely,

(v)′ $a_j(t) = 0$ on each interval of constancy of the function $\delta_j(t)$, $1 \leq j \leq 2$.

We note that hypotheses (i) - (iii), (iv)′ are not artificial, since all type (6) functional equations that have up to now arisen in various problems in integral geometry (see [5]) and also in the theory of boundary problems for partial differential equations (see [3], [5]) satisfy these conditions.

Let Φ_δ be the semigroup of maps of I into itself generated by δ_1 and δ_2 as in Sec. 2. Due to (iv)′, the new triple $(\Phi_\delta, \mathcal{T}_1, \mathcal{T}_2)$ satisfies conditions (i)-(iv) (see Sec. 2). This makes it possible to introduce the above notions like $\mathcal{T}$- proper and $\mathcal{T}$- guided orbits, $(\mathcal{T}, \delta)$ - attractors, and to define the corresponding set $\mathfrak{N}_\delta^{\mathcal{T}}$. What is important is that under condition (v)′ the result of Theorem 1 remains true for the triple $(\Phi_\delta, \mathcal{T}_1, \mathcal{T}_2)$ in question.

Theorem 3. *Assume that the guiding sets $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy condition* (4).

1° *Let*

$$a_1(t) + a_2(t) = 1, \qquad t \in I. \tag{7}$$

If

$$\mathfrak{N}_\delta^{\mathcal{T}} = \varnothing, \tag{8}$$

then all solutions of the homogeneous equation

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = 0 \tag{9}$$

are constant functions.

2° *Let the set*

$$\mathcal{B} = \{t \mid a_1(t) + a_2(t) < 1\}$$

contain some deleted [1] *neighborhood of the boundary* ∂I*. Then, under hypothesis* (8)*, equation* (6) *has a unique solution* $F \in C(I)$ *for an arbitrary function* $H \in C(I)$*.*

Remark More general results including those describing some qualitative properties of solutions F to equation (6) are contained in [5] (see also Theorem 4).
Proof: 1° Let F be a solution of equation (9) and let

$$M = \max_I F.$$

Introduce the set

$$\mathcal{M} = \{t \in I \mid F(t) = M\}.$$

If $F(t_1) = M$ for some $t_1 \in I$, then

$$F(\delta_1(t_1)) = M - \varepsilon_1, \qquad F(\delta_2(t_1)) = M - \varepsilon_2$$

for some nonnegative ε_1 and ε_2. Substituting t_1 for t into equation (9) we obtain

$$a_1(t_1)\varepsilon_1 + a_2(t_1)\varepsilon_2 = 0.$$

It follows that if $t_1 \in \mathcal{M}\backslash\mathcal{T}$, then $\varepsilon_1 = \varepsilon_2 = 0$ and hence

$$\delta_1(t_1) \in \mathcal{M}, \qquad \delta_2(t_1) \in \mathcal{M}.$$

But if $t_1 \in \mathcal{M} \cup \mathcal{T}_1$, then $\varepsilon_1 = 0$, and consequently $\delta_1(t_1) \in \mathcal{M}$. In just the same way, if $t_1 \in \mathcal{M} \cup \mathcal{T}_2$), then $\delta_2(t_1) \in \mathcal{M}$. Combining these observations with the definition of a $\mathcal{T}$ - proper orbit generated by the semigroup Φ_δ, we conclude that along with every point t_1 the set $\mathcal{M}$ contains a next point of an *arbitrary* $\mathcal{T}$- proper orbit (t_1, t_2). Since δ_1 and δ_2 are maps of I into itself, this argument can be applied to the point t_2. As a result, we obtain one or two points t_3 of the form

$$t_3 = \delta_{j_2}(t_2) = \delta_{j_2} \circ \delta_{j_1}(t_1)$$

belonging to the set $\mathcal{M}$, and the orbit $\mathcal{O} = (t_1, t_2, t_3)$ turns out to be $\mathcal{T}$- proper. Proceeding in the same manner, we arrive at the conclusion that *along with every point* $t_1 \in \mathcal{M}$ *its every* $\mathcal{T}$*- proper orbit* $\mathcal{O} = (t_1, t_2, \ldots)$ *lies in the set* $\mathcal{M}$, (the first appearance of a $\mathcal{T}$- proper orbit!). By Theorem 1, one of these orbits converges either to the point $t = -1$ or to the point $t = 1$. By virtue of the continuity of the function F, one of the numbers $F(1)$, $F(-1)$ is equal to M. What has been said about the spread of the maximum value of a solution along $\mathcal{T}$ - proper orbits remains true for the minimal value m of the same solution. As a result, one of the numbers $F(1)$ and $F(-1)$ is equal to m. To complete the proof of part 1° it only remains to note that

$$F(-1) = F(1)$$

(to see this it suffices to substitute consecutively the points $t = -1$ and $t = 1$ for t in (9) and use relations (2)). Therefore $M = m$, whence F=const.

2° Introduce a linear operator L

$$L : \, F \mapsto a_1 F \circ \delta_1 + a_2 F \circ \delta_2.$$

in the space $C(I)$. By (iv)$'$, the norm $\|L\|$ of this operator is not greater than 1. We are now going to prove that $\|L^m\| < 1$ *for some integer* $m > 0$. This results

---

[1] If $\mathcal{U}$ is a neighborhood of a point x, then $\mathcal{U}\backslash\{x\}$ is the corresponding deleted neighborhood of this point.

(in view of the well known result in functional analysis) in the invertibility of the operator

$$F \to F - LF,$$

which is equivalent to assertion 2° of Theorem 3.

We first of all note that, for an arbitrary integer $N > 0$, the function $L^N F$ can be represented in the form

$$L^N F(t) = \sum_{j_1,\ldots,j_N=1}^{2} a_{j_1}(t) a_{j_2}\Big(\delta_{j_1}(t)\Big) \ldots a_{j_N}\Big(\delta_{j_{N-1}} \circ \ldots \circ \delta_{j_1}(t)\Big) F\Big(\delta_J(t)\Big)$$

where $J = (j_1, \ldots, j_N)$. It follows that for an arbitrary function $F \in C(I)$ with $\|F\| = 1$ the inequality

$$|L^N F(t)| \leq \sum_{j_1,\ldots,j_N=1}^{2} a_{j_1}(t) a_{j_2}\Big(\delta_{j_1}(t)\Big) \ldots a_{j_N}\Big(\delta_{j_{N-1}} \circ \ldots \circ \delta_{j_1}(t)\Big)$$

holds at each point $t \in I$. Let us prove that, given any fixed value t, there is a positive integer N and a constant $\gamma < 1$ such that the inequality

$$|L^N F(t)| < \gamma \tag{10}$$

holds for the same functions F. For points $t \in \mathcal{B}$ this is true according to the definition of $\mathcal{B}$. If $t \notin \mathcal{B}$, then we consider some $\mathcal{T}$- proper orbit $\mathcal{O} = (t_1, t_2, \ldots)$ beginning at the point $t_1 = t$ which converges to one of the boundary points of I. (The second appearance of $\mathcal{T}$ - proper orbits!) The existence of such an orbit is guaranteed by Theorem 1. By the hypothesis of the theorem, for some sufficiently large natural N, the point $t_N = \delta_{j_{N-1}} \circ \ldots \circ \delta_{j_1}(t)$ in orbit $\mathcal{O}$ belongs to the set $\mathcal{B}$, and hence

$$a_1(t_N) + a_2(t_N) < 1. \tag{11}$$

At the same time, by the definition of $\mathcal{T}$- proper orbits, the inequalities

$$a_{j_1}(t) \neq 0, \;\; a_{j_2}\Big(\delta_{j_1}(t)\Big) \neq 0, \;\; \ldots, \;\; a_{j_N}\Big(\delta_{j_{N-1}} \circ \ldots \circ \delta_{j_1}(t)\Big) \neq 0 \tag{12}$$

are valid. From (11) and (12) it follows, by induction, that inequality (10) is true. Furthermore, note that by virtue of the continuity, this inequality holds at all points of some neighborhood $\mathcal{U}$ of the point t in question probably with a larger constant $\gamma < 1$. The collection of these neighborhoods forms an open covering of the closed set $I \backslash \mathcal{B}$. Let $\{\mathcal{U}_j\}_{j=1}^k$ be a finite subsystem of these neighborhoods and let N_j and γ_j be the corresponding constants. Setting $m = \max N_j$ and $\gamma = \max \gamma_j$, we arrive at the desired inequality $\|L^m\| < 1$. This completes the proof of Theorem 3.

This is one more result relating to the solvability of the functional equation (6) whose proof becomes almost trivial by using the dynamic approach.

Theorem 4. *Let the conditions* (7) *and* (8) *be fulfilled. Then equation* (6) *has no solution if the right-hand side h does not change sign on I and $h \neq 0$ in an arbitrarily small deleted neighborhood of the boundary ∂I.*

Proof: Given a function h, let F be a solution of equation (6) and let $F(\widehat{t}\,) = M$ and $F(\tilde{t}\,) = m$ be the maximal and minimal values of F, respectively. If we substitute consecutively $\widehat{t}$ and $\tilde{t}$ for t in (6) we obtain the inequalities $h(\widehat{t}\,) \geq 0$ and $h(\tilde{t}\,) \leq 0$. Thus, the maximal and minimal values of h can not be of the same sign. Assume that $h \geq 0$ on I and $h > 0$ in a deleted neighborhood $\mathcal{U}$ of the boundary ∂I. Then $h(\tilde{t}\,) = 0$. But as we know from the previous proof, the minimal value m of a solution F spreads *along* $\mathcal{T}$*- proper orbits* of the point $\tilde{t}$. Therefore, $h = 0$ at all points of any such orbit. By virtue of Theorem 1, the function h vanishes at points lying arbitrarily close to the boundary ∂I, in contradiction with the choice of h. This completes the proof of Theorem 4.

5. **On Solvability of Some Problems in Integral Geometry.**

One of the typical problems in integral geometry is the problem of reconstructing a function in a given domain D from the values of its integrals over a family $\{D_q\}$ of subdomains in D. In this section a new problem of this kind will be studied. The peculiarity of the problem in question is that we consider bounded domains D with a boundary ∂D. Although at first sight the problem looks completely stationary, it admits an adequate formulation in which the semigroup Φ_δ introduced in Sec. 2 turns out to be connected with the problem. This makes it possible to use the dynamic approach developed in Sec. 2 and to obtain the desired results by avoiding a hard analytic work.

Let B be a disk in $\mathbb{R}^2$ and let $\mathbf{l}_1$ and $\mathbf{l}_2$ be smooth nonsingular transversal vector fields in B. Introduce a curvilinear triangle $D = OA_1A_2$ whose sides OA_1 and OA_2 are trajectories of the vector fields $\mathbf{l}_1$ and $\mathbf{l}_2$, respectively. The side $\Gamma = A_1A_2$ is assumed to be an arbitrary smooth curve without singularities transversal to $\mathbf{l}_1$ and $\mathbf{l}_2$ at the points A_1 and A_2. In addition, the closure $\overline{D}$ of a domain D is assumed to satisfy the following hypotheses:

1° For any point $p \in \overline{D}$, a trajectory of $\mathbf{l}_j$ passing through p meets OA_k at a point $\pi_k p$, $j \neq k,\ j, k = 1, 2$.

2° The set $\overline{D}$ is $\mathbf{l}_j$ - convex, $j = 1, 2$. This means that if given points p and q in $\overline{D}$ lie on some trajectory γ_j of the field $\mathbf{l}_j$, then all the points $r \in \gamma_j$ between p and q belong to $\overline{D}$.

Given an arbitrary point $q \in \Gamma$, let D_q be a curvilinear parallelogram qq_1Oq_2, where $q_j = \pi_j q,\ j = 1, 2$. The above conditions 1° and 2° guarantee the inclusion $\overline{D}_q \subset \overline{D}$ for all $q \in \Gamma$.

We formulate the above-mentioned geometric problem in the following form:

> *Given a function* $h(q) \in C(\Gamma)$*, find a function* $f \in C(\overline{D})$ *satisfying the integral equation*
>
> $$(\mathcal{A}f)(q) := \int_{D_q} f d\sigma = h(q), \qquad q \in \Gamma, \tag{13}$$
>
> *with* σ *being a measure on* B.

By dimensional considerations no uniqueness results are possible in general. Hence, following [1] we make more precise formulation, namely, for what spaces of functions f and h is the map $\mathcal{A} : f \mapsto h$ one-to-one, and what functions $h(q)$ can be represented by the integral in (13).

As to the second question, it follows from (13) that any such function h belongs to the space $\mathcal{H}(\Gamma) = (C^2 \cap C_0)(\Gamma)$ of all twice continuously differentiable functions vanishing on the boundary $\partial\Gamma$. Therefore, the best possible solution of the problem consists in describing subspaces $\mathcal{F}(D)$ of the space $C(\overline{D})$ for which the map

$$\mathcal{A} : \mathcal{F}(D) \to \mathcal{H}(\Gamma)$$

is one-to-one. One of the possible classes of such subspaces is introduced below.

Definition Given a smooth nonsingular vector field $\mathbf{l}$ in B, we denote by $C_{\langle \mathbf{l} \rangle}(D)$ the subset of all functions in $C(\overline{D})$ which remain constant along each trajectory of the field $\mathbf{l}$.

Within a framework of this paper we consider only the case of vector fields

$$\mathbf{l} = r_1 \mathbf{l}_1 + r_2 \mathbf{l}_2, \quad r_1 r_2 > 0,$$

with constant coefficients r_1 and r_2. [2] However, in this situation we obtain an exhaustive solution of the problem in question by stating a necessary and sufficient condition on Γ ensuring the above-mentioned property of the operator

$$\mathcal{A} : C_{\langle \mathbf{l} \rangle}(D) \to \mathcal{H}(\Gamma).$$

To state this condition, we introduce the projection $\pi_{\mathbf{l}} : \overline{D} \to \Gamma$ along the trajectories of the vector field $\mathbf{l}$. Then

$$\zeta_1 = \pi_{\mathbf{l}} \circ \pi_1 \qquad \text{and} \qquad \zeta_2 = \pi_{\mathbf{l}} \circ \pi_2$$

are two maps of Γ into itself. Denote by Φ_ζ the noncommutative semigroup of maps of Γ into itself, generated by ζ_1 and ζ_2. The analogy of Φ_ζ with the semigroup Φ_δ considered in Sec. 2 is obvious. As in the case of Φ_δ we define an orbit in Γ as a sequence of points $(q_1, \dots, q_n, \dots)$ in Γ such that

$$q_{k+1} = \zeta_{j_k}(q_k), \quad k = 1, 2, \dots,$$

and all ζ_{j_k} are equal ζ_1 or ζ_2. We introduce also the *guiding* sets[3]

$$\mathcal{T}_j^\Gamma = \{q \in \Gamma \mid \mathbf{l}_j(q) \in T_q(\Gamma)\}, \quad j = 1, 2,$$

and $\mathcal{T}^\Gamma = \mathcal{T}_1^\Gamma \cup \mathcal{T}_2^\Gamma$. It is assumed by analogy with hypothesis (4) that if both sets $\mathcal{T}_1$ and $\mathcal{T}_2$ are infinite, then arbitrary points $\tau_1 \in \mathcal{T}_1'$ and $\tau_2 \in \mathcal{T}_2'$ are situated on Γ in the order A_1, τ_1, τ_2, A_2. Repeating literally what was said in Sec. 2, we define *periodic, $\mathcal{T}^\Gamma$- guided*, and *$\mathcal{T}^\Gamma$- proper orbits* corresponding to the semigroup Φ_ζ. Finally, we introduce the set $\mathfrak{N}_\zeta^\Gamma$ of all $\mathcal{T}^\Gamma$- proper $\mathcal{T}^\Gamma$- guided periodic orbits generated by Φ_ζ (see Fig.1 - Fig.3). Now everything is ready for formulating the main result of this section.

Theorem 5. *If all the above hypotheses related to the domain D, the curve Γ, and the vector fields $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}$ are fulfilled, then, for an arbitrary function $h \in \mathcal{H}(\Gamma)$, equation* (13) *has a unique solution $f \in C_{\langle \mathbf{l} \rangle}(D)$ if and only if $\mathfrak{N}_\zeta^\Gamma = \varnothing$. The inverse operator $h \mapsto f$ from $\mathcal{H}(\Gamma)$ to $C_{\langle \mathbf{l} \rangle}(D)$ is continuous.*

To illustrate this result, consider domains D represented by figures 1, 2 and 3.

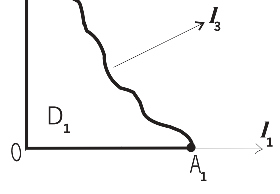

Figure 1

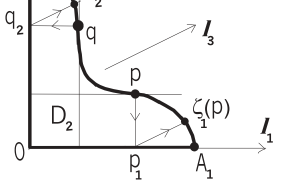

Figure 2

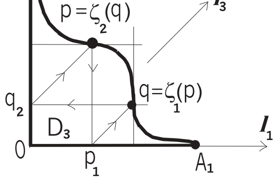

Figure 3

---

[2] The more general situation with variable r_1 and r_2 is studied in [6].

[3] $T_q(\Gamma)$ denotes the tangent space of Γ at a point q.

On these figures only the points $p \in \mathcal{T}_1^\Gamma$, $q \in \mathcal{T}_2^\Gamma$ belong to the set $\mathcal{T}^\Gamma$. The curve Γ on Fig.1 has no points in $\mathcal{T}^\Gamma$. It follows that $\mathfrak{N}_\zeta^\Gamma = \varnothing$, and by Theorem 5 problem (13) is uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. On Fig.2 the orbits $(p, \zeta_1(p))$ and $(q, \zeta_2(q))$ are the only $\mathcal{T}^\Gamma$ - proper orbits corresponding to the semigroup Φ_ζ and beginning at points p and q, respectively. Since both orbits are not $\mathcal{T}^\Gamma$- guided, the set $\mathfrak{N}_\zeta^\Gamma$ is empty in this case, and problem (13) is also uniquely solvable for all $h \in \mathcal{H}(\Gamma)$. On Fig.3, as is easily seen, the orbit (p, q, p) is a $\mathcal{T}^\Gamma$- guided $\mathcal{T}^\Gamma$- proper cycle (as well as the orbit (q, p, q)). Therefore, by Theorem 5, the operator $\mathcal{A} : C_{\langle \mathbf{l} \rangle}(D) \to \mathcal{H}(\Gamma)$ is not one-to-one.

Sketch of the proof: First of all choosing a special coordinate system (x_1, x_2) in the disk B we reduce the integral equation (13) to the form

$$\int_0^{\alpha_1(z)} \int_0^{\alpha_2(z)} f(\omega(x)) dx_1 dx_2 = h(z), \qquad z \in I_z,$$

with f being an unknown continuous function on the interval $I_t = \{t \mid -r_1 \le t \le r_2\}$. Here $I_z = \{z \mid -1 \le z \le 1\}$, $\alpha(z) = (\alpha_1(z), \alpha_2(z))$, and equalities $x_1 = \alpha_1(z)$, $x_2 = \alpha_2(z)$, $z \in I_z$, define a parametric representation of the curve Γ. It is clear that

$$\alpha_1'(z) \ge 0, \quad \alpha_2'(z) \le 0, \quad \text{and} \quad (\alpha_1' - \alpha_2')(z) > 0, \quad z \in I_z.$$

As to ω, this is a function

$$\omega(x) = r_2 x_1 - r_1 x_2$$

which does not change its values along trajectories of the vector field $\mathbf{l}$. Denote $\omega_1 = \omega(x_1, 0)$, $\omega_2 = \omega(0, x_2)$ and let

$$\sigma = \omega_\Gamma \circ \alpha \, : \, I_z \to I_t,$$

where ω_Γ is the restriction of ω to Γ. By the above, the function σ is invertible. Introducing the new unknown function

$$F(t) = -\int_0^t f(s)(t-s) ds / r_1 r_2, \quad t \in I_t,$$

we arrive at the following functional equation

$$F\big(\omega(\alpha(z))\big) - F\big(\omega_1(\alpha(z))\big) - F\big(\omega_2(\alpha(z))\big) = h(z).$$

By setting

$$\delta_1 = \omega_1 \circ \alpha \circ \sigma^{-1}, \qquad \delta_2 = \omega_2 \circ \alpha \circ \sigma^{-1},$$

we reduce this equation to the final form

$$F(t) - F(\delta_1(t)) - F(\delta_2(t)) = h(\sigma^{-1}(t)) \qquad \text{on } \ I_t. \tag{14}$$

It is remarkable that the functions δ_1 and δ_2 which are intrinsically connected with *geometric* problem (13), satisfy general conditions (i) - (iii), (see Sec. 2). Note that by definition we have

$$F(0) = 0 \tag{15}$$

and

$$\delta_1(t) + \delta_2(t) = t. \tag{16}$$

Substituting consecutively $t = -1$ and $t = 1$ for t in (14) and using (15) we conclude that for all functions F the left hand side in (14) vanishes at points $t = -1$ and $t = 1$. Consequently, differentiating equality (14) results in the *equivalent* relation

$$G(t) - \delta_1'(t) G(\delta_1(t)) - \delta_2'(t) G(\delta_2(t)) = H(t) \qquad \text{on } \ I_t, \tag{17}$$

where $G = F'$ and $H = (d/dt)h\big(\sigma^{-1}(t)\big)$. It follows from (16) that

$$\delta_1' + \delta_2' = 1 \qquad \text{in } I_t.$$

Therefore, equation (17) is none other than equation (6), where $a_1 = \delta_1'$ and $a_2 = \delta_2'$. The critical sets of the functions δ_1 and δ_2 play now a role of the guiding sets $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. It follows that hypothesis (v) (see Sec. 2) is satisfied in the case of the equation (17). Introduce the semigroup Φ_δ generated by δ_1 and δ_2, and the corresponding triple $(\Phi_\delta, \mathcal{T}_1, \mathcal{T}_2)$. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. By virtue of Theorem 3, the solvability of equation (17) (and therefore the solvability of equation (13)) depends on whether the set $\mathfrak{N}_\delta^{\mathcal{T}}$ corresponding to these Φ_δ and $\mathcal{T}$ is empty or not. Therefore, to prove Theorem 5 it only remains to show that both of the sets $\mathfrak{N}_\delta^{\mathcal{T}}$ and $\mathfrak{N}_\varsigma^{\Gamma}$ are empty or nonempty simultaneously. The corresponding proof is contained in [6]. This completes the proof of Theorem 5.

REFERENCES

[1] I.Gelfand, M.Graev and N.Vilenkin, *Integral Geometry and Representation Theory*, New York, Academic Press, 1966.

[2] M.Kuczma, B.Choczewski and R.Ger, *Iterative Functional Equations*, Encyclopedia of Mathematics and its applications 32, Cambridge University Press, Cambridge, 1990.

[3] B.Paneah, On a Problem in Integral Geometry Connected to the Dirichlet Problem for Hyperbolic Equations, International Mathematical Research Notes (IMRN), (1997), No.5, 213 - 222.

[4] B.Paneah, On solvability of functional equations relating to dynamical systems with two generators, Functional Analysis and Applications, 2003, v.37, No.1, 46-60.

[5] B.Paneah, Dynamic Methods in the General Theory of Cauchy Type Functional Equations, Israel Math.Conf.Proc. (IMCP), v.17, 2003.

[6] B.Paneah, Dynamic Approach to Some Problems in Integral Geometry, to be published in Trans.Amer.Math.Soc.

[7] M.Zdun, On the Uniqueness of Solutions of the Functional Equation $\varphi(x + f(x)) = \varphi(x) + \varphi(f(x))$, Aequationes Math. (1972), 8, 229 - 232.

Received May 2002; revised February 2003.

E-mail address: `peter@tx.technion.ac.il`