Support Dependent Weighted Norm Estimates for
Fourier Transforms

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1. INTRODUCTION

Consider a bounded convex domain \( \Omega \subseteq \mathbb{R}^n, \Omega \neq 0 \), with the support function

\[ \Re_\Omega(\tau) = \sup_{x \in \Omega} \langle x, \tau \rangle, \]

where \( x = (x_1, x_2, \ldots, x_n) \in \Omega, \tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}^n, |\tau|^2 = (\sum \tau_j^2)^{1/2} = 1, \)
and \( \langle x, \tau \rangle = x_1\tau_1 + x_2\tau_2 + \cdots + x_n\tau_n \). For \( p \geq 1 \) define the linear space \( L_{\Re_\Omega} \) of continuous complex-valued functions \( u \) on \( \mathbb{R}^n \) with the norm

\[ |u|_p = \left( \int_{\mathbb{R}^n} |u(\xi)|^p d\xi \right)^{1/p}, \]

for which the Fourier transform

\[ \hat{u}(x) = (2\pi)^{-n} \int u(\xi) e^{ix \cdot \xi} d\xi \]

is compactly supported in \( \Omega \). For \( p = \infty \) we interpret the norm \( |\cdot|_p \) as the ess sup-norm. It is well known that for finite \( p \) the elements of \( L_{\Re_\Omega} \) vanish

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at infinity in \( \mathbb{R}^n \). In addition they may be extended into the complex space \( \mathbb{C}^n \) as entire functions of exponential type.

In this work we propose a new general method for obtaining support dependent inequalities of the type

\[
|Q u|_p \lesssim K(\Omega)|u|_s, \quad 1 \leq p \leq \infty, \tag{1}
\]

for a wide class of weight functions \( Q \). The bound \( K(\Omega) \) depends on \( Q \) (and \( p \)) but is independent of the functions \( u \in L_{p,0} \) for which \( Q u \in L_p \).

The essential features of these inequalities are:

1. The function \( K(\Omega) \) is explicitly expressed in terms of the support function \( \mathcal{H}_n(\cdot) \).

2. This function grows to infinity as the domain \( \Omega \) shrinks in the weakest sense (see below).

We postpone the detailed discussion of the results until Section 3 and now we make two general remarks. First, note that one can treat the inequality (1) as a lower estimate for the convolution operator \( \tilde{u} \to \tilde{Q} \ast \tilde{u} \), where \( \tilde{Q} \) is a distribution whose Fourier transform is \( Q \). The class of operators \( \tilde{Q} \ast \cdot \) under consideration includes some classical potentials and all linear partial differential operators with constant (complex) coefficients. In particular, we obtain good estimates for the constants in the well known inequalities of Hörmander and Gårding for such operators. On the other hand, it is obvious that \( K(\Omega) \) may tend to infinity as the domain \( \Omega \) shrinks only if \( Q \) is unbounded. This allows us to interpret the inequality (1) as some form of an uncertainty principle. Such a physical interpretation has been discussed in detail in numerous papers in the past 30–35 years (see, e.g., [13] and the references therein).

Although this paper deals with functions of several variables it also includes some new one-dimensional inequalities (see Section 3). We will not elaborate on such results (with the exception of one example) because the one-dimensional case is discussed in detail in [12].

The proposed method is based on some special inequalities in certain entire function spaces. Actually, the applicability of this technique is not restricted either by the type of weight functions \( Q \) considered or by the class of the problems themselves. For instance, some applications of our method are connected with uniqueness theorems for initial value problems for evolution equations with multidimensional time (cf. Agmon and Nirenberg [1]). We expect to return to these questions later.

2. Basic Notation and Definitions

In addition to the notation of the Introduction, for an arbitrary measurable set \( U \subset \mathbb{R}^n \) and unit vector \( r \in \mathbb{R}^n \) we denote by \( d_r(U) \) the diameter
and by $\delta_{r}(U)$ the width of the set $U$ in the direction $r$. In other words,

$$
\delta_{r}(U) = \sup_{x \in U} \{ t \in \mathbb{R} : x + tr \in U \}
$$

(2)

where $\text{mes}$ is the Lebesgue measure and for any convex domain $\Omega$,

$$
\delta_{r}(\Omega) = \mathcal{H}(r) + \mathcal{H}(-r).
$$

Let $U$ be a set in $\mathbb{R}^n$ and $\omega$ be an orthogonal transformation on $\mathbb{R}^n$. Then we put

$$
\omega \cdot U = \{ z \in \mathbb{R}^n : z = \omega y, y \in U \}.
$$

It is worth pointing out that for arbitrary $r$, $\omega$, and $U$ as defined above,

$$
\delta_{r}(\omega \cdot U) = \delta_{\omega \cdot \omega^*}(U) \tag{3}
$$

where $\omega^*$ denotes the usual conjugate matrix. This follows from the definition of the width $\delta_{r}(U)$ and from the fact that orthogonal transformations preserve parallelism and distance in $\mathbb{R}^n$.

Let $\frak{R}$ be the Stiefel manifold of all orthonormal bases $w = (w_1, w_2, ..., w_n)$ in the space $\mathbb{R}^n$ and let $e = (e_1, e_2, ..., e_n) \in \frak{R}$ be the standard basis, $e_j = (0, ..., 1, ..., 0)$. We denote by $\omega$ the (orthogonal) transition matrix from the basis $e$ to the basis $w$. Put

$$
\delta_{\omega}(\Omega) = (\delta_{w_1}(\Omega), \delta_{w_2}(\Omega), ..., \delta_{w_n}(\Omega))
$$

and for an arbitrary $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \mathbb{R}^n$ put

$$
\delta_{\omega^}\sigma(\Omega) = \delta_{w_1}(\Omega) \delta_{w_2}(\Omega) \cdots \delta_{w_n}(\Omega).
$$

**Definition 1.** We say that the domain $\Omega$ shrinks if there exists a system of convex domains $\Omega_t$, with unit vectors $e(t)$, $t \geq 0$, such that $\Omega_0 = \Omega$, $\Omega_t \supset \Omega$, for $t < \varepsilon$ and

$$
\delta_{e(t)}(\Omega_t) \to 0 \text{ as } t \to \infty.
$$

In other words, the domain $\Omega$ deforms into a lower dimensional convex set $\Omega_t \subset \Omega$.

We denote by $x = (x_1, x_2, ..., x_n)$ a point in the space $\mathbb{R}^n$. If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is an arbitrary multi-index of nonnegative integers $\alpha_j$, its length is
\[ |x| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ and we put } \]
\[ \xi^m = \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_n^{m_n} \quad \text{and} \quad a! = \alpha_1! \alpha_2! \cdots \alpha_n! . \]

Further, for any nonnegative \( n \)-tuple \( \delta = (\delta_1, \delta_2, \ldots, \delta_n) \) we set
\[ (\delta / \alpha)^m = (\delta_1 / \alpha_1)^m (\delta_2 / \alpha_2)^m \cdots (\delta_n / \alpha_n)^m , \]

where the factor \((\delta_j / \alpha_j)^m\) is omitted if \( \alpha_j = 0 \).

An arbitrary polynomial \( P(\xi) = P(\xi_1, \xi_2, \ldots, \xi_n) \) of degree \( m \) may be written in the form \( P(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha \) where \( a_\alpha \neq 0 \) for at least one multi-index \( \alpha \) with \(|\alpha| = m \).

For arbitrary \( w \in \mathbb{N} \) the polynomial
\[ P \circ \phi : \xi \rightarrow P(\phi(\xi)) \]
is also of degree \( m \).

For every unit vector \( \tau \in \mathbb{R}^n \) let \( \partial_i P(\xi) \) denote the derivative of \( P(\xi) \) in the direction of \( \tau \). If \( w \in \mathbb{N} \) is one of the bases in \( \mathbb{R}^n \) we put \( \partial_\tau^w P(\xi) = \partial_{\tau_1} \partial_{\tau_2} \cdots \partial_{\tau_n} P(\xi) \).

The following definition plays an important role in our work.

**Definition 2.** Given a polynomial \( P(\xi) = \sum a_\alpha \xi^\alpha \), we call a multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) a leading multi-index of \( P(\xi) \) with respect to a basis \( w \in \mathbb{N} \) if

(i) \( \partial_{\tau_1}^w P(\xi) = \text{const} \neq 0 \),

(ii) \( \partial_{\tau_2}^w \partial_{\tau_3}^w \cdots \partial_{\tau_n}^w P(\xi) = 0 \) for all \( j = 1, 2, \ldots, n \) such that \( \alpha_j \neq 0 \).

The set of all leading multi-indices of \( P(\xi) \) with respect to a basis \( w \) will be denoted by \( \mathbb{A}_w(P) \).

**Example.** Consider the polynomial \( P(\xi) = 5 \xi_1 + \xi_2 + \xi_1 \xi_2 \) in \( \mathbb{R}^2 \). Among the three multi-indices \( (3, 0), (0, 3), (2, 2) \), the first two are leading multi-indices of the \( P(\xi) \) with respect to the basis \( e \). The third is not although the corresponding term \( \xi_1 \xi_2 \) is the principal part of the polynomial \( P(\xi) \).

The following lemma summarizes those facts related to the set \( \mathbb{A}_w(P) \) that we shall use later.

**Lemma 1.** For an arbitrary polynomial \( P \) and the basis \( w \in \mathbb{N} \),

(i) \( \mathbb{A}_w(P) = \mathbb{A}_e(P \circ \phi^w) \) and \( \partial_{\tau_1}^w P = \partial_{\tau_1}^e (P \circ \phi^w) \) for all \( e \in \mathbb{A}_w(P) \),

(ii) the set \( \mathbb{A}_w(P) \) is nonempty,

(iii) if \( \alpha \in \mathbb{A}_w(P) \) and \( P(\phi^w) = \sum \alpha_\beta \phi^w \xi^\beta \), then \( \phi^w = \sum \alpha_\beta \phi^w \xi^\beta \).
Proof. (i) By the chain rule,

$$(\partial^\alpha P)(\omega^a \xi) = \partial^\beta (P \circ \omega^a)(\xi).$$

and the polynomial on the right is identically equal to a constant iff the polynomial on the left is.

(ii) According to (i) it is sufficient to prove that for an arbitrary polynomial $P$ the set $\mathfrak{N}_d(P)$ is nonempty.

We define $\beta \in \mathfrak{N}_d(P)$ by an inductive procedure: let $I_1 = \{\alpha: \alpha \not\equiv 0\}$ and put $\beta_1 = \max \{\alpha: \alpha \in I_1\}$. Having chosen $\beta_1, \beta_2, \ldots, \beta_{k-1}$, put $I_k = \{\alpha = (\beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_k, \ldots, \alpha_n) \cap I_{k-1} \text{ and put } \beta_k = \max \{\alpha: \alpha \in I_k\}, k = 1, 2, \ldots, n.$

(iii) For any $\alpha$,

$$\hat{e}^a(P \circ \omega^a)(\xi) = \sum \alpha_k! \alpha_1! \ldots \alpha_n! \beta!/(\beta - \alpha)!$$

where the summation is taken over all indices $\beta$ with $\beta \equiv \alpha$ (i.e., $\beta_k \equiv \alpha_k$, $1 \leq k \leq n$). According to (i), with $\alpha \in \mathfrak{N}_d(P)$ this function coincides with the number $\partial^\alpha P$. It follows that $\partial^\alpha P = \alpha! \alpha_1! \ldots \alpha_n!$.

This completes the proof of the lemma.

3. Statement and Discussion of the Results

Let $P(\xi) = \sum a_n \xi^n$ be an arbitrary complex valued polynomial on $\mathbb{R}^n$ with constant coefficients $a_n$. The weight functions $Q$ we deal with in the course of this work have the form $Q(\xi) = \Phi(|P(\xi)|)$, where $\Phi: [0, \infty) \to [0, \infty)$ is an arbitrary nondecreasing function. We introduce the constant

$$K_d = \sup_{0 \leq \xi \leq \Psi} \delta^d(\xi) \partial^\alpha P.$$

The following theorem contains the main result of the work.

Theorem 1. For an arbitrary polynomial $P(\xi)$ of degree $d$ and for every $p \geq 1$ there exists a constant $c = c(p, n, m)$ such that for all functions $u \in L_p$, the inequality

$$|\Phi(|P|)u|_p \geq c \Phi(K_d(\Omega))|u|_p$$

holds.
If $p = \infty$, then

$$\sup_{\xi \in \mathbb{R}^d} \Phi(|P(\xi)|) |u(\xi)| \geq c \Phi\left(K_p(\Omega) \sup_{\xi \in \mathbb{R}^d} |u(\xi)|\right)$$

(5)

where $c_1 = c(\infty, n, m) < \infty$.

Assuming that $\Phi(\infty) = \infty$ and $\partial_t P \neq 0$ for every non-zero vector $t$, then

$$\Phi(K_1(\Omega)) \to \infty$$

as the domain $\Omega$ shrinks.

Remark. The condition $\partial_t P \neq 0$ for all $t \neq 0$ means that the polynomial $P$ really depends on all variables.

Let us consider some special cases of this theorem (cf. the Introduction). The two first examples are related to the general theory of differential operators. We use the usual notation $D_t = -i\partial_t$, $D_j = (D_1, D_2, \ldots, D_n)$, and for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers $\alpha_j$ we put $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be an arbitrary partial differential operator with constant coefficients (PDO) with characteristic polynomial $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$.

(a) Take $p = 2$, $\Phi(t) = t$. Then the inequality (4) and Parseval's equality give us a support dependent form of Hörmander's inequality for arbitrary PDO,

$$\|P(D)u\|_{L_2} \leq c K_p(\Omega) \|u\|_{L_2}, \quad u \in L_{2,\Omega}.$$  (6)

(b) If $p = 2$, $\Phi(t) = t^{12}$, and $P(\xi) \geq 0$, then (4) is nothing else than a support dependent form of Gårding's inequality

$$\text{Re}(P(D)u, \bar{u}) \geq c K_p(\Omega) \|u\|_{L_2}^2, \quad u \in L_{2,\Omega},$$

for arbitrary PDO, $P(D)$ with a nonnegative symbol.

In these inequalities the dependence $\Omega \to K(\Omega)$ was not previously investigated.

(c) When $p = 1$, Theorem 1 makes it possible to obtain a good estimate for the constants in inequalities of another kind, namely,

$$|\Phi(P)|u| \leq c \Phi(K(\Omega)) |u| \leq c \Phi(K(\Omega)) \sup_n |\bar{u}|, \quad u \in L_{1,\Omega}.$$

(6)
In the one-dimensional case and for \( P(\xi) - \xi \) it follows that for an arbitrary increasing function \( \Phi \) the inequality

\[
|\Phi u| \geq c \Phi(\sigma^{-1}) \sup |u|
\]

is valid for all functions \( u \in L_{(1-\sigma,\infty)} \). If \( \Phi(\sigma) = \infty \) (the most interesting case) then under some condition on the function \( \Phi \) this constant \( \Phi(\sigma^{-1}) \) cannot be improved [12]. It means that if for \( \sigma \to 0 \) the last estimate is true with some \( \phi(\sigma) \) instead of \( \Phi(\sigma^{-1}) \), then \( \phi(\sigma) \leq \text{const} \Phi(\sigma^{-1}) \), \( \sigma \geq 1 \). In particular, for \( \Phi(\xi) = 1 + |\xi|^{\alpha} \), \( \alpha \geq 0 \), the unimprovable estimate

\[
|\Phi u| \geq c \xi^{-\alpha} \sup |u|, \quad u \in L_{(1-\sigma,\infty)},
\]

holds. In the recent work [2] this inequality was obtained with the constant \( c \xi^{-\alpha} \).

(d) When \( p = \infty \) one can interpret the result of Theorem 1 (i.e., the inequality (5)) as a support dependent lower estimate for the multiplication operator in the space of analytic functions. The estimate (5) seems to be new even for PDO, \( P(D) \) and we expect that it will have applications. One of them is the following. Consider the inequality (5) in the one-dimensional case with \( \Phi(t) = t, \Omega = \{| - \alpha < x < \alpha \} \) and \( P(D) = -id/dx \). Combining (3) with the well-known Bernstein inequality for entire functions of exponential type, we conclude that for all \( \delta(x) \in C^\infty(\Omega) \),

\[
\sigma^{-1} \sup_{\xi \in \Omega} \left| \frac{d}{d\xi} u(\xi) \right| \leq \sup_{\xi \in \Omega} |u(\xi)| \leq \sigma \sup_{\xi \in \Omega} |\xi^{-1} u(\xi)|.
\]

This relation between the operator \( d/d\xi \) and the operator of multiplication on \( \xi \) in the space of entire functions of exponential type suggests the following question: Is it true that for every polynomial \( P(\xi) = P(z_1, z_2, \ldots, z_n) \) there exists a constant \( c > 0 \) (which does not depend on \( \Omega \)) such that for all \( u(\xi) \in C^\infty(\Omega) \),

\[
c K_\delta(\Omega) \sup_{\xi \in \Omega} |P(D_\xi) u(\xi)| \equiv \sup_{\xi \in \Omega} |u(\xi)| \leq c^{-1} K_\delta^2(\Omega) \sup_{\xi \in \Omega} |P(\xi) u(\xi)|
\]

where \( D_\xi = (\partial/\partial \xi_1, \partial/\partial \xi_2, \ldots, \partial/\partial \xi_n) \)?

The proof of Theorem 1 is based on two auxiliary propositions which are of independent interest and may have other applications. The first is related to the problem of describing domains in the space \( \mathbb{R}^n \) on which the absolute value of a polynomial \( P(\xi) \) is bounded away from zero by a positive constant \( \delta \). The size and shape of these domains are very important in various applications. There are a number of results in the one-
dimensional case by Polya, Bernstein, and Cartan [3] which associate this constant $\delta$ with the distance between the domain and the zeros of a polynomial. One of the best estimates is given by the following.

**Cartan’s Theorem [3]**. Let $a_1, a_2, \ldots, a_n$ be arbitrary complex numbers. Given any positive $\delta$ one can find in the complex plane $\mathbb{C}$ a system $M_\delta$ of circles with radii $\delta_1, \delta_2, \ldots, \delta_n$ where $\delta_1 + \delta_2 + \cdots + \delta_n = 2\delta$ such that for all points $z \in \mathbb{C} \setminus M_\delta$ the inequality

$$|z - a_1| |z - a_2| \cdots |z - a_n| > (\delta/m)^m m! > (\delta/e)^n$$

holds.

It is obvious that no results of such kind are possible in the space $\mathbb{R}^n$, $n > 1$, because in the general case even the set $\{\xi \in \mathbb{R}^n; P(\xi) = 0\}$ can be unbounded. Nevertheless, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an arbitrary leading multi-index of a polynomial $P(\xi)$, then for any $n$-tuple $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \geq 0$, where $\delta_j = 0$ if $\alpha_j = 0$, we are able to describe some strip-shaped domains $M_\delta \subset \mathbb{R}^n$ such that the inequality $|P(\xi_1, \xi_2, \ldots, \xi_n)| > (\delta/\alpha)^n$ holds outside $M_\delta$. Here $\Gamma_\delta$ is a constant which does not depend on $\delta$. This is done in the following.

**Theorem 2**. Let $P(\xi)$ be an arbitrary complex-valued polynomial of degree $m$ in the space $\mathbb{R}^n$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any leading multi-index of $P(\xi)$ with respect to a basis $w = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{N}^n$, i.e., $\alpha \in \mathbb{N}^n(P)$. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ be a nonnegative $n$-tuple with $\delta_j = 0$ if $\alpha_j = 0$, $1 \leq j \leq n$. Then there is a system of mutually disjoint sets $M_1, M_2, \ldots, M_n$ in $\mathbb{R}^n$ with $\omega$-diameters $d_\omega(M_j) \leq 4\delta_j$, $j = 1, 2, \ldots, n$, such that

$$|P(\xi)| > |P|/\omega(\delta/\alpha)^n$$

for all points $\xi \in M_\delta = \mathbb{R}^n \setminus \bigcup_{j=1}^n M_j$.

For $n = 1$ this is exactly the $\mathbb{R}$-variant of Cartan’s Theorem. A generalization of Cartan’s Theorem to the complex space $\mathbb{C}^n$ is given in [11].

Another step (in providing Theorem 1) which is a crucial part of our method involves some inequalities in the space of entire functions.

We recall that we are dealing with functions $u(\xi)$ whose Fourier transforms $\hat{u}(\xi)$ are supported on a bounded convex domain $\Omega$. Therefore every such function $u(\xi)$ can be extended to an entire analytic function $u(\xi) = u(\xi + m)$. It follows that any subset $M \subset \mathbb{R}^n$ of positive measure generates in the space $L^p_x$ a norm

$$|u|^M_p = \left(\int_M |u(\xi)|^p \, d\xi\right)^{1/p}, \quad 1 \leq p \leq \infty.$$
This norm is obviously weaker than the norm $|\nu|_{p}$. But it turns out that for a wide class of sets $M$

\[ |\nu|_{p} \text{ and } |\nu|_{p}^{M} \text{ are equivalent.} \]

That is, there exists a positive constant $T = T(\Omega, M, p, n)$ such that for all $\nu \in L_{p,0}$,

\[ \left( \int_{\Omega} |\nu(\xi)|^{p} \, d\xi \right)^{\frac{1}{p}} \leq T \left( \int_{\Omega} |\nu(\xi)|^{p} \, d\xi \right)^{\frac{1}{p}}. \]

The problem of describing all such sets $M$ was first formulated (for $p = 2$) in [9] (in connection with a priori estimates for general systems of PDO). The following answer was suggested in [9]:

The set $M$ has the desired property if and only if

\[ \text{there exist two positive constants } \ell \text{ and } r \text{ such that for each ball } S_{\ell} \subset \mathbb{R}^{n} \text{ of radius } \ell, \text{ the measure } \text{mes}(M \cap S_{\ell}) \geq r. \] (7)

The necessity of (7) for any $n \geq 1$ and its sufficiency for $n = 1$ was proved for $p = 2$ in [9] (see also [10] for more general norms and for applications to the theory of PDO). For arbitrary $1 \leq p < \infty$ and $n \geq 1$ the sufficiency was proved in [5, 8]. Recently Gorin [4] found a simple and elegant proof of the sufficiency for all $p$ and $n$, including $p = \infty$. What is important is that he also gave an estimate for the constant $T = T(\Omega, M, p, n)$ (cf. also [6, 7, 14]). The next theorem generalizes the result of Gorin to more general norms,

\[ \|\nu\|_{s,0} = \left( \int_{\Omega} \left| k_{s}(\xi) \nu(\xi) \right|^{p} \, d\xi \right)^{\frac{1}{p}} \quad \|\nu\|_{p} = \left( \int_{\Omega} \left| k_{s}(\xi) \nu(\xi) \right|^{p} \, d\xi \right)^{\frac{1}{p}} \]

where

\[ k_{s}(\xi) = (1 + |\xi|^{s})^{-n}. \]

**Theorem 3.** The condition (7) is necessary and sufficient if $s \geq 0$, $1 \leq p < \infty$ and sufficient if $s = 0$, $p = \infty$ for the inequality

\[ \|\nu\|_{p}^{M} \leq T(\Omega, M, n, p, n)\|\nu\|_{s,0}, \]

to be valid for all functions $\nu \in L_{p,0}$ whose norm $\|\nu\|_{p,0}$ is finite.

Let $w = \{w_{1}, w_{2}, \ldots, w_{n} \} \subset \mathbb{R}$ be an arbitrary basis. Assume that $\text{mes } (M \cap B_{d}) > r$ for every box $B_{d}$ whose sides of length $\varepsilon_{d}(M)$, $\varepsilon_{d}(M)$, ...
\(\sigma_n(M)\) are parallel to the corresponding vectors \(\omega_1, \omega_2, \ldots, \omega_n\). Put \(\sigma_n(M) = (\sigma_1(M), \sigma_2(M), \ldots, \sigma_n(M))\). Then one can choose the constant \(T(\Omega, M, n, p, s)\) to equal

\[c(n, p, s) \exp(-\beta(\omega_\omega(\Omega), \sigma_n(M))),\]

where \(\beta = 3\pi^n \text{vol } \mathfrak{B}_W/r\). If \(s = 0\) we can take \(c(n, p, s) = 2^{-p/n}\).

The assumption \(s \geq 0\) is essential in our proof. But the condition (7) is necessary when \(s < 0\) as well. As for the sufficiency of (7) when \(s < 0\), the proof shows it only for the case \(n = 1\).

4. Proof of Theorem 1

Let \(w = \{w_1, w_2, \ldots, w_n\} \subset \Omega\) be an arbitrary basis and \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) a leading multi-index of the polynomial \(P(\xi)\) with respect to the basis \(w\), i.e., \(\alpha \in \mathfrak{A}_w(P)\). Let \(\delta = (\delta_1, \delta_2, \ldots, \delta_n)\) be any nonnegative \(n\)-tuple with \(\delta_j = 0\) if \(\alpha_j = 0, 1 \leq j \leq n\). According to Theorem 2 there is a set \(M \subset \mathbb{R}^n \cup \bigcup_{j=1}^n M_j\) such that

\[|P(\xi)| \geq |w_\omega P| \cdot (\delta/\alpha)^\omega, \quad \xi \in M,\]

and \(\omega_j\)-diameters \(d_{\omega_j}(M_j)\) of the sets \(M_j\) satisfy the inequalities \(d_{\omega_j}(M_j) \leq 4\delta_j\), for all \(j = 1, 2, \ldots, n\). It follows that

\[\Phi(|P(\xi)|) \equiv \Phi(|w_\omega P|) \cdot (\delta/\alpha)^\omega, \quad \xi \in M, \tag{8}\]

because the function \(\Phi\) is monotonic.

Renumbering the vectors of the basis \(w\), if necessary, one can assume that \(\delta_j \neq 0\) if \(j = 1, 2, \ldots, k, k \leq n\). Let us verify that for every box \(\mathfrak{B}_w\) whose sides of length \(\sigma_j = \sigma_j(M)\) are parallel to the corresponding vectors \(\omega_j, 1 \leq j \leq n\), and \(\sigma_j = 2kd_{\omega_j}(M_j)\) if \(j = 1, 2, \ldots, k\), the measure \(\text{mes } (M \cap \mathfrak{B}_w) > \text{vol } \mathfrak{B}_w/2\). In fact, denote by \(\chi(G)\) the characteristic function of a set \(G \subset \mathbb{R}^n\). Then \(\chi(\mathbb{R}^n \setminus M) = \sum \chi(M_j)\) and therefore

\[\text{mes } (M \cap \mathfrak{B}_w) = \text{mes } \mathfrak{B}_w - \int_{\mathfrak{B}_w} \chi(\mathbb{R}^n \setminus M) = \text{vol } \mathfrak{B}_w - \sum_{j=1}^k \int_{\mathfrak{B}_w} \chi(M_j) \]

\[\geq \text{vol } \mathfrak{B}_w - \sum_{j=1}^k \left( \prod_{j \neq j} \sigma_j \right) d_{\omega_j}(M_j) \]

\[= \text{vol } \mathfrak{B}_w - \text{vol } \mathfrak{B}_w/2 - \text{vol } \mathfrak{B}_w/2.\]
From Theorem 3 it follows that for all functions \( u \in L_{\rho,3} \),

\[
\left( 2 \int_{\mathscr{H}} |u(\xi)|^p \, d\xi \right)^{1/p} \leq \exp(-\beta \langle \sigma_u(\Omega), \sigma_u(M) \rangle) \|u\|_p
\]  

(9)

where \( \sigma_u(M) = (\sigma_1, \sigma_2, \ldots, \sigma_n) \). Further, since \( \sigma_j \leq 8k\delta_j, 1 \leq j \leq k \), for an arbitrarily small \( \varepsilon > 0 \) we can choose \( \sigma_j, j > k \), such that the inequality

\[
\langle \sigma_u(\Omega), \sigma_u(M) \rangle \geq 8k \langle \sigma_u(\Omega), \delta \rangle + \varepsilon \beta
\]

(10)

holds. Using successively inequalities (8)–(10) we obtain

\[
|\Phi(P)|u|_{p,3} = \left( \int_{\mathscr{H}} |\Phi(P(\xi))u(\xi)|^p \, d\xi \right)^{1/p} \\
\geq \left( \int_{\mathscr{H}} |\Phi(P(\xi))u(\xi)|^p \, d\xi \right)^{1/p} \\
\geq \Phi(\sigma_u^{-1}(\delta/\alpha)^p) \left( \int_{\mathscr{H}} |u(\xi)|^p \, d\xi \right)^{1/p} \\
\geq 2^{-1p} \Phi(\sigma_u^{-1}(\delta/\alpha)^p) \exp(-\gamma(\sigma_u(\Omega), \delta) - \varepsilon) |u|_{p,3},
\]

(11)

where \( \gamma = 8k\beta = 48\pi^2k \). To complete the proof of the first assertion of Theorem 1 we substitute in (11) \( \delta = \alpha/\sigma_u(\Omega) \). It gives the estimate

\[
|\Phi(P)|u|_{p,3} \geq e^{\Phi(\sigma_u^{-1}(\delta/\alpha)^p)P} |u|_{p,3}
\]

(12)

with \( e^* = 2^{-1p} e^{-\gamma m} \). It remains to replace the constant \( |\Phi(P)|\sigma_u^*(-\alpha)^p(\Omega) \) by its maximum over all \( w \in \mathscr{W} \) and \( \alpha \in \mathfrak{A}(P) \) and to use the monotonic property of the function \( \Phi \).

To verify the inequality (5) it suffices to let \( p \to \infty \) in (4) and to use the fact that \( c^{\Phi,\infty} \leq c^{\Phi,\infty, \rho} < \infty \) (Theorem 3).

To prove the last assertion of Theorem 1 let us consider a set of domains \( \Omega_k \) and unit vectors \( \tau(t) \) which satisfy Definition 1, i.e., \( \delta_{\tau(t)}(\Omega_k) \to 0 \) as \( t \to \infty \). For an arbitrary sequence \( t_k \to \infty \) we can find a subsequence \( t_{k(k)} \) such that the vectors \( \tau(t_{k(k)}) \) tend to a unit vector \( \xi \) as \( k \to \infty \). Put \( \tau(t_{k(k)}) = \tau_k, \Omega(t_{k(k)}) = \Omega_k \). Then \( \delta_{\tau_k}(\Omega_k) \to 0 \). On the other hand,

\[
\delta_{\tau_k}(\Omega_k) = \sup_{x \in \Omega_k} \langle x, \tau_k \rangle + \sup_{x \in \Omega_k} \langle x, -\tau_k \rangle \\
= \sup_{x \in \Omega_k} (\langle x, \xi \rangle + \langle x, \tau_k - \xi \rangle) + \sup_{x \in \Omega_k} (\langle x, -\xi \rangle + \langle x, \tau_k - -\xi \rangle) \\
\leq \delta_{\tau_k}(\Omega_k) - 2 \sup_{\xi \in \Omega_k} \|x\|_{\infty} \|\tau_k - \xi\|_{\infty}.
\]
Since $\Omega \subset \Omega$ and $z_1 \rightarrow \xi_1$ we find that the second term of the right side of
the last inequality tends to zero as $k \rightarrow \infty$. This means that $\delta_t(\Omega_0) \rightarrow 0$, as
$k \rightarrow \infty$. Let $w = (\xi_1, \xi_2, ..., \xi_n)$ be an arbitrary basis in $\mathbb{R}^n$, which includes
the vector $\xi_n$, and let $(\xi_1, \xi_2, ..., \xi_n)$ be corresponding coordinates in $\mathbb{R}^n$.
Then our polynomial $P$ can be written in the form $P(\xi) = \sum a_{\alpha} \xi^\alpha$ and the
set of indices $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, with $\alpha_1 \neq 0$, is not empty by the
condition $\delta_t P \neq 0$. This means that one of these indices is a leading index
and therefore

$$K_{P}(\Omega_0) \geq \mu_1 := |\partial^{\alpha} P| \delta^{\alpha}(\Omega_0)\delta_{\alpha_1}(\Omega_0) \cdots \delta_{\alpha_n}(\Omega_0).$$

Since all sets $\Omega_t$ are included in $\Omega$ we find that for all $t$

$$\delta^{\alpha}(\Omega_t)\delta_{\alpha_1}(\Omega_t) \cdots \delta_{\alpha_n}(\Omega_t) > \text{const} > 0.$$

Hence $\mu_1 \rightarrow \infty$ as $k \rightarrow \infty$ and a fortiori $K_{P}(\Omega_0) \rightarrow \infty$. Thus we have proved
that from an arbitrary sequence $K_{P}(\Omega_j), j \rightarrow \infty$, one can choose a sub-sequence
$K_{P}(\Omega_{j,0})$ which tends to $\infty$ together with $k$. This means that
$K_{P}(\Omega_t) \rightarrow \infty$ as $t \rightarrow \infty$, and Theorem 1 is proved.

5. Proof of Theorem 2

Let us first consider the case of the basis $w = e$. Assume initially that all
components $\alpha$ of the leading index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ are positive. From
the conditions of Theorem 2 and from the definition of leading multi-index
it follows that the polynomial $P(\xi)$ may be represented in the form

$$P(\xi_1, \xi_2, ..., \xi_n) = A \xi_1^{\alpha_1} P_2(\xi_2, \xi_3, ..., \xi_n) + \sum_{\alpha \neq 0} \xi^{\alpha-\alpha} P_{\alpha}(\xi_1, \xi_2, ..., \xi_n)$$

where $A = \partial^{\alpha} P(\alpha)$, and

$$P_2(\xi_2, \xi_3, ..., \xi_n) = \xi_2^{\alpha_2} P_2(\xi_3, ..., \xi_n) + \sum_{\alpha \neq 0} \xi^{\alpha-\alpha} P_{\alpha}(\xi_2, ..., \xi_n) \cdots$$

$$P_{\alpha}(\xi_1, \xi_2, ..., \xi_n) = \xi_\alpha^{\alpha_\alpha} P_{\alpha}(\xi_\alpha + \sum_{\alpha \neq 0} \xi^{\alpha-\alpha} P_{\alpha}(\xi_\alpha, ..., \xi_\alpha)$$

$$P_d(\xi_\alpha) = \xi_\alpha^{\alpha_\alpha} + \sum_{\alpha \neq 0} P_{\alpha} \xi^{\alpha-\alpha}.$$

According to Cartan's theorem in the one-dimensional space $\mathbb{R}^1$ of the
variable $\xi_\alpha$ one can find a set $K_{\alpha}$ such that

$$\{\xi_\alpha \in \mathbb{R}, |P_d(\xi_\alpha)| < (\delta_{\alpha}(\alpha_n)^\alpha) \subset K_{\alpha}, \quad \text{and} \quad d_{\alpha}(K_{\alpha}) \leq 48_{\alpha},$$
Put \( \hat{K}_n = \mathbb{R} \setminus K_n \). Then
\[
|P_n(\xi_n)| \geq (\delta_0/\alpha_0)^n \quad \text{for} \quad \xi_n \in \hat{K}_n.
\]
In the space \( \mathbb{R}^2 \) of the variables \((\xi_n, \xi_n')\) for arbitrary fixed \( \xi_n \in \hat{K}_n \) let us consider the straight line
\[
\mathbb{R}^2(\xi_n) = \{(\xi_n, \xi_n') \in \mathbb{R}^2; -\infty < \xi_n < \infty\}.
\]
Applying Cartan's Theorem again we define a set \( K_{n-1}(\xi_n) \in \mathbb{R}^2(\xi_n) \) of the points \((\xi_{n-1}, \xi_n)\) such that
\[
|P_{n-1}(\xi_{n-1}, \xi_n)| < (\delta_{n-1}/\alpha_{n-1})^{n-1} \quad \text{in} \quad K_{n-1}(\xi_n)
\]
and \( d_{n-1}(K_{n-1}(\xi_n)) = 4\delta_{n-1} \). Put \( \hat{K}_{n-1}(\xi_n) = \mathbb{R}^2(\xi_n) \setminus K_{n-1}(\xi_n) \) and define in \( \mathbb{R}^2 \) the sets
\[
K_{n-1} = \bigcup_{\xi_n \in \hat{K}_n} K_{n-1}(\xi_n) \quad \text{and} \quad \hat{K}_{n-1} = \bigcup_{\xi_n \in \hat{K}_n} \hat{K}_{n-1}(\xi_n).
\]
It is clear that \( d_{n-1}(K_{n-1}) = 4\delta_{n-1} \) and the inequality
\[
|P_{n-1}(\xi_{n-1}, \xi_n)| \geq (\delta_{n-1}/\alpha_{n-1})^{n-1} (\delta_0/\alpha_0)^n
\]
hold for all points \((\xi_{n-1}, \xi_n) \in \hat{K}_{n-1} \).

Continuing this procedure we find after \( n - 1 \) steps the sets \( K_1 \in \mathbb{R}^n \) and \( \hat{K}_1 \in \mathbb{R}^n \) consisting of the points \((\xi_1, \xi_2, ..., \xi_n)\) such that \( d_n(K_1) \leq 4\delta_1 \),
\[
|P(\xi_1, \xi_2, ..., \xi_n)| > |A|\delta_1^\alpha \delta_2^\alpha \cdots \delta_n^\alpha
\]
for all \((\xi_1, \xi_2, ..., \xi_n) \in \hat{K}_1 \) and \( K_1 \cup \hat{K}_1 = \{(\xi_1, \xi) \in \mathbb{R}^n; \xi' \in \hat{K}_1, -\infty < \xi_1 < \infty\} \). Consider the sets
\[
M_1 = K_1, \quad M_j = K_j \times \mathbb{R}^{n-j}, \quad j = 2, 3, ..., n,
\]
where \( \mathbb{R}^\ell \) is the \( \ell \)-dimensional space of variables \((\xi_1, \xi_2, ..., \xi_\ell)\), \( \ell = 1, 2, ..., n - 1 \). It is obvious that the set
\[
M_{n,0} = \mathbb{R}^n \setminus \bigcup_{j=1}^n M_j
\]
satisfies all the required conditions:

\[ M_k \cap M_{\ell} = \emptyset \text{ if } k \neq \ell; \ d_{\gamma}(M_k) \leq 4\delta_j, \quad \text{if } 1 \leq j \leq n, \]

and finally,

\[ |P(\xi)| > |\partial_{\alpha}^\nu P(\delta/\alpha)^\nu \quad \text{for } \xi \in M_{n,h}. \]

This proves Theorem 2 if all \( \alpha_j \) are positive. If some \( \alpha_j \) are equal to zero and only the components \( \alpha_k, \alpha_{k+1}, \ldots, \alpha_h \) of \( \alpha \) do not vanish then we introduce in the space \( \mathbb{R}^a \) a new coordinate \( \xi = (\xi_1, \xi_2, \ldots, \xi_a) \) where \( \xi_j = \xi_{j'} \) for \( j = 1, 2, \ldots, q \) and for \( j > q \) the coordinates \( \xi_j \) coincide with one of \( \xi_{j'} = \xi_{k_j} \). We next construct the set \( \tilde{M} \subset \mathbb{R}^q \), as above, and define the desired set \( M_{n,h} \) as

\[ M_{n,h} = \tilde{M} \times \mathbb{R}^{a-q} \]

where \( \mathbb{R}^{a-q} \) is the space of variables \( (\xi_{q+1}, \xi_{q+2}, \ldots, \xi_a) \). This proves Theorem 2 for \( w = e \).

If \( w \in \mathbb{N} \) is an arbitrary basis, we introduce in \( \mathbb{R}^a \) new coordinates \( \lambda = \omega \xi \). Let \( \mathcal{P}(\lambda) = P(\omega^\nu \lambda) \). According to Lemma 1 (i), if \( \alpha \in \mathcal{F}(P) \) then \( \alpha \in \mathcal{F}(\lambda) = \mathcal{F}(\mathcal{P}) \). It is obvious that the set \( M_{n,h} \) which relates to the polynomial \( \mathcal{P} \) and the basis \( e \) (in the space \( \mathbb{R}^a \) of the variable \( \lambda \)) satisfies all needed conditions. In fact, each orthogonal transformation in \( \mathbb{R}^a \) preserves parallelism and distance. Using the fact that \( \omega = \omega^\nu e \), it follows that the \( \omega \)-diameters \( d(M_j) \) of all sets \( M_j, j = 1, 2, \ldots, n \) (see the definition of the \( M_{n,h} \)) coincides with given numbers \( \delta_j \). On the other hand, for all points \( \xi \in M_{n,h} \) after the change of variables \( \xi = \omega^\nu \lambda \), we find that

\[ |P(\xi)| = |P(\omega^\nu \lambda)| = |\mathcal{P}(\lambda)| = |\partial_{\alpha}^\nu \mathcal{P}(\delta/\alpha)^\nu \]

\[ = |\partial_{\alpha}^\nu (\partial_{\alpha}^\nu \mathcal{P})(\delta/\alpha)^\nu| = |\partial_{\alpha}^\nu \mathcal{P}(\delta/\alpha)^\nu|, \]

where the last equality is based on Lemma 1 (i). This completes the proof of Theorem 2.

6. Proof of Theorem 3

We begin with a lemma giving us, in particular, a simple example of a set \( M \) of the considered type (see the remark following the lemma).

**Lemma 2.** For arbitrary \( s \in \mathbb{R}^l \) and finite \( p \geq 1 \) there exists a constant \( T(s, p, \Omega) \) such that if \( \text{mes } G < T(s, p)e \) for some \( e > 0 \) then for all
functions \( \hat{u}(x) \in C_0(\Omega) \) the inequality
\[
\int_{\Omega} |k_\epsilon(\xi)u(\xi)|^p \, d\xi \leq c \int_{\Omega} |k_\epsilon(\xi)u(\xi)|^p \, d\xi
\]
holds.

To prove this assertion we choose a \( C^\infty \)-function \( \varphi(x) \) in \( \mathbb{R}^\ell \) which is equal to 1 in \( \Omega \). Then \( \hat{u}(x) = \varphi(x)u(x) \) for all \( \hat{u}(x) \in C_0(\Omega) \) and
\[
k_\epsilon(\xi)u(\xi) = \int_{\Omega} k_\epsilon(\xi)\varphi(x)u(x)e^{-i(x,\xi)} \, dx
= \int_{\Omega} k_\epsilon(\eta)u(\eta)\varphi(\xi - \eta)g(\xi, \eta) \, d\eta
\]
where \( g(\xi, \eta) = ((1 + |\xi|^2)/(1 + |\eta|^2))^{\tau/2} \). By the triangle inequality \( g(\xi, \eta) \leq 2^{\tau/2}(1 + |\xi - \eta|^2)^{\tau/2} \), and using Hölder's inequality we can conclude that
\[
|k_\epsilon(\xi)u(\xi)|^p \leq 2^{\tau/2}||u||_{L^{p,\ell}}^p ||\varphi||_{L^{1,\ell}}^p
\]
with \( p^{-1} + p^{*,-1} = 1 \). Integrating this inequality over \( \Omega \) gives us the desired result with the constant \( T = (2^{\tau}||\varphi||_{L^{1,\ell}})^{-1} \).

Remark. It follows that if \( 0 < \epsilon < 1 \) and \( \text{mes} \, G < T \epsilon \) then for the set \( M = \mathbb{R}^\ell \setminus G \),
\[
\int_M |k_\epsilon(\xi)u(\xi)|^p \, d\xi > \epsilon(1 - \epsilon) \int_{\Omega} |k_\epsilon(\xi)u(\xi)|^p \, d\xi, \quad u \in L^{p,\ell}.
\]

Proof of Necessity. Assume that condition (7) is violated. We will prove that for arbitrary \( \epsilon > 0 \) one can find a function \( \bar{u} \in C_0(\Omega) \) such that
\[
\int_M |k_\epsilon(\xi)\bar{u}(\xi)|^p \, d\xi \leq 2\epsilon \int_{\Omega} |k_\epsilon(\xi)\bar{u}(\xi)|^p \, d\xi. \tag{13}
\]
By assumption, for an arbitrarily large \( N \) one can find a ball \( S_N(z) \) of radius \( N \) with center at a point \( z \in \mathbb{R}^\ell \) such that
\[
\text{mes}(M \cap S_N(z)) < T \epsilon \tag{14}
\]
(the number \( T \) is defined in Lemma 2). One may assume that \( |z| > N^2 \). Otherwise, for all balls \( S_N(z) \) with \( |z| > N^2 \) the inequality
\[
\text{mes}(M \cap S_N(z)) \geq T \epsilon
\]
holds. It follows that the same inequality is valid for all balls of radius \( N^2 + N \) with arbitrary center. But this then contradicts our assumption.
Fix some function $\tilde{u} \in C_0^\infty(\Omega)$, $\tilde{u} \neq 0$, and choose a number $N_0$ such that for all $N > N_0$ the inequalities
\[
\int_{|\xi| > N} |u(\xi)|^p \, d\xi \leq c(s) \varepsilon (1 - \varepsilon)^{-1} \int_{|\xi| < N} |u(\xi)|^p \, d\xi
\]
\[
\int_{|\xi| > N} |u(\xi)||\xi|^p \, d\xi \leq c(s) \varepsilon (1 - \varepsilon)^{-1} \int_{|\xi| < N} |u(\xi)||\xi|^p \, d\xi
\]
hold with the constant $c(s) = 8^{-1} \inf_{a,b} \psi_n(a, b)/\sup_{a,b} \phi_n(a, b) \phi_n(a, b) = (1 + a^2 + b^2)^{p(n-1)/2}(1 + |a|^n + |b|^n)$. Then for any point $z \in \mathbb{R}^n$ we have
\[
\int_{|\xi| > N} |u(\xi)|^p(1 + |\xi|^p + |z|^p) \, d\xi 
\leq c(s) \varepsilon \int_{|\xi| < N} |u(\xi)||\xi|^p(1 + |\xi|^p + |z|^p) \, d\xi
\]
and hence
\[
\int_{|\xi| > N} |u(\xi)|^p(1 + |\xi|^p + |z|^p)^{n/2} \, d\xi 
\leq 8^{-n} \varepsilon (1 - \varepsilon)^{-1} \int_{|\xi| > N} |u(\xi)|^p(1 + |\xi|^p + |z|^p)^{n/2} \, d\xi.
\]

Note that if $N$ is sufficiently large and $|z| > N^2$, then for all points $\xi$ from the ball $|\xi| < N$ we have the inequality
\[
|z| + \varepsilon^2 \geq |z|^2/2 - \varepsilon^2 \geq (|z|^2 + N^2)/4 - N^2
\]
\[
\geq (|z|^2 + N^2)/4 \geq (|\xi|^2 + |z|^2)/4.
\]

On the other hand, for all $\xi, z$,
\[
|\xi + z|^2 \leq 2(|\xi|^2 + |z|^2).
\]

Combining these three last inequalities we find that
\[
\int_{|\xi| > N} |u(\xi)|^p(1 + |\xi + z|^p)^{n/2} \, d\xi 
\leq \varepsilon (1 - \varepsilon)^{-1} \int_{|\xi| < N} |u(\xi)|^p(1 + |\xi + z|^p)^{n/2} \, d\xi.
\]

For the new function $w(\xi) = u(\xi - z)$ this inequality takes the form
\[
\int_{|\xi| > N} |k_n(\xi)|^p w(\xi)|^p \, d\xi \leq \varepsilon \int_{|\xi| < N} |k_n(\xi)|^p w(\xi)|^p \, d\xi.
\]

Choose a number $N$ and a point $z$ such that both inequalities (14) and (15) are valid. Then apply Lemma 2 to the set $G = M \cap \Delta_N(z)$. Combining this
result with (15) we obtain the inequality (13) for the function \( \psi \). This completes the proof of the necessity.

**Remark.** If \( s < 0 \) and the condition (7) is violated we define an integer \( N \) such that \( s' - 2N + \varepsilon \geq 0 \). Let \( \psi \in C^\infty(\Omega) \) be a function for which the inequality (13) is valid with \( s' \) (instead of \( s \)). But then for the function \( \psi = (1 - \Delta)^{s'} \psi \) (here \( \Delta \) is the Laplace operator) the same inequality is valid with \( s \). This proves the necessity of the condition (7) for negative \( s \).

**Proof of Sufficiency.** We begin with the case \( w = e \). Because of the convexity of the logarithmic function it is easy to verify that if \( \mu \) is a probability measure on the space \( \mathbb{R}^n \) and \( M_1, M_2, \ldots, M_n \) are arbitrary subsets of \( \mathbb{R}^n \) such that \( \mu(M_j) = \mu_j \), and

\[
M_i \cap M_j = \phi, \quad i \neq j, \quad \bigcup_{j=1}^n M_j = \mathbb{R}^n,
\]

then for all nonnegative functions \( f \) on \( \mathbb{R}^n \)

\[
\exp \int_{\mathbb{R}^n} \log f \, d\mu \leq m \prod_{j=1}^n \left( \int_{M_j} f \, d\mu_j \right)^{\mu_j}.
\]  \hspace{1cm} (16)

For \( m = 1 \) this is the well-known Jensen inequality. For \( m = 2 \) from this inequality it follows that

\[
\text{if } \mu(M) > \nu \text{ then } \exp \int_{\mathbb{R}^n} \log f \, d\mu \leq 2 \left( \int_{\mathbb{R}^n} f \, d\mu \right) \left( \int_{\mathbb{R}^n} f \, d\nu \right)^{1-\nu}.
\]  \hspace{1cm} (17)

The following Gorin's inequality is the key fact needed in the proof of sufficiency.

**Lemma 3.** If \( M \) is a subset of \( \mathbb{R}^n \) and \( \mu(M + \xi) \geq 0 \) for all \( \xi \in \mathbb{R}^n \), then for all nonnegative functions \( f \in L^1(\mathbb{R}^n) \) the inequality

\[
\int_{\mathbb{R}^n} \exp \int_{\mathbb{R}^n} \log f(\xi + \tau) \, d\mu(\tau) \, d\xi \geq 2 \left( \int_{\mathbb{R}^n} f \, d\xi \right) \left( \int_{\mathbb{R}^n} f \, d\xi \right)^{1-\nu}
\]  \hspace{1cm} (18)

holds.

\(^1\) This proof, except for the details connected with norms \( \| \cdot \|_{\infty}, \| \cdot \|_s \neq 0 \), repeats Gorin's proof \([4]\) which in turn uses some ideas from \([8]\). We give a complete proof because \([4]\) is undoubtedly not available to the Western reader.
(The set $M + \xi$ is the union of all point $\xi \in \mathbb{R}^*$ of the form $\xi = \lambda + \xi$ with $\lambda \in M$.)

**Proof.** To prove (18) we introduce a probability measure $\mu_\xi$ such that for all integrable functions $g$,

$$\int g(\xi + \tau) \, d\mu(\tau) = \int g(\tau) \, d\mu_\xi(\tau).$$

Then $\mu_\xi(M) \equiv \nu$ and by inequality (17) we find that

$$\exp \int_{\mathbb{R}^*} \log f(\xi + \tau) \, d\mu(\tau) = \exp \int_{\mathbb{R}^*} \log f(\tau) \, d\mu_\xi(\tau)$$

$$\leq 2 \left( \int_{\mathbb{R}^*} f(\tau) \chi_M(\tau) \, d\mu_\xi(\tau) \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\eta) \, d\mu_\xi(\eta) \right)^{1-\nu}$$

$$- 2 \left( \int_{\mathbb{R}^*} f(\xi + \tau) \chi_M(\xi + \tau) \, d\mu(\tau) \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\xi + \eta) \, d\mu(\eta) \right)^{1-\nu},$$

where $\chi_M$ is the characteristic (indicator) function of the set $M$. Using Hölder's inequality

$$\int g_1 \, d\xi \leq \left( \int g_1 \, d\xi \right)^{1/\nu} \left( \int g_2 \, d\xi \right)^{1-\nu}$$

for nonnegative functions $g_1$, $g_2$ and Fubini's theorem, we find that

$$\int_{\mathbb{R}^*} \left( \int_{\mathbb{R}^*} f(\xi + \tau) \chi_M(\xi + \tau) \, d\mu(\tau) \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\xi + \eta) \, d\mu(\eta) \right)^{1-\nu} \, d\xi$$

$$\leq \left( \int_{\mathbb{R}^*} f(\xi + \tau) \chi_M(\xi + \tau) \, d\mu(\tau) \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\xi + \eta) \, d\mu(\eta) \right)^{1-\nu}$$

$$= \left( \int_{\mathbb{R}^*} f(\xi) \chi_M(\xi) \, d\xi \, d\mu(\tau) \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\eta) \, d\mu(\xi) \right)^{1-\nu}$$

$$= \left( \int_{\mathbb{R}^*} f(\xi) \, d\xi \right)^{1/\nu} \left( \int_{\mathbb{R}^*} f(\xi) \, d\xi \right)^{1-\nu},$$

and Lemma 3 is proved.
In what follows we use some properties of polyanalytic functions. The upper semi-continuous real-valued function \( u(\xi), \xi = \xi + i\eta \in \mathbb{C}^n \), is called polyanalytic if it is subharmonic with respect to any variable \( \xi_j, 1 \leq j \leq n \), with the others held fixed. Analogously, polyanalytic functions can be defined on the product \( \mathbb{C}^n \) of half-planes \( \eta_j \geq 0, 1 \leq j \leq n \). Introduce a probability measure

\[ P(\xi, \tau) \, d\tau = \prod_{k=1}^n \frac{\eta_k}{\pi |\xi_k - \tau_k|^2} \]

which is the product of the one-dimensional Poisson forms on the half-spaces \( \mathbb{C}^n \). It is well known that if \( g(\xi) \) is a polyanalytic function on \( \mathbb{C}^n \) for which

\[ g(\xi) \leq \sum_{k=1}^n \sigma_k |\xi_k| + \lambda \]

for some nonnegative \( \sigma_k \) and \( \lambda \) and

\[ \int_{\mathbb{R}^n} g^*(\xi) \prod_{k=1}^n \frac{d\xi_k}{1 + \xi_k^2} < \infty \]

where, as usual, \( g^* = \max\{g, 0\} \), then

\[ g(\xi) \leq \sum_{k=1}^n \sigma_k \eta_k + \int_{\mathbb{R}^n} g(\tau)P(\xi, \tau) \, d\tau. \]

If \( g = \ln|G(\xi)| \) then it follows from (21) that

\[ |G(\xi)|^p \leq e^{p\sigma_\infty} \exp \int_{\mathbb{R}^n} \ln|G(\tau)|^p P(\xi, \tau) \, d\tau. \]

Let \( d\mu(\tau) \) be the Poisson measure

\[ P(i\eta, \tau) \, d\tau = \prod_{i=1}^n \frac{n_k d\tau_k}{\pi(\tau_k^2 + \eta_k^2)}. \]

After a change of variable \( \tau \to \tau + \xi \) we rewrite the last inequality in the form

\[ |G(\xi)|^p \leq e^{p\sigma_\infty} \exp \int_{\mathbb{R}^n} \ln|G(\tau + \xi)|^p d\mu(\tau). \]
Since $d\mu(\tau)$ is a probability measure, it follows from this inequality and (16) that
\[ \int_{\mathbb{C}} |G(\xi + i\eta)|^p \, d\xi \leq c^{p(\alpha, \eta)} \int_{\mathbb{C}} |G(\xi)|^p \, d\xi. \]  
(23)

We prove now that if $u(\xi)$ is an arbitrary function from the space $L_{p, 0}^\alpha$ with the finite norm $\|u\|_{p, 0}$ and $u(\xi) = u(\xi + i\eta)$ is the analytic extension of $u(\xi)$ on the space $C^\alpha$, then the inequality (21) is valid for any function $F_{j, s}(\xi) = \ln|u(\xi)(\xi_j + i\eta)|^s$, $s \in \mathbb{R}$, $1 \leq j \leq n$.

First of all the functions $F_{j, s}$ are poly-subharmonic functions in $C^\alpha$ because $\ln|u|$ is a poly-subharmonic function in $C^\alpha$ for every analytic function $u$ in $C^\alpha$. Further, according to the Paley-Wiener Theorem, for every function $u(\xi) \in L_{p, 0}^\alpha$ there is some constant $B$ such that
\[ |u(\xi)| \leq B \exp\left(\sum_{i=1}^n \delta_i |\xi_i|\right) \]
where $\delta_i = \delta_{\alpha, i}$. But for an arbitrarily small $\epsilon > 0$ one can find a constant $B_{\epsilon, s}$ such that for all $\xi_j \in C^\alpha$,
\[ |\xi_j + i\eta| \leq B_{\epsilon, s} e^{\delta_{\alpha, j}|\eta|}. \]

Using the last two inequalities and putting $BB_{\epsilon, s} = e^{\epsilon}$, we find that for all $\xi \in C^\alpha$,
\[ F_{j, s}(\xi) = p(1 + \epsilon) \sum_{i=1}^n \delta_i |\xi_i| + \lambda. \]

Finally, the inequality (20) holds for the function $g(\xi) = F_{j, s}(\xi)$ because $\ln^+ z < z$ for all $z > 0$ and therefore
\[ \int_{\mathbb{C}} F_{j, s}(\xi) \, d\xi \leq \int_{\mathbb{C}} |u(\xi)(\xi_j + i\eta)|^p \, d\xi - \|u\|^p_{L_{p, 0}^\alpha}. \]

Thus, all the conditions necessary for the validity of inequality (21) (with $g = F_{j, s}$) hold, and we find that
\[ \ln|u(\xi)(\xi_j + i\eta)|^p \leq p(1 + \epsilon) \sum_{i=1}^n \delta_i \eta_i \ln|\xi_j + i\eta| + \int_{\mathbb{C}} \ln|u(\tau)(\tau_j + i\eta)|^p P(\xi, \tau) \, d\tau. \]
It follows that
\[|u(\xi + i\eta)(\xi_j + i\eta_j + i)|^p \leq e^{p\gamma(1 \pm \delta)} \exp \int_{R^n} \ln|u(\xi + \tau(\xi_j + \gamma_j + i))|^p d\mu_{2}(\tau)\]
or after integration
\[\int_{R^n} |u(\xi + i\eta)|^p (\xi_j^2 + (\eta_j + 1)^2)^{\nu/2} d\xi \leq e^{p(1 + \delta \delta)} \int_{R^n} d\xi \exp \int_{R^n} \ln|u(\xi + \tau(\xi_j + \gamma_j + i))|^p d\mu_{2}(\tau).\]
\[(24)\]

Now we estimate the right-hand integral using the inequality (18). To do so one has, first of all, to estimate the measure \(\mu_2(M)\) of the set \(M\) of Theorem 3. But the density \(P(\eta, \tau)\) of this measure is an even function, decreasing in each argument \(\gamma_j\) for \(\tau \geq 0\). It follows that for arbitrary \(\xi \in R^n\),
\[\mu_2(M + \xi) \equiv \left(\frac{2\pi}{\rho}\right)^n r \prod_{k=1}^{n} \frac{\eta_k}{\sigma_k^2 + \eta_k^2},\]
where \(\sigma_k = \sigma_2(M)\). Choosing \(\eta_k = \sigma_k\), 1 \(\leq k \leq n\), we find the best possible estimate for \(\mu_2(M + \xi)\), namely
\[\mu_2(M + \xi) \equiv \nu; = r / \prod_{k=1}^{n} \nu \sigma_k.\]

Returning to (24) and estimating the right-hand integral by (18) we find that
\[\int_{R^n} |u(\xi + i\sigma)(\xi_j^2 + (\sigma_j + 1)^2)^{\nu/2}|^p d\xi \leq 2e^{p(1 + \delta \delta)} \left(\int_{R^n} |u(\xi)(\xi_j^2 + 1)^{\nu/2}|^p d\xi\right)^{1/\nu} \left(\int_{R^n} |u(\xi)(\xi_j^2 + 1)^{\nu/2}|^p d\xi\right)^{1/\nu},\]
\[(25)\]
where \(\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)\). We now estimate from below the left-hand integral in (25). Consider a function
\[G_{\lambda}(\xi) = u(-\xi + i\sigma)(\xi_j + i)^\nu.\]
This function is analytic in \( C^2 \) and one can apply to \( G_{\alpha}(\xi) \) the inequality\(^{(23)}\) with \( \eta = \alpha \) and \((1 + \varepsilon)p\) instead of \( p \). It gives
\[
\int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi \leq e^{\varepsilon(1 + \varepsilon)p,\alpha} \int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi.
\]
Combining this inequality with \((25)\) we obtain
\[
\int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi \leq 2e^{2\varepsilon(1 + \varepsilon)p,\alpha} \left( \int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi \right)^{1/2}
\]
and since \( \xi_j^2 + (\sigma_j + 1)^2 \geq \xi_j^2 + 1 \) it follows that
\[
\int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi \leq 2e^{2\varepsilon(1 + \varepsilon)p,\alpha} \int_{R^n} |u(\xi)\xi_j^2 + (\sigma_j + 1)^2\xi_j^2|\,d\xi.
\]
Summing these inequalities over all \( j = 1, 2, \ldots, n \) we find that
\[
\|u\|_{p,1} \leq 2^{1/2}e^{2\varepsilon(1 + \varepsilon)p,\alpha} \|u\|_{p,1}
\]
where \( c_\alpha^p = \sup_{\xi \in \mathbb{R}^n} |\chi(\xi)\hat{u}(\zeta,\sigma)| \) and \( \chi(\xi) = \Sigma(\xi_j^2 + 1)^{\alpha/2} \). This coincides with the assertion of Theorem 3 in the case \( w = e \) and for finite \( p \geq 1 \).

In the case \( p = \infty, s = 0 \) it suffices to let \( p \to \infty \) in the last inequality. Then the inequality of the theorem for \( p = \infty \) follows from the fact that \( \|u\|_{p,1} \to \text{ess} \sup u \|\) if \( p \to \infty \). This proves the sufficiency for the standard basis \( e \).

In the general case of an arbitrary basis \( w \subset \mathbb{N} \) let us note that the function \( u(\lambda) = u(\omega,\lambda) \) belongs to the space \( L_{p,1} \) if (and only if) \( u \in L_{p,0} \) and
\[
\|u\|_{p,1} = \|u\|_{p,0} \Rightarrow \|u\|_{p,1} = \|u\|_{p,0} \Rightarrow \|u\|_{p,1}.
\]
This follows from the inverse Fourier transform formula and orthogonality of the matrix \( \omega \) by the spherical symmetry of the form \( |\hat{u}(\xi)|^2 \). Further, by the condition of Theorem 3, \( \text{mes}(M \cap \beta_n) > r \) for every box \( \beta_n \) with the sides of the length \( \sigma_j(M) \) which are parallel to the corresponding vectors \( v_j, j = 1, 2, \ldots, n \).
j ≤ n. Owing to the proved part of Theorem 3,

\[ \|b_j\|_{p, s} \leq c(n, p, s) \exp(-\beta(b, \omega(2), \sigma_1(\omega(1)), \sigma_1(\omega(M)))) \|b\|_{p, s}. \]

To complete the proof of this theorem it remains only to note that \( \sigma_1(\omega(M)) = \sigma_1(M) \) according to the choice of \( \sigma_1(\omega(M)), \sigma_1(\omega(1)) = \sigma_1(\Omega) \) by (3), and to use (27).

Remark. This proof is also valid in \( \mathbb{R}^1 \) for \( s < 0 \).

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