

On the Stability of the Linear Functional Operators Structurally Associated with the Jensen Operator

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Abstract

The present work is devoted to the new approach to the stability problem for the general linear functional operator $\mathcal{P}F := \sum_{j=1}^N c_j F(a_j)$ in the closure of a domain in \mathbb{R}^n , where $F \in C(I, B)$ with I an interval in \mathbb{R} and B a Banach space. The operator we deal with structurally remind the general Jensen operator, but the functions a_j may be nonlinear, and the coefficients c_j are nonconstant. The strong and the weak stability of the operator \mathcal{P} are defined as the validity of some à priori estimates for \mathcal{P} , and the first one is significantly stronger than the Ulam stability of \mathcal{P} . Both estimates are established in the spaces $C_{(r)}(I, B)$ of continuous functions having some Hölder smoothness at the initial point of I .

In this work we continue studying the general linear functional operator

$$\mathcal{P}F := \sum_{j=1}^N c_j(x)F(a_j(x)), \quad (1)$$

where $x \in D \subset \mathbb{R}^n$, $n \geq 2$, a_j and c_j are given functions, and $F \in C(I, B)$ is a compact supported Banach - valued function of a single variable. Such operators naturally arise in particular when solving diverse problems in integral geometry and in the theory of boundary problems for hyperbolic partial differential equations. It is in this connection that the nonhomogeneous equations $\mathcal{P}F = H$ were studied for the first time in the author's papers [1]-[3]. In [4] the latter equation has been considered from the point of view of the continuity of the map $H \rightarrow F$. We have

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introduced the notions of the strong and weak stability of the operator \mathcal{P} from which the first one is much stronger than the well - known Ulam stability. The equivalency of the new types of stability to some à priori estimates for the operator \mathcal{P} makes it possible to study it in the framework of the classical functional analysis. In [4] for a wide class of the operators \mathcal{P} structurally associated with the Cauchy operator

$$\mathfrak{C}F := F(x + y) - F(x) - F(y), \quad (x, y) \in D \subset \mathbb{R}^2,$$

the strong and weak stability have been established. This leads automatically to the Ulam stability of such operators (which by no means can be obtained by traditional methods) but also demonstrates that the Ulam problem by itself is significantly overdetermined at least in the framework of the spaces of sufficiently smooth functions (see Problem 1 below).

In the present work we obtain the analogous result for a different class of the operators \mathcal{P} which structurally associated with the classical Jensen operator

$$\mathfrak{J}F := F(\alpha x + \beta y) - \alpha F(x) - \beta F(y), \quad (x, y) \in D \subset \mathbb{R}^2, \quad (2)$$

where α and β are positive constants with $\alpha + \beta = 1$.

To present the results in a self contained form we remind the main definitions from [4].

We denote by D the closure of an arbitrary domain in \mathbb{R}^n , points in D are denoted by x . The kernel and the range of any operator L are denoted by $\ker L$ and $\mathcal{R}(L)$, respectively. We denote by I the interval $\{t \mid 0 \leq t \leq 1\}$. Given a Banach space B with the norm $|\cdot|_B$, B being a module on $C(D)$, we denote by $C(I, B)$ the space of all continuous functions $F : I \rightarrow B$ with the norm $|F| = \sup_{t \in I} |F(t)|_B$. The following function spaces play a crucial role. By definition, if $0 < r < 1$, then

$$C_{\langle 1+r \rangle}(I, B) = \{f \mid f(t) = b_0 + b_1 t + t^{1+r} \varphi(t)\}$$

with $b_i \in B$ and $\varphi \in C(I, B)$ as above. The space $C_{\langle 1+r \rangle}(I, B)$ endowed by the norm

$$|f|_{\langle 1+r \rangle} = |b_0| + |b_1| + |\varphi|$$

is a Banach space.

The space $C_{\langle 1+r \rangle}(D, B)$ is defined analogously: a function $f : D \rightarrow B$ is in $C_{\langle 1+r \rangle}(D, B)$ if

$$f(x) = b_0 + b_1 x_1 + \dots + b_n x_n + |x|^{1+r} \varphi(x),$$

where all $b_j \in B$ and φ is a continuous function: $D \rightarrow B$.

In both cases above we omit the argument B in $C_{\langle 1+r \rangle}(\cdot, B)$ if $B = \mathbb{R}^n$.

Remark 1 Alternatively, $C_{(1+r)}(I, B)$ is the space of all B -valued continuous functions on I differentiable at the point $t = 0$, whose derivatives satisfy the Hölder condition of order r at $t = 0$.

It can be directly verified that if all the functions c_j and a_j in (1) lie in the space $C_{(1+r)}(D)$, then the operator \mathcal{P} maps continuously the space $C_{(1+r)}(I, B)$ into $C_{(1+r)}(D, B)$.

To simplify notation we set

$$\mathcal{B}_1 = C_{(1+r)}(D, B), \quad \mathcal{B}_2 = C_{(1+r)}(I, B).$$

Definition 1 Given an operator \mathcal{P} , a term $c_k F \circ a_k$ is called *leading term* of \mathcal{P} if the function a_k maps D onto I .

Let Γ be a one-dimensional submanifold (curve) in D and $\zeta : I \rightarrow \Gamma$ a one-to-one $C_{(1+r)}$ -map. We denote by w_Γ and $\mathcal{P}_\Gamma F$ the restriction $w_\Gamma(s) := (w \circ \zeta)(s)$, $s \in I$, of an arbitrary function $w \in C(D)$ to Γ and the operator

$$\mathcal{P}_\Gamma : F \rightarrow \sum_{j=1}^N c_{j\Gamma}(s) F(a_{j\Gamma}(s)), \quad s \in I,$$

respectively.

Definition 2 A curve $\Gamma \subset D$ is called \mathcal{P} -*admissible* if for a leading term $c_k F \circ a_k$ the $c_{k\Gamma}$ does not vanish and the $a_{k\Gamma}$ maps Γ one-to-one onto I .

Definition 3 Given a curve $\Gamma \subset D$, we say that the operator \mathcal{P} is *strongly stable (along Γ)* if the à priori estimate

$$\inf_{\varphi \in \ker \mathcal{P}} \|F - \varphi\|_{\mathcal{B}_2} < c \|\mathcal{P}_\Gamma F\|_{\mathcal{B}_2} \quad (3)$$

holds with a constant $c > 0$ not depending on $F \notin \ker \mathcal{P}$.

If we change $\ker \mathcal{P}$ for $\ker \mathcal{P}_\Gamma$ in (3), then the operator \mathcal{P} is called *weakly stable (along Γ)*.

Since $\ker \mathcal{P} \subseteq \ker \mathcal{P}_\Gamma$ for any Γ , the strong stability implies the weak one.

The main object of the present work is the following class of the operators \mathcal{P} structurally associated with the operator \mathfrak{J} .

Definition 4 Let

$$\mathcal{P}F := F \circ a - \sum_{j=1}^N c_j F \circ a_j, \quad (4)$$

with positive c_j such that

$$\sum_{j=1}^N c_j = 1. \quad (5)$$

Here $F \in C(I, B)$, $a, a_j \in C(D, I)$ and $c_j \in C(D)$. We call \mathcal{P} *Jensen type* (or \mathfrak{J} - *type*) operator if

$$a = \sum_{j=1}^N c_j a_j, \quad (6)$$

and we call \mathcal{P} *weak \mathfrak{J} - type* operator if (5) and (6) hold only on a curve $\Gamma \subset D$, i.e.

$$a_\Gamma = \sum_{j=1}^N c_{j\Gamma} a_{j\Gamma} \quad \text{and} \quad \sum_{j=1}^N c_{j\Gamma} = 1. \quad (7)$$

This definition is analogous to Definition 4 of the *Cauchy type* operators in [4].

Theorem 1 1° Let \mathcal{P} be an operator (4) with a leading term, for the definiteness, $c_1 F \circ a_1$, and let $\Gamma \subset D$ be a \mathcal{P} - admissible $C_{(1+r)}$ - curve corresponding to this term (in the sense of Definition 2) and satisfying the conditions

$$a_{j\Gamma} = 0 \quad \text{for all } j = 2, \dots, n \quad (8)$$

and

$$a_1(\zeta(0)) = 0 \quad \frac{\partial}{\partial \Gamma} a_1(\zeta(0)) \neq 0. \quad (9)$$

If \mathcal{P} is a (weak) \mathfrak{J} - type operator, then \mathcal{P} is a (weakly) strongly stable (along) Γ .

2° The kernel $\ker \mathcal{P}_\Gamma$ in both cases consists of all linear functions

$$\varphi(s) = \alpha s + \beta, \quad s \in I, \quad \alpha, \beta \in B.$$

However, in the \mathfrak{J} - type case $\ker \mathcal{P} = \ker \mathcal{P}_\Gamma$, whereas in the weak \mathfrak{J} - type case $\ker \mathcal{P} = \{0\}$, if on the set $D \setminus \Gamma$ the function $\lambda := (a - \sum_{j=1}^N c_j a_j) / (1 - \sum_{j=1}^N c_j)$ is not constant, and $\ker \mathcal{P} = \{(z - \lambda)\xi, \quad \xi \in B\}$ otherwise.

Remark 2 The first relation in (9) is not a restriction. Given an admissible curve $\Gamma = \overbrace{x_0 x_1}$, with $a_1(x_0) = 0$, $a_1(x_1) = 1$, to obtain the relation in question it suffices to define the parametrization ζ of Γ such that $\zeta(0) = x_0$.

Proof of the theorem.

1° To begin with we remind Proposition 1 from [4]:

If $L : E_1 \rightarrow E_2$ is a linear bounded operator between Banach spaces and its range $\mathcal{R}(L)$ is closed, then there is a positive constant c such that the à priori estimate

$$\inf_{\varphi \in \ker L} |F - \varphi|_{E_1} < c |LF|_{E_2} \tag{10}$$

holds for all elements $F \notin \ker L$.

As was mentioned above, the operator \mathcal{P} is a linear continuous map:

$$C_{\langle 1+r \rangle}(I, B) \rightarrow C_{\langle 1+r \rangle}(D, B).$$

By this, to obtain the desired estimate (3) in the weakly case it suffices to prove the closedness of the set $\mathcal{R}(\mathcal{P}_\Gamma)$. Assume Γ to be given by $\delta : I \rightarrow D$ with $\delta = (\delta_1, \dots, \delta_n)$, and the functions a_j to satisfy the conditions

$$\begin{aligned} (a_1 \circ \delta)(0) &= 0, \quad \text{and} \quad \frac{d}{dt} (a_1 \circ \delta)(0) > 0, \\ (a_j \circ \delta)(t) &= 0, \quad 2 \leq j \leq n, \quad \text{for all } t \in I. \end{aligned} \tag{11}$$

The relation $\mathcal{P}_\Gamma F = H_\Gamma$ takes then the form

$$F\left((c_1 a_1)(\delta(t))\right) - c_1(\delta(t))F\left(a_1(\delta(t))\right) - \sum_{j=2}^N c_j(\delta(t))F(0) = (H \circ \delta)(t), \quad t \in I,$$

or

$$F\left(\lambda_1(t)b(t)\right) - \lambda_1(t)F\left(b(t)\right) - \sum_{j=2}^N \lambda_j(t)F(0) = h(t), \quad t \in I, \tag{12}$$

with $\lambda_j = c_j \circ \delta$, $b = a_1 \circ \delta$, $h = H \circ \delta$. By (11) and (5), it follows that, for an arbitrary function $F \in C_{\langle 1+r \rangle}(I, B)$, we have

$$h(0) = 0.$$

On the other hand, differentiating (12) and using (11) and (5) again, results in

$$\begin{aligned} h'(0) &= \\ &(\lambda_1 b)'(0)F'(0) - \lambda_1'(0)F(0) - (\lambda_1 b')(0)F'(0) - \sum_{j=2}^N \lambda_j'(0)F(0) = \\ &\hspace{20em} - \sum_{j=1}^N \lambda_j'(0)F(0) = 0. \end{aligned}$$

Thus, we have proved that the range $\mathcal{R}(\mathcal{P}_\Gamma)$ lies in the closed subspace $\overset{\circ}{C}_{\langle 1+r \rangle} \subset C_{\langle 1+r \rangle}(I, B)$ of functions vanishing at $t = 0$ together with their first derivatives. Let us show that actually

$$\mathcal{R}(\mathcal{P}_\Gamma) = \overset{\circ}{C}_{\langle 1+r \rangle}.$$

To this end, consider equation (12) with an arbitrary $h \in \overset{\circ}{C}_{\langle 1+r \rangle}$, and prove that there is one and only one solution $F \in C_{\langle 1+r \rangle}(I, B)$ of this equation. By definition of the space $C_{\langle 1+r \rangle}(I, B)$, any function $F(t)$ from this space can be represented in the form

$$F(t) = F(0) + F'(0)t + t^{1+r}\varphi(t) \quad (13)$$

with φ a continuous function from I to B . Substituting this F in (12) and using (5) results in

$$(\lambda_1 b)^{1+r}\varphi(\lambda_1 b) - \lambda_1 b^{1+r}\varphi(b) = h(t), \quad t \in I. \quad (14)$$

By the condition of Theorem 1, the function $b(t)$ is strictly increasing and $b'(0) > 0$. This makes it possible to introduce the new variable

$$z = b(t), \quad 0 \leq z \leq 1,$$

and to show that $t = z\rho(z)$, for some positive continuous function ρ . As $h \in \overset{\circ}{C}_{\langle 1+r \rangle}$, we find that

$$h(t) = t^{1+r}\mu(t)$$

with a continuous function μ and, therefore,

$$h(z\rho(z)) = z^{1+r}\rho^{1+r}(z)\mu(z\rho(z)).$$

Substituting $b(t)$ for z in (14) and using the latter relation we arrive at the equation

$$\varphi(z) - \lambda_1^r(z\rho)\varphi(z\lambda_1) = -\rho^{1+r}(z)(\mu/\lambda_1)(z\rho).$$

As $0 < \lambda_1 < 1$, the norm of the operator

$$A: \varphi(z) \rightarrow \lambda_1^r(z\rho)\varphi(z\lambda_1)$$

in the space $C(I, B)$ is less than 1. Applying the classical result in functional analysis (the invertibility of the operator $E - A$, E is the identical operator in $C(I, B)$) results in the unique solvability of the equation $\mathcal{P}_\Gamma F = h$ for an arbitrary function $h \in \overset{\circ}{C}_{\langle 1+r \rangle}$. By the above reference to Proposition 1 from [4], this completes the proof of assertion 1° of Theorem 1 in the part "weakly". By (13), we have simultaneously proved that the kernel of the operator \mathcal{P}_Γ consists of only linear functions $p(t) =$

$\alpha t + \beta$. As each of these functions solves evidently the homogeneous equation $\mathcal{P}F = 0$, this completes the proof of the assertion 1° in the part "strongly".

As to assertion 2° of Theorem 1, it was already proved above, except when \mathcal{P} is a weak \mathfrak{J} -type operator. In this case, as we know, the space $\ker \mathcal{P}_\Gamma$ consists of all linear functions $\xi z + \eta$ with $\xi, \eta \in B$. If some non-trivial linear function $\xi z + \eta$ solves the equation $\mathcal{P}F = 0$, then

$$\begin{aligned} \mathcal{P}(\xi z + \eta) &= (\xi a + \eta) - \sum_{j=1}^N c_j (\xi a_j + \eta) = \\ &= \xi \left(a(x) - \sum_{j=1}^N c_j(x) a_j(x) \right) + \eta \left(1 - \sum_{j=1}^N c_j(x) \right) = 0 \text{ for all } x. \end{aligned}$$

But it is impossible if $\lambda(x) \neq \text{const}$ in $D \setminus \Gamma$. In the case $\lambda(x) = \lambda$ the above equality to zero holds if and only if $\eta + \lambda \xi = 0$. Thus, the general solution of the equation $\mathcal{P}F = 0$ is $F = (z - \lambda)\xi$, $\xi \in B$.

This completes the proof of the theorem.

In conclusion we consider two examples of the linear functional operators \mathcal{P} whose stability (strong and weak) is an immediate consequence of Theorem 1.

Example 1 Let $I_1 = [0, 6]$, $D = \{(x, y) \mid 0 \leq x, y \leq 1\}$. Let \mathcal{P} be the operator of the form

$$\mathcal{P}F = 3F(x^2 + xy + 2y^2 + 2x) - 2F(3y^2 + \frac{3}{2}xy + \frac{3}{2}x) - F(3x^2 + 3x)$$

with $F \in C_{(1+r)}(I_1, B)$ and the functions

$$a = x^2 + xy + 2y^2 + 2x, \quad a_1 = 3y^2 + \frac{3}{2}xy + \frac{3}{2}x, \quad a_2 = 3x^2 + 3x$$

are given in D . These functions are chosen so that

$$a = \frac{2}{3}a_1 + \frac{1}{3}a_2,$$

and hence $\frac{1}{3}\mathcal{P}$ is the \mathfrak{J} -type operator in D with $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$. It is obvious that the curve $\Gamma = \{(x, y) \mid x = t, y = 0, 0 \leq t \leq 1\}$ is \mathcal{P} -admissible, because $c_{2\Gamma} \neq 0$ and the functions $a_{1\Gamma}(t) = 0$, $a_{2\Gamma}(t) = 3t + 3t^2$ satisfy conditions (8) and (9), respectively. By Theorem 1, the operator \mathcal{P} is strongly stable along Γ , and for some elements α and β from B the inequality

$$|F(z) - \alpha z - \beta| < c\varepsilon, \quad 0 \leq z \leq 9,$$

holds under condition

$$|(\mathcal{P}F)_\Gamma| < \varepsilon$$

with a constant c independent of F , ε and of z .

Example 2 Let $I_2 = [0, \mu]$, $D = \{(x, y) \mid 0 \leq x, y \leq \gamma\}$. Let \mathcal{P} be the operator

$$F \rightarrow F\left(x\sqrt{1+x^2}e^{x^2y} + \sqrt{x^2y^3+1}\sin y\right) - \\ - \frac{1}{3}e^{x^2y}F\left(3x\sqrt{1+x^2}\right) - \frac{2}{3}\sqrt{x^2y^3+1}F\left(\frac{3}{2}\sin y\right).$$

At first sight, this operator has no special structure and hence, it does not make sense to discuss its stability (of any type) and to try to describe its kernel. But note that with

$$a_1 = 3x\sqrt{1+x^2}, \quad a_2 = \frac{3}{2}\sin y, \quad c_1 = \frac{1}{3}e^{x^2y}, \quad c_2 = \frac{2}{3}\sqrt{x^2y^3+1}$$

the restriction a_Γ of the function

$$a = x\sqrt{1+x^2}e^{x^2y} + \sqrt{x^2y^3+1}\sin y$$

to the curve $\Gamma = \{(x, y) \mid x = \nu t, y = 0; 0 \leq t \leq 1\}$ turns out to have a form

$$a_\Gamma = c_{1\Gamma}a_{1\Gamma} + c_{2\Gamma}a_{2\Gamma},$$

and also

$$c_{1\Gamma} + c_{2\Gamma} = 1.$$

It is easy to show that if we take

$$\gamma = (\ln 2)^{1/3}, \quad \nu = \gamma \quad \text{and} \quad \mu = 3\gamma(1 + \gamma^2),$$

then all the functions a_1, a_2 and a map the domain D into I_2 , the curve Γ lies in D , and, therefore, the \mathcal{P} is a weak \mathfrak{J} -type operator along Γ . Furthermore, the Γ is \mathcal{P} -admissible, as $c_{1\Gamma} \neq 0$ and $\mathcal{R}(a_{1\Gamma}) = [0, \mu]$. It is clear that conditions (8) and (9) hold for the $a_{2\Gamma}$ and $a_{1\Gamma}$. By Theorem 1, the operator \mathcal{P} is weakly stable. This means, as in Example 1, that the inequality

$$|(\mathcal{P}F)_\Gamma| < \varepsilon$$

implies the inequality

$$|F(z) - \alpha z - \beta| < c\varepsilon, \quad 0 \leq z \leq \mu,$$

for some elements α, β from B and constant c . This time the function $\alpha z + \beta$ solves the equation $\mathcal{P}_\Gamma F = 0$, but does not solve the equation $\mathcal{P}F = 0$. In addition, $F = 0$ is the unique element of $\ker \mathcal{P}$. Is it possible to establish this fact directly?

Concluding remarks. To discuss the results obtained we remind the notion of the Ulam stability.

Definition 5 Given a pair of Banach spaces E_1 and E_2 , the operator $\mathcal{P} : E_1 \rightarrow E_2$ is called (E_1, E_2) - stable if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|\mathcal{P}F|_{E_2} < \delta$$

implies

$$|F - f|_{E_1} < \varepsilon$$

for some $f \in \ker \mathcal{P}$.

It is clear that the strong stability of \mathcal{P} (along a curve Γ) implies the Ulam stability of \mathcal{P} .

To the best of the author knowledge the Ulam stability of the \mathfrak{J} - operator has been established only for the model operator \mathfrak{J} with $\alpha = \beta = 1/2$ (where the Hyers machinery still works). Comparing this result with Theorem 1 demonstrates, first of all, the novelty of our approach to the stability of linear functional operators (making it possible to deal with operators of a very general form). On the other hand, at least in the framework of the above spaces $C_{\langle 1+r \rangle}$, there is the fundamental distinction in results. When dealing with the Ulam $(\mathcal{B}_1, \mathcal{B}_2)$ - stability, the closeness of the almost solution to the solution of the corresponding homogeneous equation $\mathfrak{J}F = 0$ depends on the uniform smallness of $\mathfrak{J}F(x)$ for *all* $x \in D$, whereas, as it follows from Theorem 1, such a smallness is required only at points $x \in \Gamma$.

The both above Examples by no means can be considered in the framework of the Ulam stability even in spaces of smooth functions, as the Hyers machinery is not compatible neither with nonlinear arguments $a_j(x)$, nor with nonconstant coefficients $c_j(x)$ characterizing the operator \mathcal{P} .

Thus, we arrive naturally to the following interesting problems, that may shed light on the compatibility of the two approaches to the stability of the general linear functional operators.

Problem 1. To prove or to disprove that if some operator \mathcal{P} of the form (1) is Ulam stable, then there is a curve Γ for which \mathcal{P} is strongly stable (along Γ).

Problem 2. To prove or to disprove the Ulam stability of one of the studied above operators \mathcal{P} in the spaces of continuous functions.

To complete the paper I would like to call the reader's attention to some problem intimately connected with the stability of the linear functional operators \mathcal{P} , which, probably, have never been mentioned earlier. The question is the identification of a function F as an (almost) element of the subspace $\ker \mathcal{P}$ by calculating values of

the function $\mathcal{P}F(x)$ on a set of points x . The necessity of such identification quite often arises when solving diverse problems of an applied nature.

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