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MATHEMATICS

Dynamical Systems and Functional Equations Related to Boundary Problems for Hyperbolic Differential Operators

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We consider two problems of mathematical analysis, which are not formally related to each other. One of them refers to the solvability theory of functional equations, which not only are of independent interest (and have not been investigated earlier) but also have found an unexpected application in several domains of analysis. The second problem is related to the formulation of general boundary problems for high-order strictly hyperbolic differential equations in bounded domains. Both problems were stated and studied in detail under certain conditions in [3–5]. It turned out that these problems are equivalent in some sense. It was established that not only the proofs of the related theorems but even their statements essentially use new notions and results of the theory of dynamical systems generated by the noncommutative semigroup of maps of interval into itself with two generators.

In this paper, we study the same problems under conditions which differ radically from those considered in the papers cited above. However, in the new situation, the decisive role in both setting and studying of the problems is played by dynamical systems analogous to those mentioned above. One of the purposes in this paper is to further develop the theory of these systems and deepen their relation to analysis. The most typical result in this direction, Theorem 1, asserts the possibility to expand any continuous function from the space $C_{\langle 1+r\rangle}([0, 1])$ in a series in orbits of the dynamical systems under consideration. This allows us to investigate the problems described above under assumptions fundamentally different from those of [3–5]. In particular, we obtain a solution to the Cauchy type functional equation in closed form and prove its regularity on the scale of the spaces C^k.

In the second part of the paper, we consider arbitrary strictly hyperbolic operators of the third order in the plane. With any such operator we canonically associate a system of bounded domains; in each of these domains, the corresponding boundary problem with only one additional condition on the unknown function is uniquely solvable in the scale of the spaces C^k. The kernel of the inverse operator of this problem is the second derivative of the solution to one of the functional equations studied in the first part of the paper.

1. CAUCHY TYPE FUNCTIONAL EQUATIONS

Let $I = \{t \mid 0 \le t \le 1\}$. Consider two real-valued continuous mappings δ_1 and δ_2 on the interval I such that

(I) $\delta_1(t)\delta_2(t)$ for $t > 0$,

(II) $\delta_1(0) = \delta_2(0) = 0$.

We say that δ_1 and δ_2 form a Z-configuration (see [6]).

The title of this section refers to equations of the form

$$F(\delta_1(t) + \delta_2(t)) - F(\delta_1(t)) - F(\delta_2(t)) = H(t), \quad t \in I, \tag{1}$$

in which H and F are, respectively, given and unknown functions defined on the intervals I and $I' = [0, \max_I(\delta_1 + \delta_2)]$. For the first time, Eq. (1) was considered in [3], where the functions δ_1 and δ_2 were supposed to satisfy conditions quite different from (i) and (ii). In particular, the graphs of these functions were disjoint. In the terminology of [6], these functions formed a $\mathcal{P}$-configuration. The interest in Eq. (1) was caused mainly by the fact that several new problems in such diverse fields of analysis as partial differential equations, integral geometry, measure theory, and functional and integral equations, reduce to this equation. Later, it turned out (see [7]) that the same relates completely to Eq. (1) under conditions (I) and (II). In this section, we prove the unique solvability of Eq. (1) in the appropriate pairs of spaces under minimal additional restrictions and, as mentioned, the solution is given in an explicit form. As an application, in the next section, we solve the new boundary problem for an arbitrary third-order strictly hyperbolic operator in a bounded domain in the plane in several steps.

The majority of the results presented in this paper are related to the cases in which the mapping $\delta = \delta_1 + \delta_2$: I

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$\to I'$ is invertible. This allows us to make the change $\delta(t) \to t$ and reduce Eq. (1) to the normal form

$$F(t) - F(\delta_1(t)) - F(\delta_2(t)) = H(t), \quad t \in I' \tag{2}$$

of a Cauchy type functional equation, in which

$$\delta_1(t) + \delta_2(t) = t, \quad t \in I'. \tag{3}$$

Let Φ_δ denote the semigroup of maps of the interval I' into itself generated by the maps δ_1 and δ_2. The elements of this semigroup are all maps of I' having the form $\delta_{J_{\langle 0\rangle}} = id$ or $\delta_{J_{\langle n\rangle}} = \delta_{j_n} \circ \ldots \circ \delta_{j_1}$, where $J_{\langle n\rangle} = (j_1, j_2, \ldots, j_n)$ is an arbitrary multi-index with $j_k = 1$ or 2, $n = 1, 2, \ldots$, and $\circ$ denotes the composition of maps. If t is a point of the interval I', then by its orbit be mean an arbitrary sequence $(t_1, t_2, \ldots)$ of points of I' satisfying the conditions

$$t_1 = t, \quad t_{k+1} = \delta_{j_k}(t_k), \quad k = 1, 2, \ldots,$$

$$j_k = 1 \quad \text{or} \quad 2.$$

Obviously, the set of such orbits is much larger than sets of orbits of dynamical systems with one generator, because the passage from a point t_k to a point t_{k+1} can be realized in two ways. Therefore, as mentioned in [4], it is unclear how this amorphous formation can be applied as a whole. In [4, 5], the case where the functions δ_1 and δ_2 form a $\mathcal{P}$-configuration was considered, and an important role was played by the subset of $\mathcal{T}$-proper orbits, where $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are arbitrary disjoint closed subsets of I'. An orbit $(t_1, t_2, \ldots)$ is said to be $\mathcal{T}$-proper if, in the definition of an orbit given above, $\delta_{j_k} = \delta_i$ whenever $t_k \in \mathcal{T}_i$ for $i = 1, 2$. In the language of these orbits, necessary and sufficient conditions for the solvability of very diverse problems of the above type were described. In this paper, we show that, for functions forming a Z-configuration and satisfying condition (3), all orbits of the semigroup Φ_δ are equally important when solving the above-mentioned problems of analysis, which lead to Eq. (2).

Consider the spaces

$$C_{\langle k+r\rangle}(I') = \{u(t) |\; u(t) = P_k(t) + t^{k+r}\varphi(t)\},$$

where $k \geq 1$ is an integer and $0 \leq r < 1$. In this definition, $P_k(t)$ is an arbitrary polynomial of degree at most k and $\varphi(t)$ is any continuous function if $r > 0$, and, in addition, $\varphi(0) = 0$ if $r = 0$. The space $C_{\langle k+r\rangle}(I')$ with the norm

$$\|u\|_{\langle k+r\rangle} = \sup_{I'}|P_k(t)| + \sup_{I'}|\varphi(t)|$$

is Banach. Roughly speaking, the elements of $C_{\langle k+r\rangle}(I')$ are continuous functions on I' having all derivatives up to order k at the point $t = 0$ and satisfying the Hölder condition with exponent r.

It is easy to verify that, if the components F, H, and δ_j of Eq. (2) belong to the space $C_{\langle 1\rangle}(I')$, then the conditions

$$F(0) = -H(0), \quad \text{and} \quad H'(0) = 0$$

are necessary for the solvability of this equation. Thus, in what follows, without loss of generality, we consider Eq. (2) under the additional conditions

$$H(0) = H'(0) = 0, \quad F(0) = 0. \tag{4}$$

We start the presentation of results with a theorem describing an unexpected relationship between the dynamical system generated by the semigroup Φ_δ with some classical structures of analysis, on the one hand, and, on the other hand, with one of the fundamental equations of analysis, what can be undoubtedly said about, the Cauchy equation $F(x + y) - F(x) - F(y) = 0$.

Theorem 1. (i) *Let functions $\delta_1(t)$ and $\delta_2(t)$ form a Z-configuration on the interval I and satisfy condition* (3). *If both of these functions belong to the space $C_{\langle 1+r\rangle}(I)$, where $r > 0$, and satisfy the condition*

$$\delta_1'(0)\delta_2'(0) \neq 0, \tag{5}$$

then, for any function $H \in C_{\langle 1+r\rangle}(I)$, the series

$$\sum_J H(\delta_J(t)), \quad t \in I, \tag{6}$$

in which the summation is over all multi-indices $J = J_{\langle n\rangle}$ with $n = 0, 1, \ldots$, converges absolutely and uniformly.

(ii) *The sum $F(t)$ of this series is a solution to Eq.* (2) *and belongs to the space $C_{\langle 1+r\rangle}(I)$.*

It may be appropriate to call series (6) orbital series related to the dynamical system Φ_δ.

The following two theorems give positive answers to the questions of the uniqueness of the solution $F(t)$ to Eq. (2) and of its regularity subject to regularity of H.

Theorem 2. *All solutions $F \in C_{\langle 1\rangle}(I')$ to homogeneous equation* (1) *are linear functions.*

Importantly, this theorem is valid for the general functional Cauchy type equation (with no assumption about the monotonicity of the function $\delta_1 + \delta_2$). Moreover, it is valid in the space $C_{\langle 1\rangle}$; this space is larger than any of the spaces $C_{\langle 1+r\rangle}$, for which the existence theorem for a solution to Eq. (2) is proved.

Theorem 3. *If $H \in C^k(I)$, where $k = 2, 3, \ldots$, then the solution F to Eq.* (2) *belongs to the space $C^k(I)$ and series* (6) *converges in $C^k(I)$.*

This theorem plays an important role in applications of results on functional equations to the theory of boundary problems for partial differential equations (see Theorem 5 below).

The following theorem asserts that Theorem 2 is sharp; namely, it says that there are homogeneous Cauchy type functional equations having nonlinear solution in spaces that are slightly larger than $C_{\langle 1\rangle}(I)$.

Theorem 4.[1] *If positive numbers a and b satisfy the condition $a + b = 1$, then the equation*

$$F(t) - F(at) - F(bt) = 0, \quad t \in I \tag{7}$$

has infinitely many nonlinear solutions provided that $\log_b a$ *is rational. All these solutions belong to the space* $C_{\langle 1, \ln t\rangle}(I) = \{u \mid u = a_0 + t\ln t\varphi(t)\}$, *where* a_0 *is a constant and* $\varphi(t) \in C(I)$.

For brevity, we prove only Theorems 1 and 2.

Proof of Theorem 1. (i) It follows from hypothesis (4) and the definition of the space $C_{\langle 1+r\rangle}(I)$ that the function $H(t)$ can be represented as $H(t) = t^{1+r}\varphi(t)$, where φ is a continuous function on I. Therefore,

$$F(t) = \sum_J \delta_J^{1+r}(t)\varphi(\delta_J(t))$$

$$= \sum_{n=0}^{\infty}\sum_{J_{\langle n\rangle}} \delta_{J_{\langle n\rangle}}^{1+r}(t)\varphi(\delta_{J_{\langle n\rangle}}(t)).$$

Let $A_n(t) = \sum_{J_{\langle n\rangle}} \delta_{J_{\langle n\rangle}}^{1+r}(t)$ for $n = 1, 2, \ldots$. Taking into account the relation $\delta_{J_{\langle 0\rangle}}(t) = t$, we can rewrite $A_n(t)$ in the form

$$A_n(t)$$

$$= t^{1+r}\sum_{j_1,\ldots,j_n=1}^{2}\left(\frac{\delta_{j_n}\circ\ldots\circ\delta_{j_1}(t)}{\delta_{j_{n-1}}\circ\ldots\circ\delta_{j_1}(t)}\right)^{1+r}\circ\ldots\circ\left(\frac{\delta_{j_1}(t)}{t}\right)^{1+r}.$$

Note that relations (I), (3), and (5) imply the existence of a constant $q < 1$ such that the inequality

$$\left(\frac{\delta_1(t)}{t}\right)^{1+r} + \left(\frac{\delta_2(t)}{t}\right)^{1+r} \le q$$

holds for all $t \in I$. Therefore, for any k such that $1 \le k \le n$, we have

$$\sum_{j_k=1}^{2}\frac{(\delta_{j_k}\circ\ldots\circ\delta_{j_1}(t))}{(\delta_{j_{k-1}}\circ\ldots\circ\delta_{j_1}(t))^{1+r}} \le q,$$

where $\delta_{j_0} = \mathrm{id}$. This yields $A_n(t) \le t^{1+r}q^n$, which implies the uniform convergence of the series $\sum_J\left(\frac{\delta_J(t)}{t}\right)^{1+r}$; hence, the function $F(t)$ can be represented as

$$F(t) = t^{1+r}f(t), \quad t \in I,$$

where $f(t)$ is a continuous function. This completes the proof of assertion (i) of the theorem.

(ii) The fact that the function $F(t)$ satisfies the Cauchy type functional equation is verified directly with the use of representation (6). This completes the proof of Theorem 1.

Proof of Theorem 2. Let us divide both parts of homogeneous equation (1) by $(\delta_1 + \delta_2)(t)$. Setting

$$\Phi(z) = \frac{F(z)}{z}, \quad \rho_j(z) = \frac{\delta_j(z)}{(\delta_1+\delta_2)(z)}$$
$$\text{for} \quad z \in I'\backslash\{0\},$$

$$\Phi(0) = 0, \quad \rho_j(0) = \frac{\delta_j'}{(\delta_1+\delta_2)'(0)}, \quad j = 1, 2,$$

we obtain the equivalent functional equation

$$\Phi(\delta_1+\delta_2) - \rho_1(t)\Phi(\delta_1(t)) - \rho_2(t)\Phi(\delta_2(t)) = 0, \quad t \in I', \tag{8}$$

in which all of the functions δ_j, ρ_j, and Φ are continuous on the interval I'. Let us show that any continuous solution Φ to Eq. (8) is constant. Let $\mathcal{M} = \max_{I'}\Phi$, and let $t_0 = \min\{t \mid \Phi(t) = \mathcal{M}\}$. Assume that $t_0 > 0$. Then, by continuity, $\Phi(t_0) = \mathcal{M}$; hence, for any point t_1 from the nonempty set $\{t \mid (\delta_1 + \delta_2)(t_1) = t_0\}$, we have

$$\Phi(\delta_1(t_1)) = \Phi(\delta_2(t_1)) = \mathcal{M},$$

since $(\rho_1 + \rho_2)(t) = 1$ for all t. However, this contradicts the definition of t_0, because $\delta_1(t_1) < t_0$ [by (I)]. Thus, we have proved that $t_0 = 0$, whence $\Phi(0) = \mathcal{M}$. A word-for-word repetition of the same argument for the point of minimum of Φ leads to the relation $\Phi(0) = \min_{I'}\Phi$, which proves the theorem.

Remarks. (i) The problem of the solvability of Eq. (2) for an arbitrary function $H \in C_{\langle 1\rangle}(I')$ remains open.

(ii) Of course, all of the results obtained above are transferred (with obvious modifications) to the nonhomogeneous multiplicative functional Cauchy type equations

$$F(\delta_1(t)\delta_2(t)) = F(\delta_1(t))F(\delta_2(t))H(t), \quad t \in I.$$

2. BOUNDARY PROBLEM FOR HYPERBOLIC DIFFERENTIAL OPERATORS

The results of the preceding section are new in the theory functional equations. Moreover, equations of the form (1) have never been considered in the framework of this theory, although it has a long history (see, e.g., [1]). In this section, we show that these equations, among other things, are a necessary technical tool for studying boundary problems on certain bounded domains for hyperbolic differential operators in the plane. Other applications of Eq. (1) in analysis are considered in, e.g., [5].

Consider a third-order strictly hyperbolic operator $P(\partial) = P(\partial_x, \partial_y)$ on the space $\mathbb{R}^2$ of the variables x and y

[1] For the first time, this result was obtained by a different method in the thesis of Orr Shalit (Technion, Israel).

(see [2]). According to [7], any such operator can be represented as

$$P(\partial) = \frac{\partial}{\partial \bar{l}_1}\frac{\partial}{\partial \bar{l}_2}\frac{\partial}{\partial \bar{l}_3} + Q(\partial),$$

where $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ are mutually transversal vector fields and Q is a second-order differential operator. Take any point O and consider the six semitrajectories of the fields l_1, l_2, and l_3 that start at O and are the characteristics of the operator $P(\partial)$. Take any triple of neighboring rays l_1, l_2, and l_3 lying on the trajectories of the fields $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$, respectively, and such that the ray l_2 is between l_1 and l_3; in other words, $\bar{l}_2 = \lambda_1\bar{l}_1 + \lambda_3\bar{l}_3$, where $\lambda_1 > 0$ and $\lambda_3 > 0$. Take arbitrary points A_1 and A_2 on the rays l_1 and l_2, respectively, and consider the domain D_1 that is the characteristic parallelogram $OA_1O'A_2$ in which the sides A_1O' and A_2O' are the trajectories of the vector fields $\bar{l}_2$ and $\bar{l}_1$, respectively.

Let $\Gamma = OO'$ be a quasidiagonal in D_1 being a nonsingular C^2-curve satisfying the following hypotheses:

(i) the curve Γ is transversal to the vector field $\bar{l}_3$ everywhere and to the vector fields $\bar{l}_1$ and $\bar{l}_2$ at the point O;

(ii) the curve Γ intersects the trajectories OA_1 and OA_2 only at the point O.

The first boundary problem for the operator $P(\partial)$ is stated as follows: For any functions f (on D_1) and g (on $\mathcal{M} = OA_1 \cup OA_2 \cup \Gamma$), define a function u on $\bar{D}_1$ for which

$$P(\partial)u = f \quad \text{on} \quad D_1, \quad u = g \quad \text{on} \quad \mathcal{M}. \tag{9}$$

Before stating results concerning this problem, we turn our attention to the compatibility conditions for the data. Let g_1, g_2, and g_3 denote the restrictions of the function g to the curves OA_1, OA_2, and Γ, respectively. If the relations

$$x = x_j(t), \quad y = y_j(t), \quad t \in I, \quad j = 1, 2, 3$$

determine parametric representations of these curves and $O = (x_j(0), y_j(0))$ for each j, then

$$u(x_j(t), y_j(t)) = g_j(t), \quad t \in I.$$

Let $\bar{\tau}$ denote a unit tangent vector to the curve Γ at O. The choice of the vectors $\bar{l}_1$ and $\bar{l}_2$ and their transversality imply the existence of positive constants λ_1 and λ_2 for which

$$\bar{\tau} = \lambda_1\bar{l}_1 + \lambda_2\bar{l}_2.$$

This leads to the first necessary compatibility condition

$$g_3'(0) = \lambda_1 g_1'(0) + \lambda_2 g_2'(0). \tag{10}$$

The second necessary condition is a reformulation of the fact that the point O is common for the three curves OA_1, OA_2, and Γ:

$$g_1(0) = g_2(0) = g_3(0, 0) = u(0, 0). \tag{11}$$

Theorem 5. *Let $P(\partial)$ be an arbitrary third-order strictly hyperbolic operator with constant coefficients in the domain D_1.*

Then, for any functions $f \in C(\bar{D}_1)$ and $g \in C^2(\mathcal{M})$ satisfying hypotheses (10) *and* (11), *problem* (9) *has a unique solution $u \in C^2(\bar{D}_1)$. The inverse operator $(f, g) \mapsto u$ is bounded, and if $(f, g) \in C^k(\bar{D}_1)C^{k+2}(\mathcal{M})$, then $u \in C^{k+2}(\bar{D}_1)$ for $k = 1, 2, \ldots$.*

The relation $g \in C^k(\mathcal{M})$ means that the functions g_1, g_2, and g_3 are k times continuously differentiable on the curves OA_1, OA_2, and OO', respectively.

Proof. For brevity, we consider only the case of the homogeneous operator $P = \frac{\partial}{\partial \bar{l}_1}\frac{\partial}{\partial \bar{l}_2}\frac{\partial}{\partial \bar{l}_3}$. In this situation, a suitable change of variables reduces problem (9) to the problem

$$(\partial_x - \partial_y)\partial_x\partial_y u = f \quad \text{on} \quad D_1, \quad u = g \text{ on } \mathcal{M}. \tag{12}$$

Here, $D_1 = \{(x, y)|\ 0 \le x \le a,\ 0 \le y \le b\}$ and the set $\mathcal{M}$ consists of the curves

$$\mathcal{M}_1 = \{(x, y)|\ 0 \le x \le a,\ y = 0\},$$

$$\mathcal{M}_2 = \{(x, y)|\ x = 0,\ 0 \le y \le b\},$$

$$\Gamma = \{(x, y)|\ x = \delta_1(t),\ y = \delta_2(t),\ t \in I\}.$$

The relations $x = \delta_1(t)$ and $y = \delta_2(t)$ determine a parametric representation of the curve Γ and satisfy the following conditions, which are dictated by properties (i) and (ii) of this curve:

(j) $\delta_1(0) = \delta_2(0) = 0,\ \delta_1(1) = a,\ \delta_2(1) = b;$

(jj) $\delta_1(t)\delta_2(t) > 0,\ 0 < t \le 1;$

(jjj) $\delta_1'(0)\delta_2'(0) > 0,\ \delta_1'(t) + \delta_2'(t) > 0,\ t \in I.$

Condition (jjj) makes it possible to choose a new parameterization on Γ, under which the (new) function $\delta_1(t)$ and $\delta_2(t)$ satisfy the condition

(jv) $\delta_1(t) + \delta_2(t) = t,\ t \in I' = [0, a + b]$

and, therefore, the condition $\delta_1'(t) + \delta_2'(t) = 1$. Let

$$g = G_1(x) \quad \text{on} \quad \mathcal{M}_1,$$

$$g = G_2(y) \quad \text{on} \quad \mathcal{M}_2, \quad g = G_3(x, y) \quad \text{on} \quad \Gamma.$$

Consider the function $\varphi(t) = G_3(\delta_1(t), \delta_2(t))$. Obviously, when passing to problem (12), conditions (10) and (11) transform into the conditions

$$G_1(0) = G_2(0) = \varphi(0) \quad \text{and}$$
$$\varphi'(0) = \delta_1'(0)G_1'(0) + \delta_2'(0)G_2'(0). \tag{13}$$

First, suppose that $f = 0$. Consider the function

$$u(x, y) = \int_0^x \int_0^y F(s+t)dtds + G_1(x) + G_2(y) - G_1(0)$$

on the domain D_1. Here, F is an arbitrary continuous function on the interval I'. According to (13), this function satisfies the boundary condition $u = g$ on the curves OA_1 and OA_2. On the other hand, $u(x, y)$ is a solution to the differential equation in (12) for $f = 0$ [the operator $\partial_x - \partial_y$ is interpreted as the vector field $\frac{\partial}{\partial \bar{l}}$ with $\bar{l} = (1, -1)$]. Thus, to solve problem (12) with $f = 0$, it remains to find a function F so that $u(x, y) = G(x, y)$ on Γ. Thus, we obtain the integral equation

$$\int_0^{\delta_1(t)} \left(\int_0^{\delta_2(t)} F(x+y)dy \right) dx = H(t), \quad t \in I' \tag{14}$$

for the unknown function $F \in C(I')$. As to the given function $H(t)$, it has the form

$$H(t) = -G_1(\delta_1(t)) - G_2(\delta_2(t)) + G_3(\delta_1(t), \delta_2(t)) + G_1(0)$$

and, according to (13), satisfies hypothesis (4). Let us show that Eq. (14) has a unique solution for any function $H(t) \in C^2(I')$ with property (4). For this purpose, we introduce a new unknown function Φ such that

$$\Phi''(z) = F(z), \quad \Phi(0) = 0, \quad \Phi'(0) = 0, \quad z \in I'$$

(it is uniquely determined by F). Substituting $\Phi''(z)$ for F in Eq. (14) and twice integrating by parts, we obtain the functional equation

$$\Phi(t) - \Phi(\delta_1(t)) - \Phi(\delta_2(t)) = H(t), \quad t \in I'$$

for Φ. By (jv), this is nothing but the Cauchy type functional equation studied in Section 1. The conditions (j)–(jv) make it possible to apply Theorems 1 and 3 and to conclude that the equation is uniquely solvable in the space $C^2(I')$ for any function $H \in C^2(I')$. This means that problem (12) with $f = 0$ has the same property. To complete the proof of Theorem 5, it remains to present some solution $u(x, y)$ to the differential equation in (12) and to show that this solution belongs to the space $C^k(\bar{D}_1)$ provided that $f \in C^k(\bar{D}_1)$. It can be verified directly that the function

$$u(x, y) = \frac{1}{2}\int_0^x \left(\int_0^y \left(\int_{(\delta_2 - \delta_2)(x+y)}^{x-y} f\left(\frac{s+t+z}{2}, \frac{s+t-z}{2} \right) dz \right) dt \right) ds$$

possesses all the required properties. This completes the proof of Theorem 5.

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REFERENCES

1. M. Kuczma, B. Choczewski, and R. Ger, in *Encyclopedia of Mathematics and Its Applications* (Cambridge Univ. Press, Cambridge, 1990), Vol. 32.
2. L. Hörmander, *The Analysis of Linear Partial Differential Operators* (Springer-Verlag, Berlin, 1983), Vol. 2.
3. B. Paneah, Funct. Anal. Its Appl. **37** (1), 46–60 (2003).
4. B. Paneah, Discrete Continuous Dyn. Syst. **9**, 1411–1420 (2003).
5. B. Paneah, Trans. Am. Math. Soc. **356**, 2757–2780 (2003).
6. B. Paneah, Contemp. Math. **364**, 205–223 (2004).
7. B. Paneah, Russ. J. Math. Phys. **11** (4), 311–330 (2004).