

On the general theory of the Cauchy type functional equations with applications in analysis.

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SUMMARY. The work deals with the questions of the solvability of the Cauchy type functional equations introduced recently and proved to be not only very much interesting objects of functional analysis but also important technical tool in solving quite different problems in such diverse fields as Integral geometry, Partial differential equations, the measure theory, ergodic theory, functional equations. The major part of the results is obtained on the base of the new dynamical method which has been work out especially for studying functional equations in question. Two of the possible applications in Analysis of the results obtained (in Integral geometry and in the theory of hyperbolic differential equations) are discussed in the final part of the paper.

Introduction.

In the last 30-40 years functional equations have grown to be a large, independent branch of mathematics with its own methods, circle of problems and, what is of great importance, abounding in applications. If on an early stage of development (XVIII - XIX centuries) functional equations played some auxiliary, may be even decorative role, describing in an abstract form various fundamental functions from Analysis, then nowadays functional equations turn out to be a powerful tool when solving *analytical problems* in quite different fields of mathematics. One of the first and most important functional equations was considered by Cauchy. He proved that *if a continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the relation*

$$F(x + y) - F(x) - F(y) = 0 \tag{0.1}$$

for all $(x, y) \in \mathbb{R}^2$, then $F(z) = F(1)z$, $z \in \mathbb{R}$. Repeating literally his proof leads to the local version of this result: any continuous function $F : [-1, 1] \rightarrow \mathbb{R}$ satisfying relation (0.1) for all points

$$(x, y) \in K := \{(x, y) \mid |x \pm y| \leq 1\}$$

is a linear function. It looks surprisingly, but as was clarified recently, a great deal of various problems in such diverse fields as integral and functional equations, measure theory, integral geometry, boundary problems for hyperbolic partial differential equations turn out to be reducible (sometimes in an equivalent manner) to the *Cauchy type functional equation*

$$F((\delta_1 + \delta_2)(t)) - F(\delta_1(t)) - F(\delta_2(t)) = h(t), \quad t \in I := (-1, 1), \quad (0.2)$$

with respect to an unknown continuous function $F : I \rightarrow \mathbb{R}$. Here $\delta_1, \delta_2 : I \rightarrow I$ and $h : I \rightarrow \mathbb{R}$ are given continuous functions, so that all the points $(x, y) = (\delta_1(t), \delta_2(t))$, $t \in I$, lie in K but do not exhaust all the K . Therefore, relation (0.2) contains considerably less information about F than relation (0.1) even when $h(t) = 0$. Precisely this equation is the main object of the present paper. It turns out that the solvability properties of equation (0.2) such as existence and uniqueness of a solution, fredholmness of the operator, defined by the left hand side of the equation, explicit solutions of the nonhomogeneous equation and full description of the kernel of the above operator and the methods of their deriving are determined by some mutual properties of the functions $\delta_j(t)$, $j = 1, 2$. These properties are formalized by means of the notion "configuration". Each pair (δ_1, δ_2) in question generates a semigroup Φ_δ of maps: $I \rightarrow I$ and the corresponding dynamical system (which has never been studied earlier). In the terms of orbits of this system we not only formulate all the main results related to equation (0.2), but also prove them using the existence of some specific attractors. What was described above is the subject of the first part of the paper, whereas the second part is mainly devoted to the applications of the results obtained in Integral geometry and in PDE. The existence of such deep and unexpected connection between these parts of analysis is the main novelty related to the general theory of functional equations.

1 The main notations and definitions

Throughout this paper all maps are supposed to be real and continuous (unless otherwise stipulated), and we denote by I and I' the intervals

$$I = \{t \mid -1 \leq t \leq 1\} \quad \text{and} \quad I' = \{t \mid 0 \leq t \leq 1\}.$$

We also use the notation \mathcal{I} if a corresponding statement is equally related to both intervals I and I' .

Definition 1. Given some functions H , and α_1, α_2 on the interval \mathcal{I} , the equation

$$F((\alpha_1 + \alpha_2)(t)) - F(\alpha_1(t)) - F(\alpha_2(t)) = H(t), \quad t \in \mathcal{I} \quad (1)$$

where F is an unknown function with the domain

$$\mathcal{D}_F = \mathcal{R}(\alpha_1 + \alpha_2) \cup \mathcal{R}(\alpha_1) \cup \mathcal{R}(\alpha_2)$$

is said to be *Cauchy type functional equation on \mathcal{I}* .

Introducing a linear operator $\mathcal{B}_\alpha: C(\mathcal{I}) \rightarrow C(\mathcal{D}_F)$ of the form

$$\mathcal{B}_\alpha(F)(t) := F(\alpha_1(t) + \alpha_2(t)) - F(\alpha_1(t)) - F(\alpha_2(t))$$

makes it possible to rewrite (1) in a short form as

$$\mathcal{B}_\alpha F = H.$$

As will be seen, both solvability properties of this equation and the methods of their receiving depend essentially on some mutual properties (configurations) of the functions α_1 and α_2 . In this paper we restrict ourselves to two different configurations formed by α_1 and α_2 , and we fix this difference in the following definitions.

Definition 2. We say that maps α_1 and α_2 from I into itself form a \mathcal{P} -configuration if

- 1° both functions do not decrease on I ;
- 2° the ranges of the maps α_1 and α_2 are the closed intervals $[0, 1]$ and $[-1, 0]$, respectively.

It follows that

$$\alpha_1(-1) = \alpha_2(1) = 0, \quad \alpha_1(1) = 1, \quad \alpha_2(-1) = -1. \quad (2)$$

Definition 3. We say that functions α_1 and α_2 on the interval I' form a Z -configuration if

- 3° $\alpha_k(0) = 0, \quad k = 1, 2$;
- 4° $\alpha_k(t) > 0, t > 0, \quad k = 1, 2$.

Figures 1 and 2 (see below) represent typical examples of Z - and \mathcal{P} -configurations, respectively. Dotted lines in both figures are the graphs of the functions $z = \alpha_1(t) + \alpha_2(t)$ in (t, z) -plane.

We note that the above conditions determining the two configurations are not artificial in the sense that all functional equations of the type (1) that have up to now arisen in various problems in applications satisfy these conditions (see [P1] - [P3]). Some confirmations of this fact the reader will find in Sec. 4.

It turns out that an essential information related to the solvability of equation (0.1), especially, in the case of a \mathcal{P} -configuration, may be derived

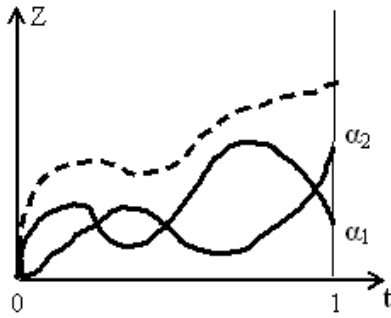


Fig. 1

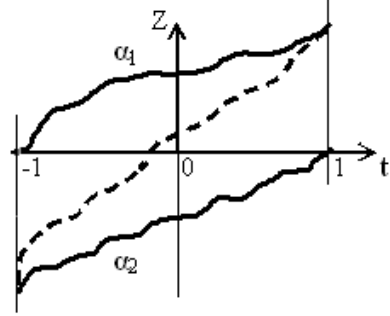


Fig. 2

by means of new dynamical methods, introduced in the author's papers [P5]. The application of these methods becomes possible, if we associate this equation with a noncommutative semigroup of maps from the interval \mathcal{I} into itself, generated by α_1 and α_2 . Pass to the exact formulations.

Definition 4. Given maps α_1, α_2 from the interval \mathcal{I} into itself, we denote by Φ_α the noncommutative semigroup of maps from \mathcal{I} into itself generated by these maps. The elements of this semigroup are all the maps of \mathcal{I} into itself of the form

$$\alpha_{J_{(0)}} = id, \quad \alpha_{J_{(n)}} = \alpha_{j_n} \circ \dots \circ \alpha_{j_1}, \quad n = 1, 2, \dots,$$

where $J_{(n)} = (j_1, \dots, j_n)$ is an arbitrary n -tuple with all j_k equal 1 or 2, and \circ denotes the composition of maps. This semigroup naturally determines a dynamical system.

When dealing with a Z -configuration and the corresponding functions α_j are defined on the interval I' , it is convenient to work with the spaces

$$C_{\langle k+r \rangle}(I') = \{u \mid u(t) = P_k(t) + t^{k+r}\phi(t)\},$$

where $k \geq 1$ is an integer, and $|r| < 1$. Here $P_k(t)$ is an arbitrary polynomial of degree $\leq k$, and ϕ is an arbitrary continuous function on I' if $r > 0$ and, in addition, $\phi(0) = 0$, if $r = 0$. With the norm

$$\|u\|_{\langle k+r \rangle} = \sup_{I'} |P_k(t)| + \sup_{I'} |\phi(t)|$$

the space $C_{\langle k+r \rangle}(I')$ becomes a Banach space. Roughly speaking, elements of this space are continuous functions on I' whose derivatives $u^{(k)}$ at the point $t = 0$ satisfy the Hölder condition of order r . It is obvious that $C_{\langle k-r \rangle}(I') \supset C_{\langle k \rangle}(I') \supset C_{\langle k+r \rangle}(I')$.

2 Statement of the problems and main results

2.1 The case of a Z -configuration

In this subsection we deal with the functional equation (1), where $t \in I'$, and the functions α_1, α_2 form a Z -configuration. It is assumed, unless otherwise stipulated, that $\alpha_j \in C_{(1)}(I')$, $j = 1, 2$. As to the unknown function F , it has to belong to the space $C_{(1)}(I')$ and to be defined on the interval

$$\widehat{I} = \{t \mid 0 \leq t \leq \tau\} \quad \text{with} \quad \tau = \max_{t \in I'} \sum_j \alpha_j(t).$$

It is obvious that the homogeneous equation $\mathcal{B}_\alpha F = 0$ has nontrivial solutions $F(t) = \lambda t$, $\lambda \in \mathbb{R}$. On the other hand, differentiating both part of equation (1) at the point $t = 0$ leads to the relation (independently of $F!$)

$$H'(0) = 0, \tag{3}$$

which is thus a *necessary condition* for solvability of this equation. These observations enable us to formulate the following problem for equation (1) which is (as will be seen) well posed (in the operator terminology).

Given some real number λ and an arbitrary function $H \in C_{(1)}(I')$ satisfying condition (3), to find a function $F \in C_{(1)}(\widehat{I})$ such that

$$\mathcal{B}_\alpha F = H \text{ on } I', \quad F'(0) = \lambda. \tag{4}$$

The exposition of the results related to this problem we begin with the theorem which describes an unexpected connection of the dynamical system generated by the semigroup Φ_α with some classical structures of analysis, on the one hand, and, on the other hand, with the Cauchy equation $F(x + y) - F(x) - F(y) = 0$, which is undoubtedly one of fundamental equations in analysis.

Theorem 1 *Let the functions α_1 and α_2 satisfy the hypotheses*

$$\alpha_1(t) + \alpha_2(t) = t, \quad t \in I', \tag{5}$$

and

$$\alpha_1'(0)\alpha_2'(0) \neq 0. \tag{6}$$

1° *If both functions belong to the space $C_{(1+r)}(I')$, $0 < r < 1$, then, for an arbitrary function $H \in C_{(1+r)}(I')$ satisfying the boundary conditions*

$$H(0) = H'(0) = 0, \tag{7}$$

the series

$$\sum_J H(\alpha_J(t)), \quad t \in I',$$

with the summation over all multi-indices $J = J_{\langle n \rangle}$, $n = 0, 1, \dots$, converges on I' absolutely and uniformly.

2° The sum $F = \sum_J H \circ \alpha_J$ of this series is a solution of equation $\mathcal{B}_\alpha F = H$ and belongs to the space $C_{\langle 1+r \rangle}(\widehat{I})$.

The complete solution of problem (4) is given in the following two theorems.

Theorem 2 *If the functions α_1 and α_2 satisfy condition (6) and $H = 0$ on I' , then, for a given real number λ , the linear function $F(t) = \lambda t$ is a unique solution of the problem (4) lying in the space $C_{\langle 1 \rangle}(\widehat{I})$.*

Remark In the case of increasing functions $\alpha_1(t)$ and $\alpha_2(t)$ this result is contained in [KCG, p.237].

It is worth noting that this uniqueness result is established, in contrast to the previous theorem, in the wider space $C_{\langle 1 \rangle}(I')$ and without restriction (5).

Theorem 3 *Assume that, in addition to condition (6) the function $\alpha = \alpha_1 + \alpha_2$ is invertible. Then, for an arbitrary function $H \in C_{\langle 1+r \rangle}(I')$, satisfying condition (3), and for any real number λ , there is a unique solution $F \in C_{\langle 1+r \rangle}(\widehat{I})$ of problem (4).*

Thus, by using the operator terminology, the linear operator \mathcal{B}_α is a Fredholm operator: $C_{\langle 1+r \rangle}(I') \rightarrow C_{\langle 1+r \rangle}(\widehat{I})$ of index zero.

The following theorem relates to the regularity of the solution F to problem (4).

Assume that the functions $\alpha_j(t)$, $j = 1, 2$, belong to the space $C^k(I')$ of k times continuously differentiable functions, for some natural k .

Theorem 4 *If the functions α_1 and α_2 satisfy hypotheses (5) and (6), and $H \in C^k(I')$, $k = 1, 2, \dots$, then all solutions of the equation $\mathcal{B}_\alpha F = H$ belong to the space $C^k(\widehat{I})$. If the function $H(t)$ satisfies hypothesis (7), then the series $\sum_J H \circ \alpha_J$ converges in the topology of $C^k(I')$.*

This result plays a crucial role when applying the technique of functional equations in the theory of boundary problems for partial differential equations (where such a result is usually said to be *increasing in smoothness*) (see Sec. 4).

2.2 The case of a \mathcal{P} -configuration

In this subsection we deal with the Cauchy type functional equation $\mathcal{B}_\alpha F = H$, but now the functions α_1 and α_2 form a \mathcal{P} -configuration.

As it follows from Definition 2, for an arbitrary function $F \in C(I)$, the relation

$$(\mathcal{B}_\alpha F)(-1) = \mathcal{B}_\alpha F(1) = -F(0) \quad (8)$$

holds. It follows *that the periodicity condition*

$$H(-1) = H(1) \quad (9)$$

is necessary for solvability of the equation in question and, hence, the cokernel $\text{coker}\mathcal{B}_\alpha$ of the operator \mathcal{B}_α is not empty. As the kernel $\ker\mathcal{B}_\alpha$ is also nonempty (as we already know), the best possible result related to the solvability of equation (1) is the fredholmness of \mathcal{B}_α and, in addition, the relations $\dim\ker\mathcal{B}_\alpha = \dim\text{coker}\mathcal{B}_\alpha = 1$. This result will be achieved, if we prove the unique solvability of the following boundary problem:

given a number $\lambda \in \mathbb{R}$ and a function H satisfying hypothesis (9), to find a function F on I such that

$$\mathcal{B}_\alpha F = H \quad \text{on } I \quad \text{and} \quad F(1) = \lambda. \quad (10)$$

The conditions for solvability of this problem are related to the smoothness of the functions F and H , and to some nontrivial features of the dynamical system generated by the semigroup Φ_α . The following notions were introduced in [P1].

- 1) An ordered set $\mathcal{O} = (t_1, t_2 \dots)$ of points in I is said to be *orbit* (or *orbit of the point t_1*) generated by Φ_α if

$$t_{k+1} = \alpha_{j_k}(t_k), \quad (11)$$

for an arbitrary natural $k \geq 1$, $j_k = 1$ or 2 .

- 2) The subsets in I

$$\mathcal{T}_1 = \{t \mid \alpha'_2(t) = 0\}, \quad \mathcal{T}_2 = \{t \mid \alpha'_1(t) = 0\}, \quad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$$

are said to be *guiding sets*.

- 3) An orbit $\mathcal{O} = (t_1, t_2, \dots)$, is said to be \mathcal{T} -*proper* if in (11)

$$\alpha_{j_k} = \alpha_1, \quad \text{when } t_k \in \mathcal{T}_1, \quad \text{and} \quad \alpha_{j_k} = \alpha_2, \quad \text{when } t_k \in \mathcal{T}_2.$$

- 4) If all points of an orbit \mathcal{O} belong to the guiding set \mathcal{T} , then \mathcal{O} is said to be \mathcal{T} -*guided orbit*.

- 5) If the end points of a finite orbit $\mathcal{O} = (t_1, \dots, t_{n+1})$ coincide, i.e. $t_1 = t_{n+1}$, then this orbit is said to be *periodic orbit* or *cycle*.

Definition. We denote by $\mathfrak{N}_\alpha^\mathcal{T}$ the set of all \mathcal{T} -proper \mathcal{T} -guided periodic orbits in I .

Now everything is ready to formulate the main results treated the solvability properties of equation (1) in the case of a \mathcal{P} -configuration.

Denote by \mathcal{T}'_j the set of all limit points of the set \mathcal{T}_j , $j = 1, 2$, and let

$$\tau_1 = \min\{t \mid t \in \mathcal{T}'_1\}, \quad \tau_2 = \max\{t \mid t \in \mathcal{T}'_2\}.$$

Theorem 5 *Assume that the functions α_1 and α_2 belong to $C^1(I)$ and satisfy inequalities*

$$\alpha_1'(t) + \alpha_2'(t) > 0 \quad \text{on } I \quad \text{and} \quad \alpha_1'(-1)\alpha_2'(1) > 0 \quad (12)$$

Let also one of the sets $\mathcal{T}_1, \mathcal{T}_2$ be finite or both \mathcal{T}_j be infinite and in the latter case

$$\tau_2 < \tau_1, \quad (13)$$

Then, all solutions $F \in C^1(I)$ of the homogeneous equation

$$F \circ (\alpha_1 + \alpha_2) - F \circ \alpha_1 - F \circ \alpha_2 = 0 \quad (14)$$

are linear functions $F(z) = \lambda z$, if

$$\mathfrak{N}_\alpha^{\mathcal{T}} = \emptyset.$$

Theorem 6 *Assume that in addition to hypotheses (12) and (13) both functions α_1 and α_2 belong to $C^2(I)$. Then, for all real λ and for an arbitrary function $H \in C^2(I)$, satisfying condition (9), there is a unique solution $F \in C^2(I)$ of problem (10) if and only if*

$$\mathfrak{N}_\alpha^{\mathcal{T}} = \emptyset. \quad (15)$$

The inverse operator $H \mapsto F$ from the space $C^2(I)$ into itself is continuous.

3 Proofs of Theorems 1-7

3.1 Proof of Theorem 1

1° Set $F(t) = \sum_J H(\alpha_J(t))$. The function H , as it follows from condition (7) and from the definition of the space $C_{(1+r)}(I')$, can be represented as $H(t) = t^{1+r}\varphi(t)$ with φ being a continuous function on I' . For this reason,

$$F(t) = \sum_J \alpha_J^{1+r}(t)\varphi(\alpha_J(t)) = \sum_{n=0}^{\infty} \sum_{J_{\langle n \rangle}} \alpha_{J_{\langle n \rangle}}^{1+r}(t)\varphi(\alpha_{J_{\langle n \rangle}}(t)).$$

Introduce the notation

$$A_n(t) = \sum_{J_{\langle n \rangle}} \alpha_{J_{\langle n \rangle}}^{1+r}(t), \quad n = 1, 2, \dots$$

As $\alpha_{J_{\langle 0 \rangle}}(t) = t$, one has

$$A_n(t) = t^{1+r} \sum_{j_1, \dots, j_n=1}^2 \left(\frac{\alpha_{j_n} \circ \dots \circ \alpha_{j_1}(t)}{\alpha_{j_{n-1}} \circ \dots \circ \alpha_{j_1}(t)} \right)^{1+r} \dots \left(\frac{\alpha_{j_1}(t)}{t} \right)^{1+r}, \quad t \in I.$$

By 3°, (5) and (6), there is a constant $q < 1$ such that

$$(\alpha_1(t)/t)^{1+r} + (\alpha_2(t)/t)^{1+r} \leq q, \quad t \in I.$$

Therefore, for all k , $1 \leq k \leq n$, we have

$$\sum_{j_k=1}^2 \left(\frac{\alpha_{j_k} \circ \dots \circ \alpha_{j_1}(t)}{\alpha_{j_{k-1}} \circ \dots \circ \alpha_{j_1}(t)} \right)^{1+r} \leq q,$$

where $\alpha_{j_0} = id$. This leads immediately to the relation $A_n(t) \leq t^{1+r} q^n$ which implies the uniform convergence of the series $\sum_J (\alpha_J(t)/t)^{1+r}$. This results in the representation of the function $F(t)$ in the form

$$F(t) = t^{1+r} f(t), \quad t \in I',$$

with f a continuous function, and completes the proof of assertion 1°.

2° By (5), it can be directly shown that the function F satisfies the Cauchy type functional equation (1). This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

Dividing both parts of the homogeneous equation (1) by $(\alpha_1 + \alpha_2)(t)$, we arrive at the *equivalent* functional equation

$$\Phi(\alpha_1(t) + \alpha_2(t)) - \rho_1(t)\Phi(\alpha_1(t)) - \rho_2(t)\Phi(\alpha_2(t)) = 0, \quad t \in I', \quad (16)$$

where

$$\begin{aligned} \Phi(z) &= F(z)/z, & \rho_j(z) &= \alpha_j(z)/(\alpha_1 + \alpha_2)(z), \quad z \in I' \setminus \{0\}, \\ \Phi(0) &= 0, & \rho_j(0) &= \alpha'_j/(\alpha_1 + \alpha_2)'(0), \quad j = 1, 2, \end{aligned}$$

and all the functions α_j , ρ_j , Φ are continuous on the interval I' . Show that any continuous solution Φ of equation (16) is a constant. Indeed, let $\mathcal{M} = \max_{I'} \Phi$ and let $t_0 = \inf\{t \mid \Phi(t) = \mathcal{M}\}$. Assume that $t_0 > 0$. Then, by continuity, $\Phi(t_0) = \mathcal{M}$, and, therefore, the relation

$$\Phi(\alpha_1(t_1)) = \Phi(\alpha_2(t_1)) = \mathcal{M},$$

is valid for an arbitrary point t_1 belonging to the nonempty subset $\{t \mid \alpha_1(t) + \alpha_2(t) = t_0\}$, as $\rho_1(t) + \rho_2(t) = 1$ for all values $t \in I'$. However, this contradicts to the definition of the point t_0 , as $\alpha_1(t_1) < t_0$ (by Definition 3). Thus, we have proved that $t_0 = 0$, and hence, $\Phi(0) = \mathcal{M}$. Repeating literally these arguments in connection with the point of minimum of the function Φ , we arrive at the relation $\Phi(0) = \min_{I'} \Phi$, and this proves the theorem.

3.3 Proof of Theorem 3

The result follows immediately from the previous two theorems. To see this, it suffices to change variable $\alpha(t) \rightarrow t$. Then we arrive at the functional equation

$$G(t) - G(\rho_1(t)) - G(\rho_2(t)) = \widehat{H}(t), \quad t \in \widehat{I},$$

where

$$\rho_1 = \alpha_1 \circ \alpha^{-1}, \quad \rho_2 = \alpha_2 \circ \alpha^{-1}, \quad \widehat{H} = H \circ \alpha^{-1}$$

and $\widehat{I} = (0, r)$, $r = \max_{I'}(\alpha_1(t) + \alpha_2(t))$. It is clear that the above change of variable preserves the $C_{(1+r)}$ -smoothness of all involving functions, and, in addition, the functions ρ_j , $j = 1, 2$ and H satisfy conditions (5) and (3), respectively, on \widehat{I} . Obviously, the function $\mathcal{G}(t) = G(t) + \widehat{H}(0)$ is a solution of the equation

$$\mathcal{B}_\rho \mathcal{G} := \mathcal{G}(t) - \mathcal{G}(\rho_1(t)) - \mathcal{G}(\rho_2(t)) = \mathcal{H}(t),$$

where $\mathcal{H}(t) = \widehat{H}(t) - \widehat{H}(0)$. By Theorem 1, the function

$$\mathcal{G}(t) = \sum_J \mathcal{H}(\rho_J(t))$$

solves this equation, and using Theorem 2 we find that the function

$$G(t) = -\widehat{H}(0) + \sum_J \mathcal{H}(\rho_J(t)) + \lambda t \quad (17)$$

is a unique (in the space $C_{(1+r)}(\widehat{I})$) solution of the problem

$$\mathcal{B}_\rho G = \widehat{H} \quad \text{on } \widehat{I}, \quad G'(0) = \lambda \alpha'(0).$$

It remains to substitute $\alpha(t)$ for t in (17), and to obtain the solution to problem (4) in an explicit form.

3.4 Proof of Theorem 4

For the sake of brevity we restrict ourselves to the case $k = 2$. The general case can be considered analogously. Note, first of all, that, as it follows from Definition 3 and relation (6),

for an arbitrary value T_1 , $0 < T_1 < 1$, there is a number $N = N(T_1)$ such that the relation

$$\alpha_{J(n)}(t) < T_1 \quad \text{for all } n \geq N$$

is valid at any point $t \in I'$.

Further, as it follows from (5) and (6), there is a number T_2 such that the relation

$$0 < \alpha'_j(t) < \gamma < 1, \quad j = 1, 2,$$

holds at all points t , $0 \leq t \leq T_2$, for some constant γ . Take an arbitrary number $T_1 < 1$ and let $T = \min(T_1, T_2)$. Now introduce a number $N > 1$ such that at all points $t > 0$ and for $n \geq N$ the relations

$$\alpha_{J_{\langle n \rangle}}(t) < T \quad \text{and, consequently,} \quad \alpha'_j(\alpha_{J_{\langle n \rangle}}(t)) < \gamma \quad (18)$$

are valid. Let us prove the differentiability of the function F . To this end consider the series $\sum_J |H'(\alpha_J(t))\alpha'_J(t)|$ and write down it in the form

$$\sum_{n=0}^{N-1} \sum_{J_{\langle n \rangle}} |H'(\alpha_{J_{\langle n \rangle}}(t))\alpha'_{J_{\langle n \rangle}}(t)| + \sum_{n=N}^{\infty} \sum_{J_{\langle n \rangle}} |H'(\alpha_{J_{\langle n \rangle}}(t))\alpha'_{J_{\langle n \rangle}}(t)|, \quad (19)$$

with $\sum_{J_{\langle n \rangle}}$ being the summation over all multiindices $J_{\langle n \rangle} = (j_1, \dots, j_n)$, $n = 1, 2, \dots$. If we use condition (3) and the inequality $|H'(t)| < ct$ with c a positive constant, which follows (3), then we conclude that the second term in (19) is majorized by the function

$$\sum_{m=1}^{\infty} A_m(t) = \sum_{m=1}^{\infty} \sum_{J_{\langle m \rangle}} \sum_{J_{\langle N \rangle}} \alpha_{J_{\langle m \rangle}}(\alpha_{J_{\langle N \rangle}}(t)) \left| \left(\alpha_{J_{\langle m \rangle}}(\alpha_{J_{\langle N \rangle}}(t)) \right)' \right|.$$

We are now going to estimate from above the function $A_m(t)$, $m = 1, 2, \dots$. By virtue of the positiveness of the functions $\alpha_1(t)$ and $\alpha_2(t)$ for $t > 0$, we find that

$$\begin{aligned} A_m &= \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \alpha_1 \circ \alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \left| \left(\alpha_1 \circ \alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)' \right| \\ &\quad + \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \alpha_2 \circ \alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \left| \left(\alpha_2 \circ \alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)' \right| \\ &= \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \left(\alpha_1(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \alpha'_1(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right) \left| \left(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)' \right| \\ &\quad + \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \left(\alpha_2(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \alpha'_2(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right) \left| \left(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)' \right| \\ &\leq \gamma \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \left| \left(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)' \right| = \gamma A_{m-1}. \end{aligned}$$

On the last step the relations (18) and (5) were used. This proves the uniform convergence of the series (19) and differentiability of the function F .

It remains to prove that $F \in C^2(I')$, if $H \in C^2(I')$. To do this, it suffices, as above, to establish the uniform convergence of the series

$$\begin{aligned} \sum_{n=N}^{\infty} \sum_{J_{\langle n \rangle}} \left(H(\alpha_{J_{\langle n \rangle}}(t)) \right)'' &= \sum_{n=N}^{\infty} \sum_{J_{\langle n \rangle}} H''(\alpha_{J_{\langle n \rangle}}(t)) \left(\alpha'_{J_{\langle n \rangle}}(t) \right)^2 \\ &+ \sum_{n=N}^{\infty} \sum_{J_{\langle n \rangle}} H'(\alpha_{J_{\langle n \rangle}}(t)) \alpha''_{J_{\langle n \rangle}}(t) := \sum_1 + \sum_2, \end{aligned}$$

where the relation

$$(f(g))'' = f''(g)g'^2 + f'(g)g'' \quad (20)$$

has been used. To prove the convergence of the series \sum_1 consider the series $\sum_{m=1}^{\infty} \mathcal{B}_m(t)$ majorizing \sum_1 , where

$$\mathcal{B}_m = \sum_{J_{\langle m \rangle}} \sum_{J_{\langle N \rangle}} \left(\alpha_{J_{\langle m \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)^2.$$

As $\alpha_{J_{\langle N \rangle}}(t) \in (0, T)$ for all $t > 0$, we find that $\alpha'_j(\alpha_{J_{\langle N \rangle}}(t)) \leq \gamma$, and hence

$$\begin{aligned} \mathcal{B}_m &= \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \left(\alpha_1(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right)^2 + \left(\alpha_2(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right)^2 \\ &= \sum_{J_{\langle m-1 \rangle}} \sum_{J_{\langle N \rangle}} \left(\alpha'_1(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right)^2 \left(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)^2 \\ &+ \left(\alpha'_2(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}})) \right)^2 \left(\alpha_{J_{\langle m-1 \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)^2 \leq \gamma^2 \mathcal{B}_{m-1} \end{aligned}$$

This proves the convergence of the series $\sum_{m=1}^{\infty} \mathcal{B}_m$ and, therefore, the convergence of the series \sum_1 . It remains to establish the uniform convergence of the series \sum_2 , or, remembering that $|H'(t)| \leq ct$, the analogous convergence of the series $\sum_{m=1}^{\infty} \mathcal{C}_m(t)$, where

$$\mathcal{C}_m = \sum_{J_{\langle m \rangle}} \sum_{J_{\langle N \rangle}} \alpha_{J_{\langle m \rangle}}(\alpha_{J_{\langle N \rangle}}) \left| \left(\alpha_{J_{\langle m \rangle}}(\alpha_{J_{\langle N \rangle}}) \right)'' \right|.$$

Using, as above, relation (20) and the identity $\alpha_{J_{\langle m \rangle}} = \alpha_1 \circ \alpha_{J_{\langle m-1 \rangle}} + \alpha_2 \circ \alpha_{J_{\langle m-1 \rangle}}$, we arrive at the relation

$$\begin{aligned}
\mathcal{C}_m(t) &= \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \alpha_1 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \left| \left(\alpha_1 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right)'' \right| \\
&+ \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \alpha_2 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \left| \left(\alpha_2 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right)'' \right| \\
&\leq \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \left(\alpha_1 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right) \left| \alpha_1'' \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right| \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)^2 \\
&+ \alpha_2 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \left| \alpha_2'' \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right| \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)^2 \\
&+ \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \left(\alpha_1 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \right) \alpha_1' \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \left| \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)'' \right| \\
&+ \alpha_2 \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \alpha_2' \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right) \left| \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)'' \right| \\
&\leq c \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)^2 \\
&+ \gamma \sum_{J_{\langle m-1 \rangle}, J_{\langle N \rangle}} \alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \left| \left(\alpha_{J_{\langle m-1 \rangle}} \left(\alpha_{J_{\langle N \rangle}} \right) \right)'' \right| = c\mathcal{B}_{m-1} + \gamma\mathcal{C}_{m-1},
\end{aligned}$$

where $c = \max\{\sup |\alpha_1''(t)|, \sup |\alpha_2''(t)|\}$. Combining with the uniform convergence of the series $\sum_{m=1}^{\infty} \mathcal{B}_m(t)$ already proved, this proves the uniform convergence of the series $\sum_{m=1}^{\infty} \mathcal{C}_m(t)$ and, thus, completes the proof of Theorem 4 for $k = 2$.

3.5 Proof of Theorem 5

Without loss of generality we can restrict ourselves to the case of the equation

$$F(t) - F(\delta_1(t)) - F(\delta_2(t)) = 0, \quad t \in I, \quad (21)$$

where

$$\delta_1'(t) + \delta_2'(t) = 1, \quad t \in I, \quad \text{and} \quad \delta_1'(-1)\delta_2'(1) > 0. \quad (22)$$

Indeed, by the first relation in (12) the map $\alpha = \alpha_1 + \alpha_2$ from I into itself is invertible. In addition, the boundary ∂I is invariant with respect to this map. The change of variable $\alpha(t) \rightarrow t$ reduces equation (14) to equation (21) with

$$\delta_1 = \alpha_1 \circ \alpha^{-1}, \quad \delta_2 = \alpha_2 \circ \alpha^{-1}.$$

The new guiding sets $\mathcal{T}_j = \{t \mid \delta_j'(t) = 1\}$, $j = 1, 2$, are nothing but the sets $\alpha^{-1}\mathcal{T}_j$ with the "old" \mathcal{T}_j . The second relation in (12) and relation (13)

remain true for the new variable, due to relation $\alpha'(t) > 0$. To avoid some technical details we consider the only case, where *the set \mathcal{T}_2 is finite*. The general case can be studied on the basis of some specific results related to the attractors of dynamical systems in question (see [P5]).

Introduce the notation

$$(\mathcal{B}F)(t) = F(t) - F(\delta_1(t)) - F(\delta_2(t)).$$

Substituting successively the values $t = -1$ and $t = 1$ in the relation $(\mathcal{B}F)(t) = 0$ we find, remembering the properties of the functions $\delta_j(t)$, that

$$(\mathcal{B}F)(0) = 0.$$

This makes it possible to conclude that, for an arbitrary function $F(t) \in C^1(I)$, relation (21) is *equivalent* to the relation

$$F'(t) - \delta_1'(t)F'(\delta_1(t)) - \delta_2'(t)F'(\delta_2(t)) = 0, \quad t \in I. \quad (23)$$

To simplify notation, let

$$a_j(t) = \delta_j'(t), \quad j = 1, 2, \quad \text{and} \quad F' = G.$$

Then equation (23) becomes

$$G(t) - a_1(t)G(\delta_1(t)) - a_2(t)G(\delta_2(t)) = 0, \quad t \in I, \quad (24)$$

where, by (22),

$$a_1(t) + a_2(t) = 1 \quad a_j(t) \geq 0, \quad j = 1, 2; \quad t \in I, \quad (25)$$

$$a_1(-1)a_2(1) \neq 0. \quad (26)$$

Note that now

$$\mathcal{T}_j = \{t \mid a_j(t) = 1\}, \quad j = 1, 2.$$

To prove Theorem 5 it remains to show that *the only solution of equation (24) is an arbitrary constant $F(t) = c$* . We will do this in several steps.

I. Let us show that *the extremal values of any solution $G \in C(I)$ of equation (24) spread only along \mathcal{T} -proper orbits*.

To this end, let $M = \max_I G$ and let $\mathfrak{M} = \{t \in I \mid F(t) = M\}$. If $G(\hat{t}) = M$, then $G(\delta_1(\hat{t})) = M - \varepsilon_1$, $G(\delta_2(\hat{t})) = M - \varepsilon_2$ for some nonnegative numbers ε_1 and ε_2 . Substituting \hat{t} for t in equation (24) we find that $a_1(\hat{t})\varepsilon_1 + a_2(\hat{t})\varepsilon_2 = 0$. It follows that if $\hat{t} \in \mathfrak{M} \setminus \mathcal{T}$, then $\varepsilon_1 = \varepsilon_2 = 0$, and hence $\delta_1(\hat{t}) \in \mathfrak{M}$, $\delta_2(\hat{t}) \in \mathfrak{M}$. But if $\hat{t} \in \mathfrak{M} \cap \mathcal{T}_1$, then $\varepsilon_1 = 0$, and consequently $\delta_1(\hat{t}) \in \mathfrak{M}$. In just the same way, if $\hat{t} \in \mathfrak{M} \cap \mathcal{T}_2$, then $\delta_2(\hat{t}) \in \mathfrak{M}$. Combining these observations with the definition of \mathcal{T} -proper orbit we conclude that together with every point $t_1 \in \mathfrak{M}$ the next point of *any* \mathcal{T} -proper orbit

(t_1, t_2) also belongs to \mathfrak{M} . As δ_1 and δ_2 are maps in I , this argument can be applied to the point t_2 . As a result we obtain one or two points t_3 of the form $t_3 = \delta_{j_2}(t_2) = \delta_{j_2} \circ \delta_{j_1}(t_1)$ lying in the set \mathfrak{M} , so that the orbit $\mathcal{O} = (t_1, t_2, t_3)$ turns out to be \mathcal{T} -proper. Arguing in the same way, we conclude that *together with every point $t_1 \in \mathfrak{M}$ any its \mathcal{T} -proper orbit $\mathcal{O} = (t_1, t_2, \dots)$ is situated in the set \mathfrak{M}* . This proves the above assertion with respect to the maximal value of the function G . But what has been said above with respect to spreading maximal value M of a solution F along \mathcal{T} -proper orbits remains valid for a minimal value m of the same solution. This completes the proof of assertion.

II. Assume that *for an arbitrary point $t \in I$ there is a \mathcal{T} -proper orbit (t, t_1, t_2, \dots) converging to one of the boundary points of I* . Then, by continuity of the function G , it attains its extremal values at the points $t = -1$ and $t = 1$. Substituting consecutively these points for t in equation (24) and using relations (2), (25) and (26) we arrive at the relation

$$G(-1) = G(1) = G(0).$$

This means that the maximal and minimal values of G coincide, and hence, G is a constant. But this immediately implies the relation $F(t) = \lambda t$ and, thus, completes the proof of Theorem 5.

III. We now pass to the main technical part of the proof of Theorem 5: the proof of the assertion formulated at the beginning of II.

Lemma 7 *If $t \neq 0$ and $(t_1, t), (t_2, t)$ are two \mathcal{T} -proper orbits, then $t_1 = t_2$.*

Proof. If $t_1 \neq t_2$ and

$$t = \delta_{j_1}(t_1), \quad t = \delta_{j_2}(t_2),$$

then $j_1 = j_2$, as the ranges of the maps δ_1 and δ_2 have the only common point $t = 0$. By the monotonicity properties of the maps δ_j , it is possible only if $\delta_{j_1}(t) = \text{const}$ at all points t between t_1 and t_2 . But then $\delta'_{j_1}(t) \equiv 0$ for these t , and, in particular, $\delta'_{j_1}(t_1) = 0$. However, this contradicts to the definition of \mathcal{T} -proper orbit: according to this definition, only the orbit $(t_1, \delta_{j'}(t_1))$ with $j' \neq j_1$ is \mathcal{T} -proper.

Remark From a geometrical point of view this fact means that two different \mathcal{T} -proper orbits cannot *enter* at the same point $t \in I \setminus \{0\}$.

Lemma 8 *If a cyclic orbit Σ is a part of a \mathcal{T} -proper orbit $\mathcal{O} = (t_1, t_2, \dots)$, then $t_1 \neq 0$ and $t_1 \in \Sigma$.*

Proof. If $t_1 \notin \Sigma$, we let t_q , $q \geq 2$, be the first point in \mathcal{O} , situated in Σ , so that

$$\Sigma = (t_q, t_{q+1}, \dots, t_{q+n}),$$

and $t_q = t_{q+n}$. But then $t_{q-1} \neq t_{q+n+1}$, as $t_{q-1} \notin \Sigma$ and $t_{q+n+1} \in \Sigma$, and, by Lemma 7, for none of indices j_1, j_2 the relation

$$t_q = \delta_{j_1}(t_{q-1}) = \delta_{j_2}(t_{q+n-1})$$

is possible. This proves that $t_1 \in \Sigma$.

To complete the proof of the lemma it remains to note that the relation $t_1 = 0$ is impossible due to (2): only the points $t = -1$ and $t = 1$ can precede the point $t_1 = 0$, but both of them have no predecessors on a cyclic orbit.

We now note that, if both sets \mathcal{T}_1 and \mathcal{T}_2 are infinite, then, by (13), for a sufficiently small $\varepsilon > 0$, there is a neighborhood $U_1 = (-1, -1 + \varepsilon)$ such that $\delta_2'(t) > 0$ for all points $t \in U_1$. It follows that for any such t the orbit $\mathcal{O} = (\delta_2(t), \delta_2^2(t), \dots, \delta_2^k(t), \dots)$ is \mathcal{T} -proper and converges to the point $t = -1$. In just the same way, there is a neighborhood $U_2 = (1 - \varepsilon, 1)$ such that the orbit $\mathcal{O}_1 = (t, \delta_1(t), \delta_1^2(t), \dots)$, remaining \mathcal{T} -proper, converges to the point $t = 1$, for an arbitrary $t \in U_2$.

However, if one of the sets, say \mathcal{T}_2 , is finite, than this property is guaranteed only for the neighborhood U_2 .

Lemma 9 *If the above $\varepsilon > 0$ is sufficiently small, then there is an integer ν such that*

$$\delta_1^\nu(t) > 1 - \varepsilon \quad \text{and} \quad \delta_2^\nu(t) < -1 + \varepsilon$$

for all points $t \in I$.

Proof. To begin with, we note that the relation

$$\delta_2(t) < t < \delta_1(t), \quad t \neq \pm 1, \quad (27)$$

holds. For example, the second relation is trivial for $t \leq 0$, by Definition 2. If $\delta_1(\hat{t}) = \hat{t}$ for some value \hat{t} , $0 < \hat{t} < 1$, then, by (22), $\delta_2(\hat{t}) = 0$, and by the same Definition 2, $\delta_2(t) \equiv 0$ on the interval $(\hat{t}, 1)$. But then $\delta_2'(1) = 0$, what contradicts to the second relation (12). This proves the inequality $t < \delta_1(t)$ for all $t \neq 1$. In turn, the latter inequality implies the relation $\delta_1^\nu(-1) \rightarrow 1$, as $\nu \rightarrow \infty$. Indeed, the sequence $\delta_1(-1), \delta_1^2(-1), \dots$ is bounded and increases. If $\lim \delta_1^\nu(-1) = \xi$, then, by continuity of δ_1 , $\lim \delta_1^{\nu+1}(-1) = \delta_1(\xi)$, whence $\delta_1(\xi) = \xi$, and $\xi = 1$. In the same way $\delta_2^\nu(1) \rightarrow -1$, as $\nu \rightarrow \infty$. This proves the lemma for the points $t = 1$ and $t = -1$. The case of an arbitrary t follows now by condition 1° in Definition 2. This completes the proof of Lemma 9.

Corollary 10 *If the last point of some \mathcal{T} -proper orbit $\mathcal{O}(t_1, t_2, \dots, t_N)$ lies in the set U_1 (U_2 resp.) (for some $\varepsilon > 0$), then the extended orbit $\mathcal{O}' = (t_1, \dots, t_N, \delta_1(t_N), \delta_1^2(t_N), \dots)$ ($\mathcal{O}' = (t_1, \dots, t_N, \delta_2(t_N), \delta_2^2(t_N), \dots)$, resp.) converges to the point $t = -1$ ($t = 1$, resp.)*

The proof is evident.

Now everything is ready to complete the proof of Theorem 5. Let M be the number of points of the set \mathcal{T}_2 . Then, given an arbitrary point $t_1 \notin \mathcal{T}_2$, we create the \mathcal{T} - proper orbit $\mathcal{O}_1 = (t_1, \delta_1(t_1), \dots, \delta_1^N(t_1))$ which moves toward the point $t = 1$. If $N = \nu$, then the last point of \mathcal{O}_1 lies in U_2 and, by Corollary 10, Theorem 5 is proved. If for some $N < \nu$ the point $t_2 = \delta_1^N(t_1)$ lies in \mathcal{T}_2 , then we create the \mathcal{T} - proper orbit $(t_2, \delta_2(t_2), \delta_2^2(t_2), \dots, \delta_2^k(t_2))$ moving toward $t = -1$. As above, if $k = \nu$, then the proof of Theorem 5 is finished by using the extended orbit, due to Corollary 10. If $k < \nu$ and $\delta_2^k(t_2) \in \mathcal{T}_1$, we extend the sewed orbit $(t_1, \dots, \delta_1^N(t_1), \delta_2 \circ \delta_1^N(t_1), \dots, \delta_2^k \circ \delta_1^N(t_1))$ by the points $\delta_1 \circ \delta_2^k(t_2), \dots, \delta_1^{N_1} \circ \delta_2^k(t_2)$ and so on. As it follows from Lemma 8, following this procedure we will never return at one of the previous points t_j of the orbit in question, with the exception, may be, of the point t_1 . It is clear, that if we do not meet the point t_1 after $2M\nu$ described steps, the corresponding point $t_{2M\nu+1}$ turns out to be in the set $U_1 \cup U_2$. After this; it remains to sew the orbit $(t_1, \dots, t_{2M\nu+1})$ with the mentioned orbit $(\delta_j(t_{2M\nu+1}), \delta_j^2(t_{2M\nu+1}), \dots)$ with $j = 1$ or 2 .

Consider the concluding situation: moving along the orbit \mathcal{O} we achieved the point t_1 . This means that we deal with a periodic \mathcal{T} - proper orbit $\Sigma = (t_1, \dots, t_m)$ which is a suborbit of a \mathcal{T} - proper orbit \mathcal{O} we construct. But (attention!), by the crucial hypothesis $\mathfrak{N}_\delta^{\mathcal{T}} = \emptyset$, this cycle Σ is not \mathcal{T} -guided. Consequently, Σ contains a point t_q , $1 \leq q \leq m - 1$, which does not belong to the guiding set \mathcal{T} . If, when constructing the orbit \mathcal{O} , we left the point t_q by means of a map δ_j (i.e. $t_{q+1} = \delta_j(t_q)$), then now we introduce the new point $\hat{t}_{q+1} = \delta_{j'}(t_q)$, $j' \neq j$. Extending the orbit $\mathcal{O} = (t_1, \dots, t_q, \hat{t}_{q+1})$ as above, we arrive at a new \mathcal{T} - proper orbit $\tilde{\mathcal{O}} = (t_1, \dots, t_q, \hat{t}_{q+1}, \hat{t}_{q+2}, \dots)$. But this time it can't meet one of its previous points (by Lemma 8) and, consequently, after at most $2M\nu$ steps, our \mathcal{T} - proper orbit enters in the neighborhood $U_1 \cup U_2$. To complete the proof of Theorem 5 it remains to use Corollary 10.

Remark If the point t_1 (see the text straight away after Corollary 10) lies in \mathcal{T}_2 , then we consider the point $t_1^* = \delta_2(t_1)$ and repeat all the above arguments in connection with this point.

3.6 Proof of Theorem 6

As in the proof of Theorem 5, we reduce problem (10) to the equivalent problem

$$\mathcal{B}F(t) := F(t) - F(\delta_1(t)) - F(\delta_2(t)) = h(t) \quad \text{on } I, \quad F(1) = \lambda \quad (28)$$

with δ_1 and δ_2 satisfying hypothesis (22). By the invariance of the boundary ∂I with respect to the transformation $t \rightarrow \alpha(t)$, the hypotheses (8) and (9)

become the necessary conditions

$$\mathcal{B}F(-1) = \mathcal{B}F(1) = -F(0) \quad \text{and} \quad h(-1) = h(1)$$

for solvability of problem (28). Replacing $F(t)$ by $F(t) + t(\lambda + h(1)) - h(1)$ in (28) we arrive at the equivalent problem

$$\mathcal{B}F = \widehat{h} \quad \text{on } I, \quad F(0) = 0, \quad F(1) = 0, \quad (29)$$

where $\widehat{h}(t) = h(t) - h(1)$. This form is precisely the starting point in studying problem (10). As is easily seen, both functions $\mathcal{B}F$ and \widehat{h} vanish at the points $t = -1$ and $t = 1$. Therefore, differentiating successively the equation $\mathcal{B}F = \widehat{h}_\mu$, we obtain the two new problems which are both *equivalent* to the problem (29). These new problems are

$$B_1 F'(t) := F'(t) - \delta'_1(t)F'(\delta_1(t)) - \delta'_2(t)F'(\delta_2(t)) = \widehat{h}'_\mu(t), \quad (30)$$

$$B_2 F''(t) := F''(t) - \delta_1'^2(t)F''(\delta_1(t)) - \delta_2'^2(t)F''(\delta_2(t)) - KF''(t) = \widehat{h}''_\mu(t), \quad (31)$$

where F' and F'' stand for the correspondent derivatives, and K denotes a linear operator in $C(I)$

$$K : H(t) \mapsto \delta_1''(t) \int_\xi^{\delta_1(t)} H(s)ds + \delta_2''(t) \int_\xi^{\delta_2(t)} H(s)ds.$$

Here ξ is an arbitrary point of the interval $(0, 1)$ such that $F'(\xi) = 0$. The existence of such point ξ is guaranteed by the boundary conditions in (29). For the sake of brevity, we omit the conditions $F(0) = 0$, $F(1) = 0$ in (30) and (31). What is important here is that K is a compact operator in the space $C(I)$. This is easily verified by virtue of the Arzelà criterion.

Since all the three problems (29), (30) and (31) in question are equivalent, *the solvability of problem (10) under condition (15) will be proved if we establish the injectivity of the operator B_1 , and also show that the index $\text{ind}B_2$ of the operator B_2 in the space $C(I)$ is zero.*

To prove the injectivity of the operator B_1 in the space of functions F vanishing at the points $t = 0$ and $t = 1$ we set $\widehat{h}'_\mu = 0$ in (30). By relation (25), the new equation coincides with problem (24) and, by Theorem 5, all its solutions F' are constants. These constants are nothing except zero, as the function F vanishes at two points. It remains to prove that $\text{ind}B_2 = 0$.

To this end we represent the operator B_2 in the form $\mathcal{B}_2 = E - L - K$, where E is the identical operator in the space $C(I)$, and L stands for the linear operator of the form

$$L : f \rightarrow \delta_1'^2(f \circ \delta_1) + \delta_2'^2(f \circ \delta_2)$$

in the same space. The following lemma is the main technical part of the proof of Theorem 6.

Lemma 11 *Under conditions of Theorem 6 for some natural number N the inequality*

$$\|L^N\| < 1 \quad (32)$$

holds with $\|\cdot\|$ a standard norm in the space of linear operators in $C(I)$.

This lemma makes it possible to readily complete the proof of Theorem 6 (in the "if" part) in framework of functional analysis. Indeed, by (32), the operator $E - L$ is invertible in the space $C(I)$, as the series $\sum_{m=0}^{N-1} \sum_{k=0}^{\infty} L^{kN+m}$ converges and its sum is equal to $(E - L)^{-1}$. Therefore, the operator B_2 is the sum of the invertible and compact operators. The relation $\text{ind} B_2 = 0$ follows now by a slight modification (see [P2]) of the classical Riesz-Schauder theorem.

Proof of Lemma 11. Introduce the following notation

$$a_j = \delta_j'^2; \quad a_{j_2}(\delta_{j_1}) = a_{j_1 j_2}; \quad a_{j_3}(\delta_{j_2} \circ \delta_{j_1}) = a_{j_1 j_2 j_3}; \quad \dots,$$

with all subindices j, j_1, j_2, \dots equal to 1 or 2. By induction on N , it is easily verified that the function $L^N F$ can be represented in the form

$$L^N F = \sum_{j_1, \dots, j_N=1}^2 a_{j_1} a_{j_1 j_2} \dots a_{j_1 \dots j_N} F(\delta_J),$$

where $J = (j_1, \dots, j_N)$ and $\delta_J = \delta_{j_N} \circ \dots \circ \delta_{j_1}$. It follows, that, for an arbitrary function F with $\|F\| = 1$, the inequality

$$|(L^N F)(t)| \leq \sum_{j_1, \dots, j_N=1}^2 a_{j_1}(t) a_{j_1 j_2}(t) \dots a_{j_1 \dots j_N}(t)$$

holds at each point $t \in I$. Denote by $\Lambda_N(t)$ the write-hand side of this relation. We will now prove that, for any fixed point $t \in I$, there is a positive number N and a constant $\gamma < 1$ such that the relation

$$\Lambda_N(t) \leq \gamma \quad (33)$$

holds. If $t \notin \mathcal{T}$, then $a_1(t) + a_2(t) < 1$, by (22), and relation (33) is true for $N = 1$. If $t \in \mathcal{T}$, consider a \mathcal{T} -proper orbit $\mathcal{O} = (t_1, t_2, \dots, t_N)$ with $t_1 = t$ such that t_N is the first point in \mathcal{O} situated in $I \setminus \mathcal{T}$ and N is the minimal number with this property. This means, in particular, that

$$t_2 = \delta_{j_1'}(t_1), \quad t_3 = \delta_{j_2'}(t_2), \dots, t_N = \delta_{j_{N-1}'}(t_{N-1}),$$

for some fixed indices j_1', \dots, j_{N-1}' and, in addition,

$$a_{j_1'}(t_1) = 1, \quad a_{j_2'}(t_2) = 1, \quad \dots, \quad a_{j_{N-1}'}(t_{N-1}) = 1, \quad \sum_{j_N=1}^2 a_{j_N}(t_N) < 1. \quad (34)$$

The existence of such orbit has been established when proving Theorem 5 (see III, page 15). It is clear that, for an arbitrary natural m , the relation $\Lambda_m \leq 1$ is true. To see this, it suffices to represent Λ_m in the form

$$\Lambda_m = \sum_{j_1} a_{j_1} \sum_{j_2} a_{j_1 j_2} \cdots \sum_{j_m} a_{j_1 \dots j_m}$$

and to use the relation $a_1(t) + a_2(t) \leq 1$ for all $t \in I$. Setting

$$A_{j_1 \dots j_{N-1}} = a_{j_1} a_{j_1 j_2} \cdots a_{j_1 \dots j_{N-1}} \quad \text{and} \quad \mathcal{B}_{j_1 \dots j_{N-1}} = \sum_{j_N=1}^2 a_{j_1 \dots j_{N-1} j_N},$$

we can rewrite the function Λ_N in the form

$$\Lambda_N = \sum_{j_1, \dots, j_{N-1}} A_{j_1 \dots j_{N-1}} \mathcal{B}_{j_1 \dots j_{N-1}}.$$

It is evident that if, for some nonnegative numbers p_k and q_k , the relations

$$\sum_{k=1}^r p_k q_k \leq 1, \quad \sum_{k=1}^r p_k \leq 1, \quad \text{and} \quad q_k \leq 1 \text{ for all } k,$$

hold, and at least one number $q_{k'}$ is less than one, then

$$\sum_{k=1}^r p_k q_k < 1.$$

To make use this observation with respect to the function $\Lambda_N(t)$ note that, by the above,

$$\sum_{j_1, \dots, j_{N-1}} A_{j_1 \dots j_{N-1}} \leq 1.$$

Furthermore, by the definition of the multi-index (j'_1, \dots, j'_N) , and by (34), we find that

$$\mathcal{B}_{j'_1 \dots j'_{N-1}}(t) = \sum_{j_N} a_{j'_1 \dots j'_{N-1} j_N}(t) = \sum_{j_N} a_{j_N}(t_N) < 1.$$

On the other hand, by virtue the same relation (34),

$$A_{j'_1 \dots j'_{N-1}}(t) = a_{j'_1}(t_1) a_{j'_2}(t_2) \cdots a'_{j'_{N-1}}(t_{N-1}) = 1.$$

Thus, the function $\Lambda_N(t)$ has the same structure as the expression $p_k q_k$ above and, consequently, the inequality $\Lambda_N(t) < 1$ holds with a constant $\gamma = \gamma(t) < 1$.

Note now that, by virtue of the continuity of all the functions in question, relation (33) holds at all points of some neighborhood \mathcal{U} of the point under

consideration for the same number N , may be with a larger constant $\gamma < 1$. The collection of such neighborhoods forms an open covering of the closed set $I \setminus \mathcal{B}$. Let $\{\mathcal{U}_j\}_{j=1}^k$ be a finite subsystem of these neighborhoods, and N_j, γ_j the corresponding constants. Taking $m = \max N_j$ and $\gamma = \max \gamma_j$ we arrive at the desired inequality $\|L^m\| < 1$.

To prove the necessity of the hypothesis $\mathfrak{N}_\alpha^{\mathcal{T}} = \emptyset$, we assume that $\mathfrak{N}_\alpha^{\mathcal{T}} \neq \emptyset$. Then there is a \mathcal{T} - proper \mathcal{T} - guided periodic orbit $\mathcal{O} = (t_1, \dots, t_{n+1})$. If a function $F(t)$ solves problem (28), than the function $G = F'$ solves the equation

$$G(t) - \delta'_1 G(\delta_1(t)) - \delta'_2 G(\delta_2(t)) = H(t), \quad (35)$$

with $H = \widehat{h}'_\mu$ (see (30)). By definition of the orbit \mathcal{O} ,

$$t_{k+1} = \delta_{j_k}(t_k), \quad \delta'_{j'_k}(t_k) = 0 \quad \text{for } j'_k \neq j_k, \quad k = 1, \dots, n, \quad \text{and } t_{n+1} = t_1.$$

Let us substitute t_1 for t in equation (35). Then one of the numbers $\delta'_{j'_k}(t_1)$ is equal to zero, whereas $\delta'_{j'_k}(t_1) = 1$. Furthermore, $t_2 = \delta_{j'_1}(t_1)$, by the definition of \mathcal{T} - proper orbit. This leads to the relation $G(t_1) - G(t_2) = H'(t_1)$. Continuing this procedure we arrive at the chain of equalities $G(t_2) - G(t_3) = H'(t_2), \dots, G(t_n) - G(t_1) = H'(t_n)$, where the periodicity of the orbit \mathcal{O} is used on the last step. Adding all these relations together, we obtain

$$\sum_{j=1}^n H'(t_j) = 0. \quad (36)$$

Thus, in the presence of a \mathcal{T} - proper \mathcal{T} - guided cycle $\mathcal{O} = (t_1, \dots, t_n)$, this relation is a necessary condition for the solvability of problem (28). Hence, the necessity of the condition $\mathfrak{N}_\zeta^{\mathcal{T}} = \emptyset$ is proved. To complete the proof of Theorem 6 it remains to establish the continuity of the inverse operator \mathcal{B}_α^{-1} . The simplest way to do this is using Banach closed graph theorem combined with the solvability of problem (28) already proved, and with an obvious a priory estimate

$$\|\mathcal{B}F, C^2(I)\| \leq \text{const}\|F, C^2(I)\|.$$

4 Applications

In this section we show that the Cauchy type functional equations arise naturally as a necessary technical tool in solving various problems in such diverse fields as integral and functional equations, integral geometry, boundary problems for partial differential equations etc. In order not to enlarge extremely the size of this paper, some of corresponding results are given with full proofs, and the proofs are only outlined for some.

4.1 Multiplicative Cauchy type functional equations

This title was introduced in the paper [P4] as the definition for functional equations of the form

$$F(\alpha_1(t) + \alpha_2(t)) = F(\alpha_1(t))F(\alpha_2(t))H(t), \quad t \in I', \quad (37)$$

where $H > 0$ is a given function, and the functions α_1 and α_2 form one of the above configurations.

Below we prove two theorems giving an exhaustive solution of the solvability problem for this equation in both cases of Z - and \mathcal{P} - configurations.

4.1.1 The case of a Z - configuration

We begin with a very simple but crucial lemma which makes it possible to reduce problem (37) to a standard Cauchy type functional equation.

Lemma 12 *If functions α_1 and α_2 form a Z - configuration, then any non-trivial continuous solution F of equation (37) is positive on I' .*

Proof. Let $F(z) \neq 0$ on I' and

$$z_0 := \min\{z \mid F(z) = 0\}.$$

If $z_0 \neq 0$ and $F(z_0) = F(\alpha_1(t_0) + \alpha_2(t_0))$, then, by (37), $F(\alpha_1(t_0)) = 0$ or $F(\alpha_2(t_0)) = 0$. This means that for some $z_1 < z_0$ with $z_1 = \alpha_1(t_0)$ or $z_1 = \alpha_2(t_0)$, one has $F(z_1) = 0$. But this contradicts to the definition of z_0 . It remains to exclude the possibility $F(0) = 0$. If $F(0) = 0$, then for an arbitrary small $\varepsilon > 0$ (satisfying also the relation $\varepsilon H(0) < 1$), there is a point $z \in \mathcal{R}(\alpha_1 + \alpha_2)$ such that $F(z) = \varepsilon$ and $F(z') < \varepsilon$ for all $z' < z$. If $z = (\alpha_1 + \alpha_2)(t_0)$, then $F(\alpha_1(t_0)) < \varepsilon$ and $F(\alpha_2(t_0)) < \varepsilon$, as $\alpha_j(t) > 0$ for $t > 0$ and $j = 1, 2$. By (37), this leads to the impossible relation

$$\varepsilon \leq \varepsilon^2 H(0) < \varepsilon,$$

and this completes the proof of the lemma.

This lemma makes it possible to reduce equation (37), by taking the logarithm of both sides of this equation, to the standard Cauchy type functional equation

$$G(\alpha_1(t) + \alpha_2(t)) = G(\alpha_1(t)) + G(\alpha_2(t)) + H(t), \quad t \in I'.$$

The following result is a direct corollary of Theorem 2 and Theorem 3.

Theorem 13 1° Assume that the functions α_1 and α_2 satisfy condition (6) and belong to the space $C_{(1)}(I')$. Then all solutions F of the homogeneous equation

$$F(\alpha_1(t) + \alpha_2(t)) = F(\alpha_1(t))F(\alpha_2(t)), \quad t \in I', \quad (38)$$

belonging to the space $C_{(1)}(\widehat{I})$, have a form $F(z) = e^{\lambda z}$, $\lambda \in \mathbb{R}$.

2° Assume that both functions α_1 and α_2 satisfy the condition of Theorem 3. Then, for an arbitrary function $H \in C_{(1+r)}(I')$, $0 < r < 1$, satisfying condition (3), and for any positive number λ , there is a unique solution $F \in C_{(1+r)}(\widehat{I})$ of the problem

$$F((\alpha_1 + \alpha_2)(t)) = F(\alpha_1(t))F(\alpha_2(t))H(t), \quad t \in I'; \quad F(1) = \lambda. \quad (39)$$

This solution can be represented in the form

$$F(z) = e^{\mu z} \prod_J H(\delta_J(z)), \quad z \in \widehat{I},$$

with some (uniquely defined) real μ and δ_J running all elements of the semigroup Φ_δ , generated by the maps $\delta_j = \alpha_j \circ \alpha^{-1}$, $j = 1, 2$, from the interval \widehat{I} into itself.

4.1.2 The case of a \mathcal{P} - configuration

Let functions α_1 and α_2 form a \mathcal{P} - configuration. Then a new point, as compared with the case of a Z -configuration, is that solutions of equation (37) are not necessarily strictly positive on the interval I , as follows from the

Lemma 14 *The equation*

$$F(t) = F\left(\frac{t-1}{2}\right)F\left(\frac{t+1}{2}\right), \quad t \in I, \quad (40)$$

has infinite set of oscillating solution, i.e. those taking values of opposite signs.¹

Proof. Note first of all that the functions $\alpha_1 = (t-1)/2$ and $\alpha_2 = (t+1)/2$ form a \mathcal{P} - configuration on I and satisfy condition (5). As is directly verified, any function

$$F_k(t) = 2 \cos(k + 1/2)\pi t, \quad k = 0, 1, \dots$$

solves equation (40) and oscillates on I .

By this lemma, the hypothesis of positiveness of a solution to equation (37) is necessary in the following assertions.

¹This result was obtained by Dr.Galina A.Birulina from Moscow as an answer to the author's question. It would be interesting to get to know, whether this property is characteristic for a \mathcal{P} -configuration.

Theorem 15 1° Assume that the functions α_1 and α_2 belong to the space $C^1(I)$ and satisfy conditions (12), (13) and (15). Then all positive C^1 -solutions to the homogeneous equation (38) are of the form

$$F(z) = e^{\lambda z}, \quad \lambda \in \mathbb{R}.$$

2° Let α_1 and α_2 satisfy the same hypotheses and belong to the space $C^2(I)$. Then, for an arbitrary positive function $H \in C^2(I)$, satisfying hypothesis (6), and for any real λ , there is a unique positive solution $F \in C^2(I)$ to the problem (39). The inverse operator $H \mapsto F$ from $C^2(I)$ to $C^2(I)$ is continuous.

4.1.3 On a quasi-Cauchy type functional equation

Here we consider some functional equation that formally reminds the equations studied in 3.1.1. The question is the solvability of the equation

$$F(t) = F(a_1 t)F(a_2 t)H(t), \quad t \in I', \quad (41)$$

where $0 < a_1 < a_2 < 1$, and H is a given positive continuous function. The case $H(t) \equiv 1$ and $a_1^2 + a_2^2 = 1$ has been considered in [KCG, Th. 6.1.2] and the solution $F(t) = e^{\lambda t^2}$ has been obtained.

First of all, repeating word for word the proof of Lemma 12, we establish the positiveness of any solution F to equation (41). This makes it possible to reduce equation (41) in an equivalent manner to the linear equation.

$$(\mathcal{L}G)(t) := G(t) - G(a_1 t) - G(a_2 t) = \mathcal{H}(t), \quad t \in I', \quad (42)$$

where $G = \ln F$, $\mathcal{H} = \ln H$. Let κ be a unique x -root of the equation

$$a_1^x + a_2^x = 1.$$

Denote by $[\kappa]$ and $\{\kappa\}$ the integral and the fractional parts of the number κ , so that $0 < \{\kappa\} < 1$. In addition to $C_{\langle \kappa \rangle}(I')$, we introduce the linear space

$$C_{\langle \kappa, \rho \rangle}(I') = \{u \in C(I') \mid u(t) = P_k(t) + d_u t^\kappa + t^{\kappa+\rho} \gamma(t)\},$$

where $k = [\kappa]$, $0 < \rho < 1 - \{\kappa\}$, and $\gamma(t)$ is an arbitrary function from $C(I')$.

Theorem 16 1° If $H(t) = 1$, then the function

$$F(t) = e^{\lambda t^\kappa}, \quad t \in I',$$

is a unique solution of equation (41) in the space $C_{\langle \kappa \rangle}(I')$, for which $F(1) = e^\lambda$.

2° For an arbitrary function $H \in C_{\langle \kappa, \varepsilon \rangle}(I')$, $0 < \varepsilon < 1 - \{\kappa\}$, satisfying the necessary condition

$$d_H = 0,$$

and for any real λ , there is a unique solution $F \in C_{\langle \kappa, \varepsilon \rangle}(I')$ of the above problem.

Proof. 1° Let $G \in C_{\langle \kappa \rangle}(I')$ be an arbitrary solution of equation (42) with $\mathcal{H} = 0$. Then, by definition,

$$G(t) := P_k(t) + t^\kappa \gamma(t), \quad t \in I', \quad (43)$$

where $k = [\kappa]$. Assume first that $\kappa \neq k$. It is clear that if $P_k(t) = \sum_{j=0}^k c_j t^j$, then $(\mathcal{L}P_k)(t) = \sum_{j=0}^k c_j (1 - a_1^j - a_2^j) t^j$ and $\mathcal{L}(ct^\kappa) = c(1 - a_1^\kappa - a_2^\kappa) t^\kappa = 0$ for all constants c , by definition of κ . It follows that the function

$$\lambda(t) := \gamma(t) - \gamma(0)$$

satisfies the equation

$$\lambda(t) - a_1^\kappa \lambda(a_1 t) - a_2^\kappa \lambda(a_2 t) = 0.$$

As $a_1^\kappa + a_2^\kappa = 1$, we conclude, arguing as in Subsec.3.2, that the maximal and minimal values of the solution λ coincide (and equal to $\lambda(0)$). It follows, that $\lambda(t) = 0$. This proves assertion 1° for $\kappa \neq k$. If $\kappa = k$, then $\gamma(0) = 0$ in relation (43), and we arrive at the same result by repeating the same arguments word for word. This completes the proof of 1°.

2° First of all, let us show that if a function $G \in C_{\langle \kappa \rangle}$ solves the equation $\mathcal{L}G = H$ with $H \in C_{\langle \kappa \rangle}$, then $d_H = 0$. Indeed, let

$$G = \sum_{j=0}^k c_j t^j + ct^\kappa + t^\kappa \mu(t) \quad \text{and} \quad H = \sum_{j=0}^k b_j t^j + bt^\kappa + t^\kappa \nu(t)$$

where b and c are constants and $\mu(0) = \nu(0) = 0$. It is obvious that

$$b = c(1 - a_1^\kappa - a_2^\kappa) = 0,$$

if $\kappa \neq k$, and $b_k = c_k(1 - a_1^k - a_2^k) = 0$, if $\kappa = k$. To prove the assertion it remains to present a solution G of the equation $\mathcal{L}G = H$, for any given $H \in \tilde{C}_{\langle \kappa, \varepsilon \rangle}(I')$. Write down the functions G and H in the form

$$\begin{aligned} G &= \sum_1^k c_j t^j + \alpha t^\kappa + \beta t^{\kappa+\varepsilon} + t^{\kappa+\varepsilon} \mu(t), \quad \mu(0) = 0 \\ H &= \sum_1^k b_j t^j + \beta_1 t^{\kappa+\varepsilon} + t^{\kappa+\varepsilon} \nu(t), \quad \nu(0) = 0. \end{aligned}$$

If we choose $c_j = b_j / (1 - a_1^j - a_2^j)$, for all $j = 1, \dots, k$ and $\beta = \beta_1 / (1 - a_1^{\kappa+\varepsilon} - a_2^{\kappa+\varepsilon})$, then what remains to do is to choose, as $\mu(t)$, a solution of the problem

$$\mathcal{L}(t^{\kappa+\varepsilon} \mu) = t^{\kappa+\varepsilon} \nu, \quad \mu(0) = 0. \quad (44)$$

Since $\mathcal{L}(t^{\kappa+\varepsilon}\mu) = t^{\kappa+\varepsilon}[\mu(t) - a_1^{\kappa+\varepsilon}\mu(a_1t) - a_2^{\kappa+\varepsilon}\mu(a_2t)]$, problem (44) takes the form

$$\mu(t) - a_1^{\kappa+\varepsilon}\mu(a_1t) - a_2^{\kappa+\varepsilon}\mu(a_2t) = \nu(t), \quad \mu(0) = 0. \quad (45)$$

But, by definition of κ , the relation

$$a_1^{\kappa+\varepsilon} + a_2^{\kappa+\varepsilon} < 1$$

holds, and this means that the norm of the linear operator

$$\mathcal{L}_\varepsilon : \mu(t) \mapsto a_1^{\kappa+\varepsilon}\mu(a_1t) + a_2^{\kappa+\varepsilon}\mu(a_2t)$$

in the space $C(I')$ is less than one. It follows that the sum of the Neumann series $\sum_{n=0}^{\infty} \mathcal{L}_\varepsilon^n \nu$ with $\mathcal{L}_\varepsilon^0 = id$ solves the equation in (45). The boundary condition $\mu(0) = 0$ is valid, as $\nu(0) = 0$, and so $(\mathcal{L}_\varepsilon^n \nu)(0) = 0$, for all $n = 1, 2, \dots$. This completes the proof of Theorem 16.

4.2 On some integral equations relating to a geometric problem

In this subsection we deal with one of the typical problems in integral geometry: given a domain D , to reconstruct a function in D from the values of its integrals over a family $\{D_j\}$ of subdomains in D . Such problems are always of interest first of all by themselves as objects of pure analysis, but also in connection with possible applications in practical disciplines. The most remarkable example of such a connection is the famous Radon problem and tomography. A peculiarity of the considered geometric problems is that the domains D_j , $j = 1, 2$, we deal with, are bounded, and the statements of these problems are closely connected with both local and global properties of the boundaries ∂D . This connection is realized by means of a semigroups Φ_{D_j} of maps from ∂D_j into itself we associate with the problems in question. These problems, as will be seen, can be reduced to Cauchy type functional equations on certain intervals I_j , distinguishing from each other by configurations of maps generating these equations. These maps also generate corresponding semigroups Φ_{I_j} . We introduce a special isomorphism between the boundary ∂D and (each of) the intervals I_j , and we show that it can be extended up to an isomorphism of the above semigroups Φ_D and Φ_{I_j} . This makes it possible to formulate the (geometric) problems in question and the conditions of their solvability using the geometric language (related to the semigroups Φ_{D_j}), but, when solving these problems, to apply analytical methods using the semigroups Φ_{I_j} and the results related to Cauchy type functional equations, already obtained above. We consider separately two situations.

4.2.1 The case of a Z - configuration

Let \bar{l}_1 and \bar{l}_2 be smooth nonsingular transversal vector fields in a disk $B \subset \mathbb{R}^2$. Consider a curvilinear parallelogram $D = OA_1O'A_2$ in B whose sides OA_1 , $O'A_2$ and OA_2 , $O'A_1$ are trajectories of the vector fields \bar{l}_1 and \bar{l}_2 , respectively. Let $\Gamma = OO'$ be a quasidiagonal in D , i.e. a smooth curve without common points with the sides of D . Assume that Γ is a trajectory of the vector field $\bar{l} = r_1\bar{l}_1 + r_2\bar{l}_2$ with some positive constant r_1 and r_2 . It is easily seen that, given vector fields \bar{l}_1 , \bar{l}_2 and a point O , there are some points A_1 , A_2 and O' such that all the above conditions are satisfied.

Introduce in D projections

$$\pi_1: D \rightarrow OA_1, \quad \pi_2: D \rightarrow OA_2$$

along the vector fields \bar{l}_2 and \bar{l}_1 , respectively. Given an arbitrary point $q \in \Gamma$, denote by D_q the curvilinear parallelogram qq_1Oq_2 where $q_1 = \pi_1q$ and $q_2 = \pi_2q$. With any function f in D we associate the integrals

$$(\mathcal{A}f)(q) = \int_{D_q} f d\sigma, \quad q \in \Gamma,$$

where σ denotes a measure on B . The above mentioned geometric problem of reconstructing a function in \bar{D} takes now a completely definite form:

given a function h on Γ , to find a function f in D such that

$$\mathcal{A}f(q) = h(q), \quad q \in \Gamma. \quad (46)$$

For obtaining reach in content results related to equation (46) we will study it in the framework of the general problem stated in [GGV] for a wide class of integral equations: for what spaces of functions f and h is the map $\mathcal{A}: f \mapsto h$ one-to-one, and what functions $h(q)$ may be represented by the integral $(\mathcal{A}f)(q)$. Restricting ourselves to continuous functions on D and on Γ we find immediately that the kernel of the operator \mathcal{A} is infinite-dimensional. On the other hand, as will be seen, for an arbitrary function $f \in C(\bar{D})$, its image $(\mathcal{A}f)(q)$ belongs to the space $\mathcal{H}(\Gamma) = \{v \in C^2(\Gamma) \mid v(0) = 0, v'(0) = 0\}$. Therefore, the best possible solution of the problem in question consists in describing subspaces $\mathcal{F}(D) \subset C(\bar{D})$ for which the map

$$\mathcal{A}: \mathcal{F}(D) \rightarrow \mathcal{H}(\Gamma)$$

is one-to-one. Among various candidates for the role of $\mathcal{F}(D)$ we have chosen a wide class of subspaces in $C(\bar{D})$ arising naturally when solving certain boundary problems for higher order hyperbolic equations (see Subsec. 4.3).

Definition 5. Given a smooth vector field \bar{l} in B , we denote by $C_{\langle \bar{l} \rangle}(D)$ the subset of functions in $C(\bar{D})$, remaining constant along each trajectory of the field \bar{l} .

As is well known the space $C_{\bar{l}}(D)$ admits a very simple description. Let $y = (y_1, y_2)$ be a coordinate system in B and

$$\bar{l} = \{ \omega_1(y), \omega_2(y) \}, \quad y \in B,$$

be a coordinate form of \bar{l} . Denote by $\zeta(y)$ a smooth function without critical points solving the first order differential equation

$$\omega_1(y)(\partial/\partial y_1)\zeta + \omega_2(y)(\partial/\partial y_2)\zeta = 0. \quad (47)$$

Then the space $C_{\bar{l}}(D)$ consists of all functions $f(y) = F(\zeta(y))$ with F being arbitrary continuous function on the range of ζ .

Now everything is ready to formulate our first geometric result.

Theorem 17 *Let \bar{l}_1, \bar{l}_2 and \bar{l} be a triple of smooth nonsingular mutually transversal vector fields in a disk $B \in \mathbb{R}^2$ and let D be an above parallelogram constructed by \bar{l}_1 and \bar{l}_2 . Assume that the corresponding quasidiagonal Γ is a twice differentiable nonsingular curve which is transversal to \bar{l} everywhere and to \bar{l}_1 and \bar{l}_2 of the point O . Then, for an arbitrary function $H \in \mathcal{H}(\Gamma)$, there is a unique solution $f \in C_{\bar{l}}(D)$ of equation (46). The inverse operator A^{-1} is a continuous operator from $\mathcal{H}(\Gamma)$ to $C_{\bar{l}}(D)$.*

Proof. For the brevity, we consider the situation with constant vector fields \bar{l}_1, \bar{l}_2 and \bar{l} . Without loss of generality we assume that the vector fields \bar{l}_1, \bar{l}_2 are parallel to the coordinate axes x_1 and x_2 , respectively, and consider as D a square whose two sides coincide with the intervals $\{x_1 | 0 \leq x_1 \leq 1\}$ and $\{x_2 | 0 \leq x_2 \leq 1\}$ on the axes x_1 and x_2 , respectively. Let

$$x_1 = \delta_1(t), \quad x_2 = \delta_2(t), \quad t \in I',$$

be an arbitrary parametric representation of Γ , so that

$$\delta_1(0) = \delta_2(0) = 0, \quad (48)$$

If $\bar{l} = (\omega_1, \omega_2)$ with $\omega_1 > 0, \omega_2 < 0$, then equation (46) takes the form:

$$\int_0^{\delta_1(t)} \int_0^{\delta_2(t)} f(\omega_1 x_2 - \omega_2 x_1) dx_2 dx_1 = H(t), \quad t \in I, \quad (49)$$

because the function $\zeta(x_1, x_2) = \omega_1 x_2 - \omega_2 x_1$ is a solution to equation (47).

Introduce the new unknown function F as a solution to the problem

$$F''(t) = f(t), \quad F(0) = F'(0) = 0, \quad (50)$$

and substitute F'' for f in (49). After repeated integration by parts we arrive at the following functional equation for the function F :

$$F(\omega_1 \delta_2(t) - \omega_2 \delta_1(t)) - F(\omega_1 \delta_2(t)) - F(-\omega_2 \delta_1(t)) = H(t). \quad (51)$$

Since the map $f \mapsto F$ is one-to-one, to prove the theorem, it remains to establish the unique solvability of equation (51). Note that, by assumptions of the theorem, we have $(\delta'_1 \delta'_2)(0) > 0$, which means that

$$\omega_1 \delta'_2(0) - \omega_2 \delta'_1(0) > 0.$$

The transversality condition $\Gamma \pitchfork \bar{l}$ guarantees the inequality $\omega_1 \delta'_2(t) - \omega_2 \delta'_1(t) > 0$ at any point $t \in I'$. On the other hand, differentiating relation (49) leads to the differentiability of the function H and to the relation

$$\delta'_1(t) \int_0^{\delta_2(t)} f(\omega_1 x_2 - \omega_2 \delta_1(t)) dx_2 + \delta'_2(t) \int_0^{\delta_1(t)} f(\omega_1 \delta_2(t) - \omega_2 x_1) dx_1 = H'(t).$$

It follows, by (48), that $H'(0) = 0$. To see that the function H is twice differentiated, it suffices to represent the left hand-side of the late relation in the form

$$\left(-\delta'_1(t)/\omega_2\right) \int_{-\omega_2 \delta_1(t)}^{\omega_1 \delta_2(t) - \omega_2 \delta_1(t)} f(z) dz + \left(\delta'_2(t)/\omega_1\right) \int_{\omega_1 \delta_2(t)}^{\omega_1 \delta_2(t) - \omega_2 \delta_1(t)} f(z).$$

Thus, equation (51) is nothing, but a Cauchy type functional equation on I' , corresponding to a Z -configuration and satisfying all conditions of Theorem 2. By this theorem, for an arbitrary function $H \in C^2(I')$, satisfying the hypothesis $H'(0) = 0$, there is a unique solution $F \in C_{(1+r)}(I')$ of equation (51), and, by Theorem 4, this function F belongs to the space $C^2(I')$. This proves the existence and uniqueness of a function $f \in C_{(\bar{l})}(D)$, solving equation (49). As to the continuity of the inverse operator \mathcal{A}^{-1} , it follows from the Banach closed graph theorem which, in turn, is a consequence of the elementary inequality $\|\mathcal{A}f, C^2\| \leq \text{const} \|f, C\|$.

4.2.2 The case of a \mathcal{P} - configuration

In this Subsection a geometric problem analogous to that considered above is studied. But this time a dynamical system associated with this problem is much more complicated.

As above, let \bar{l}_1 and \bar{l}_2 be some nonsingular smooth transversal vector fields in a disk $B \subset \mathbb{R}^2$. Consider a curvilinear triangle $D = OA_1A_2$, whose sides OA_1 and OA_2 coincide with trajectories of the vector fields \bar{l}_1 and \bar{l}_2 , respectively, and the side $\Gamma = A_1A_2$ is an arbitrary nonsingular smooth curve transversal to both fields \bar{l}_1, \bar{l}_2 at the points A_1 and A_2 . In addition, the closure \bar{D} of the domain D is supposed to satisfy the following topological hypotheses:

- (i) For any point $p \in \bar{D}$, a trajectory of \bar{l}_j passing through p meets OA_k at a point $\pi_k p$, $j \neq k$, $1 \leq j, k \leq 2$.

(ii) The set \bar{D} is \bar{l}_j -convex, $j = 1, 2$. This means that if given points p and q in \bar{D} lie on some trajectory γ_j of the field \bar{l}_j , then all the points $r \in \gamma_j$, $j = 1, 2$, between p and q belong to \bar{D} .

Given an arbitrary point $q \in \Gamma$, let D_q be a curvilinear parallelogram qq_1Oq_2 .

Hypotheses (i) and (ii) guarantee the inclusion $D_q \subset \bar{D}$ for all $q \in \Gamma$ and something more. Namely, let $x = (x_1, x_2)$ be a coordinate system in B with the origin at the point O . Let

$$x_1 = \alpha_1(z), \quad x_2 = \alpha_2(z), \quad z \in I,$$

be an arbitrary parametric representation of Γ . Set $\alpha(z) = (\alpha_1(z), \alpha_2(z))$. Then the functions α_1 and α_2 satisfy the conditions

$$\alpha_1'(z) \geq 0, \quad \alpha_2'(z) \leq 0, \quad \text{and} \quad \alpha_1'(z) - \alpha_2'(z) > 0 \quad (52)$$

This fact is not trivial (see [P4] with respect to the proof).

The geometric problem under consideration takes this time the form of the same integral equation

$$(\mathcal{A}f)(q) := \int_{D_q} f d\sigma = h(q), \quad q \in \Gamma, \quad (53)$$

as in Subsec. 4.2.1, but with a different domain D_q . All the discussions around the statement of problem (46) are relevant to problem (53). In particular, the right-hand side $h(q)$ in (53) should be a twice differentiable function on Γ and satisfy the periodicity condition

$$h(A_1) = h(A_2) = 0.$$

The latter follows immediately from (53). We denote the space of all such functions by $\mathcal{H}_0(\Gamma)$. To formulate the result, related to the solvability of equation (53), in an invariant geometric form (not using any coordinate form of (53)) we generalize the above dynamical approach by considering certain semigroups of maps from Γ into itself. It is interesting to trace the transition from a pure geometric problem to a Cauchy type functional equation (what includes also a translation from a geometric language to an analytic one).

Introduce the vector field $\bar{l} = r_1\bar{l}_1 + r_2\bar{l}_2$, where r_1 and r_2 are positive constants. In addition to the projections π_j , $j = 1, 2$, along the vector fields \bar{l}_k , $j \neq k$, we consider the projection $\pi_{\bar{l}}: \bar{D} \rightarrow \Gamma$ along the field \bar{l} . In other words, given an arbitrary point $p \in \bar{D}$, its projection $\pi_{\bar{l}}p$ coincides with the intersection of the curve Γ and the trajectory of the field \bar{l} , passing through p . Introduce two maps from Γ into itself,

$$\zeta_1 = \pi_{\bar{l}} \circ \pi_1 \quad \text{and} \quad \zeta_2 = \pi_{\bar{l}} \circ \pi_2,$$

and denote by Φ_ζ the semigroup of maps generated by ζ_1 and ζ_2 , which is analogous to the semigroup Φ_δ considered in Sec. 2. As in Subsec. 2.2, we define, by means of the maps ζ_1 and ζ_2 , an *orbit* in Γ as a sequence of points $\mathcal{O} = (q_1, q_2, \dots, q_n, \dots)$ such that

$$q_{k+1} = \zeta_{j_k}(q_k), \quad k = 1, 2, \dots,$$

and the set of all such orbits we denote by \mathcal{O}_ζ . We introduce the *guiding* sets

$$\mathcal{T}_j = \{q \in \Gamma \mid \bar{l}_j(q) \in T_q(\Gamma)\}, \quad j = 1, 2, \quad \text{and} \quad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2,$$

where $T_q(\Gamma)$ stands for the tangent space to Γ at the point q . Repeating literally what was said in Subsec. 2.2 we define *periodic*, *\mathcal{T} -guided*, and *\mathcal{T} -proper orbits* corresponding to the semigroup Φ_ζ . Finally, we denote by $\mathfrak{N}_\zeta^\mathcal{T}$ the set of \mathcal{T} -proper \mathcal{T} -guided periodic orbits of the type in question. By analogy with condition (13), we assume that, if both sets \mathcal{T}_1 and \mathcal{T}_2 are infinite, then arbitrary points $\tau_1 \in \mathcal{T}_1'$ and $\tau_2 \in \mathcal{T}_2'$ are situated on the curve Γ in the order A_1, τ_1, τ_2, A_2 . The main result, related to the problem in question, is formulated as the following assertion.

Theorem 18 *If all the above hypotheses related to the domain D , the curve Γ , and vector fields \bar{l} , \bar{l}_1 and \bar{l}_2 are fulfilled, then, for an arbitrary function $h \in \mathcal{H}_0(\Gamma)$, there is a unique solution $f \in C_{\bar{l}}(D)$ of equation (53) if and only if $\mathfrak{N}_\zeta^\mathcal{T} = \emptyset$. The inverse operator $A^{-1} : h \mapsto f$ is continuous from $\mathcal{H}_0(\Gamma)$ to $C_{\bar{l}}(D)$.*

Proof. Without loss of generality, we assume that the vector fields \bar{l}_1 and \bar{l}_2 are parallel to the coordinate axes x_1 and x_2 , respectively, and consider as D a curvilinear triangle whose two sides coincide with the intervals $\{x_1 \mid 0 \leq x_1 \leq 1\}$ and $\{x_2 \mid 0 \leq x_2 \leq 1\}$ on the axes x_1 and x_2 , respectively. Under this assumption, the vector field \bar{l} takes the form $\bar{l} = r_1 \bar{l}_1 + r_2 \bar{l}_2$. With respect to the third side Γ of the triangle D , it is assumed to be a smooth nonsingular curve transversal to the coordinate axes. Let

$$x_1 = \alpha_1(z), \quad x_2 = \alpha_2(z), \quad z \in I,$$

be an arbitrary parametric representation of the curve Γ . Introduce the notation

$$\alpha(z) = (\alpha_1(z), \alpha_2(z)),$$

and assume that $\alpha(0) = (0, 1)$. Then, as was explained above, relations (52) are valid. It is convenient to treat α as a map: $I \rightarrow \Gamma$ which is invertible, by (52). As was explained above, the space $C_{\bar{l}}(\bar{D})$ consists of the functions $f(r_2 x_1 - r_1 x_2)$ with f being an arbitrary continuous function on the closed interval $I_T = (-r_1, r_2)$. Introduce the functions

$$\omega(x) = r_2 x_1 - r_1 x_2, \quad \omega_1(x) = \omega(x_1, 0), \quad \omega_2(x) = \omega(0, x_2)$$

in \overline{D} , and denote by ω_Γ the restriction of ω to Γ . (I do not consciously use the explicit values of the functions ω_1 and ω_2 to give to the reader a possibility to observe the main springs of the proof not connected with the constancy of the vector field \bar{l}). It is obvious, that the function

$$\sigma(z) = \omega_\Gamma \circ \alpha(z): I \rightarrow I_T$$

is invertible and surjective. Indeed, the relation

$$\sigma'(z) = \omega_{x_1}(\alpha)\alpha'_1(z) + \omega_{x_2}(\alpha)\alpha'_2(z) > 0, \quad z \in I,$$

holds, in view of the inequalities

$$\omega_{x_1} > 0, \quad \omega_{x_2} < 0, \quad \alpha'_1(z) \geq 0, \quad \alpha'_2(z) \leq 0,$$

and this proves the invertibility. As to the surjectivity, take a point $q \in \overline{D}$ such that $\omega(q) = -r_2$. The trajectory of the vector field \bar{l} passing through q meets Γ at a point q_- . Therefore $\omega_\Gamma(q_-) = -r_2$. In just the same way we determine a point $q_+ \in \Gamma$, for which $\omega_\Gamma(q_+) = -r_1$. This completes the proof of the assertion.

Now, in the coordinate form, equation (53) looks as follows:

$$\int_0^{\alpha_1(z)} \int_0^{\alpha_2(z)} f(\omega(x)) dx_2 dx_1 = h(z), \quad z \in I.$$

Applying the one-to-one change (50) of an unknown function f for F we arrive at the following equivalent functional equation for the function F :

$$F(\omega_\Gamma \circ \alpha(z)) - F(\omega_1 \circ \alpha(z)) - F(\omega_2 \circ \alpha(z)) = H(z), \quad (54)$$

Introduce the new variable $t = \sigma(z)$, $t \in I_T$. Remembering that, by the definition of the map ζ ,

$$\omega_j \circ \alpha = \omega_\Gamma \circ \zeta_j \circ \alpha, \quad j = 1, 2.$$

we can rewrite equation (54) in the form

$$F(t) - F(\omega_\Gamma \circ \zeta_1 \circ \omega_\Gamma^{-1}(t)) - F(\omega_\Gamma \circ \zeta_2 \circ \omega_\Gamma^{-1}(t)) = h(\sigma^{-1}(t)), \quad t \in I_T.$$

Let

$$\delta_j(t) = \omega_\Gamma \circ \zeta_j \circ \omega_\Gamma^{-1}(t), \quad t \in I_T. \quad (55)$$

Then, the last functional equation takes the form

$$F(t) - F(\delta_1(t)) - F(\delta_2(t)) = (h \circ \sigma^{-1})(t), \quad t \in I_T. \quad (56)$$

We are now going to show that *this equation is a Cauchy type functional equation*. This is an immediate corollary of the following lemma.

Lemma 19 1° Both maps δ_1 and δ_2 are nondecreasing functions;
 2° the range of δ_1 (resp. δ_2) is the closed interval $(0, r_1)$ (resp. $(-r_2, 0)$);
 3° the relation

$$\delta_1(t) + \delta_2(t) = t$$

holds for all $t \in I_T$.

Proof. 1° To simplify notation consider the case of δ_1 only. Note that, by the definition, $\omega_\Gamma^{-1}(t) = \alpha(z)$, so that $\delta_1(t) = \omega(\zeta_1(\alpha(z)))$, where $t = \sigma(z)$. By the definition of the map ζ_1 , it follows that $\delta_1(t) = \omega(\alpha_1(z), 0)$. Thus, $\delta_1'(t) = \omega_{x_1}(\alpha_1, 0)\alpha_1'(z) \geq 0$, by (52).

2° By the definition of the map $\zeta_1: \Gamma \rightarrow \Gamma$, we have

$$\delta_1(t) = \omega(\pi_1 \circ \alpha(z), 0), \quad \text{where } t = \sigma(z). \quad (57)$$

As the functions $\delta_1, \omega(x_1, 0)$ and $\alpha_1(z)$ do not decrease, the range of the map δ_1 coincides with the interval $(\omega(\alpha_1(0), 0), \omega(\alpha_1(1), 0))$, i.e. $\mathcal{R}[\delta_1] = (-r_1, 0)$.

3° To prove the assertion we use again relation (57). By linearity of the function ω , it follows that

$$\begin{aligned} \delta_1(t) + \delta_2(t) &= \omega(\alpha_1(z), 0) + \omega(0, \alpha_2(z)) \\ &= \omega(\alpha_1(z), \alpha_2(z)) = (\omega \circ \alpha)(\sigma^{-1}(t)) = t. \end{aligned}$$

This lemma makes it possible to use the result of Theorem 5 in studying the solvability properties of equation (56). By this theorem we have to consider the corresponding semigroup Φ_δ with $\delta = (\delta_1, \delta_2)$, to introduce the guiding sets

$$\mathcal{T}_j^\delta = \{t \in I_T \mid \delta_j'(t) = 0\}, \quad j = 1, 2,$$

and the corresponding sets of \mathcal{T}^δ -guided and \mathcal{T}^δ -proper orbits in I_T . Then, as is known from Theorem 5, the condition $\mathfrak{N}_\delta^{\mathcal{T}^\delta} = \emptyset$ is necessary and sufficient for equation (56) to be uniquely solvable.

Returning to the proof of Theorem 18, we conclude that to complete this proof it suffices to show that *the relations*

$$\mathfrak{N}_\delta^{\mathcal{T}^\delta} = \emptyset \quad \text{and} \quad \mathfrak{N}_\zeta^{\mathcal{T}^\zeta} = \emptyset$$

hold only simultaneously.

This follows immediately from the following lemma.

Lemma 20 *The ω -image of any (\mathcal{T}_ζ -guided, \mathcal{T}_ζ -proper) orbit in \mathcal{O}_ζ is a (\mathcal{T}_δ -guided, \mathcal{T}_δ -proper) orbit in \mathcal{O}_δ , and, conversely, any (\mathcal{T}_δ -guided, \mathcal{T}_δ -proper) orbit in \mathcal{O}_δ is an ω -image of a (\mathcal{T}_ζ -guided, \mathcal{T}_δ -proper) orbit in \mathcal{O}_ζ .*

Proof. It is sufficient to consider only two-element orbits. Let (q, \widehat{q}) be an orbit from \mathcal{O}_ζ , that is $\widehat{q} = \zeta_j(q)$ for some $j = 1, 2$. Then, by (55),

$$\omega(\widehat{q}) = \omega \circ \zeta_j(q) = (\delta_j \circ \omega)(q),$$

which means that the ordered pair $(\omega(q), \omega(\widehat{q}))$ is an orbit from \mathcal{O}_δ . If (t, \widehat{t}) is an orbit from \mathcal{O}_δ , i.e. $\widehat{t} = \delta_j(t)$ for some $j = 1, 2$, then $\omega_\Gamma^{-1}(\widehat{t}) = \zeta_j \circ \omega_\Gamma^{-1}(t)$, by (55), which means that (t, \widehat{t}) is the ω - image of an orbit $(\omega_\Gamma^{-1}(t), \omega_\Gamma^{-1}(\widehat{t}))$ from \mathcal{O}_ζ . This proves the lemma in the case of nonspecific orbits.

To prove the lemma it remains to show that the ω - images of the guided sets $(\mathcal{T}_\zeta)_j$ coincide with the guided sets $(\mathcal{T}_\delta)_j$, $j = 1, 2$. To this end, take an arbitrary point q from the guiding set $(\mathcal{T}_\zeta)_1$, for definiteness. This means that $q = \alpha(z)$ for a (single) $z \in I$ and $\alpha'_1(z) = 0$. Let $t = \omega(q)$. Then $t = \sigma(z) = (\omega \circ \alpha)(z)$ and, by (55),

$$\delta'_1(\sigma(z))\sigma'(z) = \omega_{x_1}(\alpha_1(z), 0) \cdot \alpha'_1(z).$$

As $\sigma' > 0$ and $\omega_{x_1} > 0$, the relations

$$\alpha'_1(z) = 0, \quad \text{and} \quad \delta'_1(t) = 0 \quad \text{for } t = \sigma(z)$$

are equivalent. This completes the proof of the lemma and, moreover, the proof of Theorem 18 in the part of the solvability of equation (53). As for the boundedness of the inverse operator \mathcal{A}^{-1} , it suffices to refer to the proof of the analogous assertion in Theorem 17. The necessity of the condition $\mathfrak{N}_\zeta^T = \emptyset$ is actually proved in proving the correspondent part of Theorem 6.

4.3 On solvability of boundary problems for hyperbolic differential equations

As another application of the results relating to a Cauchy type functional equation, we consider now certain boundary problems for an arbitrary homogeneous strictly hyperbolic differential operator $P(\partial_x, \partial_y)$ of the third order in some bounded domains in the (x, y) - plane \mathbb{R}^2 . Any such operator can be represented in the form

$$P = (\partial/\partial \bar{l}_1)(\partial/\partial \bar{l}_2)(\partial/\partial \bar{l}_3)$$

where \bar{l}_1, \bar{l}_2 and \bar{l}_3 are arbitrary mutually transversal smooth vector fields in \mathbb{R}^2 . In the framework of the classical theory of partial differential equations no boundary problems for such operators have been considered, although some particular equations of this type arisen earlier in connection with some problems relating to linearized flow of a chemically relaxing gas([Ch]).

Thus, the general theory of functional equations, its language and its methods, become an essential technical tool when solving boundary problems for partial differential equations, and hence, to be applied in different practical disciplines.

4.3.1 Formulation of boundary problems

Let \bar{l}_1, \bar{l}_2 and \bar{l}_3 be an above triple of smooth nonsingular mutually transversal vector fields in a disk $B \subset \mathbb{R}^2$. Take any point $O \in B$ and consider six semitrajectories of these vector fields, beginning at O . All this rays are nothing but characteristics of the operator P . Take an arbitrary triple of neighboring rays l_1, l_2 and l_3 lying on trajectories of the vector fields \bar{l}_1, \bar{l}_2 and \bar{l}_3 , respectively, and choose arbitrary points $A_1 \in l_1, A_2 \in l_2$, and $A_3 \in l_3$. Assume, without loss of generality, that the ray l_2 lies between l_1 and l_3 (i.e. $\bar{l}_2 = \lambda_1 \bar{l}_1 + \lambda_3 \bar{l}_3$ with $\lambda_1 > 0, \lambda_3 > 0$).

We associate with the differential operator P canonically two different bounded domains D_1 and D_2 , and they are just domains where different boundary problems for the equation $Pu = f$ will be considered.

The domain D_1 is determined as the curvilinear triangle A_1OA_3 whose third side $\Gamma = A_1A_3$ is an arbitrary nonsingular C^2 - curve.

The domain D_2 is determined as the characteristic parallelogram A_1OA_2O' , whose sides A_1O' and A_2O' are trajectories of the vector fields \bar{l}_1 and \bar{l}_2 , respectively.

4.3.2 First partly characteristic problem

This is the problem we associate with domains D_1 . In addition to the above description, this domain is assumed to satisfy the following topological conditions:

- (i) the domain D_1 is \bar{l}_1 - and \bar{l}_3 - convex²;
- (ii) the projections

$$\begin{aligned} \pi_1: \bar{D}_1 &\rightarrow l_1 \text{ along the vector field } \bar{l}_3, \\ \pi_2: \bar{D}_1 &\rightarrow \Gamma \text{ along the vector field } \bar{l}_2, \\ \pi_3: \bar{D}_1 &\rightarrow l_3 \text{ along the vector field } \bar{l}_1, \end{aligned}$$

satisfy the hypotheses

$$\pi_1 D_1 = OA_1 \quad \text{and} \quad \pi_3 D_1 = OA_3;$$

- (iii) the curve Γ is transversal to the vector field \bar{l}_2 .

The first partly characteristic problem for the operator P in question is formulated in the following way.

Given an arbitrary function f in \bar{D}_1 and g on $\partial D_1 = OA_1 \cup OA_3 \cup \Gamma$, find a function u in \bar{D}_1 such that

$$P(\partial)u = f \quad \text{in } D_1, \quad u = g \quad \text{on } \partial D_1. \tag{58}$$

²For a given smooth vector field \bar{l} in \mathbb{R}^n , a set $\omega \subset \mathbb{R}^n$ is said to be \bar{l} -convex if, for arbitrary points p and q in ω lying on a trajectory of \bar{l} , the full trajectory (p, q) belongs to ω .

We introduce two maps of Γ into itself

$$\delta_1 = \pi_2 \circ \pi_1 \quad \text{and} \quad \delta_2 = \pi_2 \circ \pi_3,$$

and consider the semigroup Φ_δ generated by δ_1 and δ_2 . Let \mathcal{O}_δ be the set of orbits in Γ corresponding to Φ_δ . Denote by $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ the guiding set in Γ , consisting of characteristic points in Γ with respect to differential operator $P(\partial)$. In other words

$$q \in \mathcal{T}_j, \quad \text{if} \quad T_q(\Gamma) \ni \bar{l}_j(q), \quad j = 1, 2,$$

where $T_q(\Gamma)$ stands for the tangent space of Γ at the point q . Now everything is ready to introduce the set $\mathfrak{N}_\delta^{\mathcal{T}}$ as the collection of all \mathcal{T} -proper periodic orbits lying in the guiding set \mathcal{T} .

Theorem 21 *Assume that the vector fields \bar{l}_1, \bar{l}_2 and \bar{l}_3 are constant and the characteristic sets \mathcal{T}_1 and \mathcal{T}_2 satisfy an ordering condition formulated before Theorem 18. Let Γ be a $(2+k)$ -times differentiable curve. Then problem (58) has a unique solution $u \in C^{2+k}(\bar{D}_1)$ for arbitrary functions $f \in C^k(\bar{D}_1)$ and $g \in C^{2+k}(\partial\bar{D}_1)$ if and only if $\mathfrak{N}_\delta^{\mathcal{T}} = \emptyset$.*

Proof. For the sake of brevity we assume the operator $P(\partial)$ to be homogeneous and restrict ourselves to the case $f = 0$. It is easily to see that a linear change of variables reduces problem (58) to the problem

$$(r_1\partial_x + r_2\partial_y)\partial_x\partial_y u(x, y) = 0 \quad \text{in } D_1, \quad u = h \quad \text{on } \partial D_1. \quad (59)$$

The boundary ∂D now consists of three parts,

$$\Gamma_1 = \{(x, y) \mid y = 0, \quad 0 \leq x \leq 1\}, \quad \Gamma_2 = \{(x, y) \mid x = 0, \quad 0 \leq y \leq 1\},$$

$$\Gamma = \{(x, y) \mid x = \alpha_1(t), \quad y = \alpha_2(t); \quad -1 \leq t \leq 1\},$$

where $\alpha_1(-1) = 0, \quad \alpha_1(1) = 1, \quad \alpha_2(-1) = -1, \quad \alpha_2(1) = 0$. Let $h = h_1(x)$ on Γ_1 , $h = h_2(y)$ on Γ_2 and $h = h_3(x, y)$ on Γ . In virtue of continuity of the function h on Γ , the following compatibility conditions are fulfilled:

$$h_1(0) = h_2(0), \quad h_1(1) = h_3(1, 0), \quad h_2(1) = h_3(0, 1). \quad (60)$$

Using the postulated properties of the domain D it is easily verified that the function

$$u(x, y) = \int_0^x \left(\int_0^y F(r_2s - r_1t) dt \right) ds + h_1(x) + h_2(y) - h_1(0), \quad 0 \leq x, y \leq 1, \quad (61)$$

satisfies the equation in (59) and the condition $u = h$ on the part $\Gamma_1 \cup \Gamma_2$ of the boundary, for any function $F \in C^1(-r_1, r_2)$. The necessity to satisfy the condition $u = h_3$ on Γ leads to the integral equation

$$\int_0^{\alpha_1(t)} \left(\int_0^{\alpha_2(t)} F(r_2x - r_1y) dy \right) dx = H(t), \quad t \in I. \quad (62)$$

Here, as in Subsec. 4.2.2, the relations $x = \alpha_1(t)$, $y = \alpha_2(t)$, $t \in I$, describe the curve Γ in a parametric form, and $H(t) = -h_1(\alpha_1(t)) - h_2(\alpha_2(t)) + h_3(\alpha_1(t), \alpha_2(t)) + h_1(0)$. What is important is that the function H , generated by *any* k times piece-wise differentiable functions h , belongs to the space $\mathcal{H}(I) := (C^2 \cap C_0)(I)$. This follows from (60). The inverse statement is also true: a function u which is defined by formula (61) with F being a solution to equation (62) is a solution of the problem (59). Thus, the latter problem turns out to be equivalent to the integral equation (62), which is nothing but equation (49). The unique solvability of the problem (59) for any function h of the required form under hypothesis $\mathfrak{N}_\zeta^T = \emptyset$ follows directly from Theorem 18.

To illustrate the result obtained let us consider the problem (59) in the domains D_1 , D_2 and D_3 (see below). On these figures the only points p and

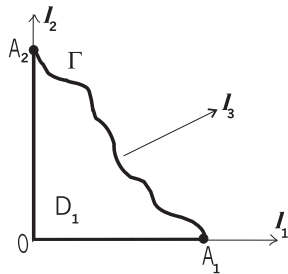


Fig. 3

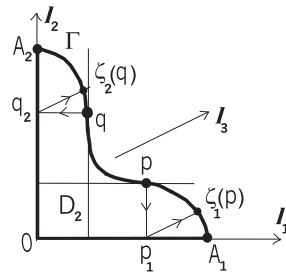


Fig. 4

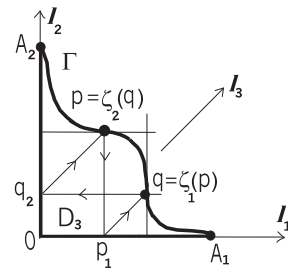


Fig. 5

q are characteristic ones. It is easily verified that $\mathfrak{N}_\zeta^T = \emptyset$ in the case of the domains D_1 and D_2 and hence, the problem (59) is well posed in D_1 and in D_2 . On the other hand, the set \mathfrak{N}_ζ^T in the case of D_3 contains the (unique) characteristic T -proper periodic orbit (p, q, p) . That is why Theorem 21 is nonapplicable here.

4.3.3 Second partly characteristic problem

Let D_2 be an arbitrary domain associated with the above operator $P(\partial)$. Denote by $\Gamma = OO'$ a quasi-diagonal of the curvilinear parallelogram A_1OA_2O' which is supposed to be a nonsingular C^2 -curve satisfying the following topological conditions:

- (j) the curve Γ is \bar{l}_1 - and \bar{l}_2 -convex;
- (jj) the curve Γ is transversal to the vector field \bar{l}_3 everywhere and to the vector fields \bar{l}_1 and \bar{l}_2 at the point O ;
- (jjj) the curve Γ intersects the trajectories OA_1 and OA_2 only at the point O .

The second partly characteristic problem for the operator $P(\partial)$ in the domain D_2 is formulated as follows:

Given arbitrary functions f in \bar{D}_2 and g on $\mathcal{M} = OA_1 \cup OA_2 \cup \Gamma$, find a function u , in \bar{D}_2 such that

$$P(\partial)u = f \quad \text{in } D_2, \quad u = g \quad \text{on } \mathcal{M}. \quad (63)$$

Proceeding with the formulation of the result related to problem (63) we first establish some compatibility conditions which must hold for the function g . Denote by g_1, g_2 and g_3 the restrictions of the function g to the curves OA_1, OA_2 and Γ , respectively. If $\{x = x_j(t), y = y_j(t); 0 \leq t \leq 1\}$, $j = 1, 2, 3$, are some smooth parametric representation of these curves and $O = (x_j(0), y_j(0))$ for all j , then

$$u(x_j(t), y_j(t)) = g_j(t), \quad 0 \leq t \leq 1$$

for $j = 1, 2, 3$. Denote by $\bar{\tau}_\Gamma$ the unit tangent vector of Γ at the point O . By virtue of transversality of the vectors \bar{l}_1 and \bar{l}_2 , there are positive constants λ_1 and λ_2 such that $\bar{\tau}_\Gamma = \lambda_1 \bar{l}_1 + \lambda_2 \bar{l}_2$. This immediately leads to the first compatibility condition

$$g'_3(O) = \lambda_1 g'_1(O) + \lambda_2 g'_2(O). \quad (64)$$

The second one

$$g_1(O) = g_2(O) = g_3(O) = u(O) \quad (65)$$

is a consequence of the coincidence of the curves OA_1, OA_2 and Γ at the point O .

Theorem 22 *Let $P(\partial)$ be the above differential operator with constant vector fields \bar{l}_j , $1 \leq j \leq 3$, defined in the domain D_2 . Then, for arbitrary functions $f \in C(\bar{D}_2)$ and $g \in C^2(\mathcal{M})$ satisfying conditions (64) and (65), there is a unique solution $u \in C^2(\bar{D}_1)$ of problem (63), and the inverse operator $(f, g) \mapsto u$ is bounded. If $f \in C^k(\bar{D}_1)$ and $g \in C^{2+k}(\mathcal{M})$, then $u \in C^{2+k}(\bar{D}_1)$, $k = 1, 2, \dots$*

The relation $g \in C^m(\mathcal{M})$ means that the three functions g_1, g_2 and g_3 are m times continuously differentiable on their domains.

Proof. We turn our attention to the main details of the proof. To begin with let $f = 0$. Then problem (63) can be reduced to the problem

$$(\partial_x - \partial_y)\partial_x\partial_y u = 0 \quad \text{in } D_2 \quad u = g \quad \text{on } \mathcal{M} \quad (66)$$

where $D_2 = \{(x, y) \mid 0 \leq x \leq X, 0 \leq y \leq Y\}$, and the set \mathcal{M} consist of parts,

$$\begin{aligned} \mathcal{M}_1 &= \{(x, y) \mid 0 \leq x \leq X, y = 0\}, \quad \mathcal{M}_2 = \{(x, y) \mid x = 0, 0 \leq y \leq Y\}, \\ \Gamma &= \{(x, y) \mid x = \delta_1(t), y = \delta_2(t), 0 \leq t \leq 1\}. \end{aligned}$$

Let

$$g = G_1(x) \text{ on } \mathcal{M}_1, \quad g = G_2(y) \text{ on } \mathcal{M}_3, \quad g = G_3(x, y) \text{ on } \Gamma,$$

and let $\varphi(t) := G_3(\delta_1(t), \delta_2(t))$, $0 \leq t \leq 1$. Then the compatibility conditions (64) and (65) become

$$\varphi'(0) = \delta_1'(0)G_1'(0) + \delta_2'(0)G_2'(0) \quad \text{and} \quad G_1(0) = G_2(0) = \varphi(0). \quad (67)$$

It is obvious that any solution u of problem (66) can be represented in D_2 in the form

$$u(x, y) = \int_0^x \left(\int_0^y F(s+t) dt \right) ds + G_1(x) + G_2(y) - G_1(0).$$

Here F is a continuous function in the closed interval $\widehat{I} = (0, X + Y)$ satisfying the integral equation

$$\int_0^{\delta_1(t)} \left(\int_0^{\delta_2(t)} F(x+y) dy \right) dx = H(t), \quad 0 \leq t \leq 1, \quad (68)$$

where

$$H(t) = -G_1(\delta_1(t)) - G_2(\delta_2(t)) + G_3(\delta_1(t), \delta_2(t)) + G_1(0).$$

What is important is that the function $H(t)$ belongs to the space $\mathcal{H}(\Gamma)$ (see Subsec. 4.2.1), since, by (67), it satisfies the conditions $H(0) = 0$ and $H'(0) = 0$. But then equation (68) coincides with equation (49) whose solvability has been demonstrated in the very end of the proof of Theorem 17. This completes the proof of Theorem 22 when $f = 0$. In the case of an arbitrary function $f \in C^k(\overline{D}_2)$ what it remains to do is to find a particular solution $u \in C^{2+k}(\overline{D}_1)$ of the differential equation

$$(\partial_x - \partial_y)\partial_x\partial_y u = f, \quad (x, y) \in D_2.$$

The reader can verify directly that the function

$$u(x, y) = \int_0^x \left(\int_0^y \left(\int_{\gamma(s+t)}^{s-t} f \left(\frac{s+t+z}{2}, \frac{s+t-z}{2} \right) dz \right) dt \right) ds,$$

where $\gamma = (\delta_1 - \delta_2) \circ (\delta_1 + \delta_2)^{-1}$, solves this equation, and this completes the proof.

CONCLUDING REMARKS

As mentioned in Introduction, the study of the Cauchy type functional equations is mostly motivated by the applications of the results obtained in

such fields of analysis very far from functional equations as Integral geometry and Partial differential equations (see Sect.4). For this reason, we have restricted ourselves to studying the solvability problems in the framework of the Z - and \mathcal{P} -configurations and sufficiently smooth functions. Another functional equation

$$F(t) - a_1(t)F(\delta_1(t)) - a_2(t)F(\delta_2(t)) = h(t) \quad (69)$$

studied in details in [P1], episodically arising in this work (see (16) and (30), for example), is intimately connected with the Cauchy type functional equations as the solvability properties of (69) represent a very important intermediate step when studying the latter equations. The homogeneous equations (69) with the functions δ_1 and δ_2 forming a configuration quite different of the Z - and \mathcal{P} -configurations were already studied by several authors (see [KCG, Sec.6]) under more free smoothness conditions than in the present paper. The results obtained play a very important role when looking for densities of measures invariant under some specific mappings of a compact interval. For this reason, the solvability of nonhomogeneous equations (69) in the case under discussion will without any doubt find some interesting applications in the theory of real functions.

Another problem which is of the great interest and importance in applications is weakening smoothness restrictions in Theorem 6 related to the \mathcal{P} -configuration. It is hard to believe that the C^2 - smoothness is necessary when solving linear functional equations. But at this moment the author has no idea about a possibility to avoid such a condition.

The dynamical systems and all accompanying notions including the orbits we apply throughout this work were introduced for the first time in [P1] and carefully studied in [P5]. These notions have nothing in common with the classical dynamical systems and the orbits, first of all, because the semigroups of maps generating the new dynamical systems are generated by two maps but not one as in the classical situation. This leads to quite different problems inside the new theory but also gives a new power technical tools when dealing with the functional equations in question. In the classical case each point of the corresponding space generates only one orbit $x, \varphi(x), \varphi(\varphi(x)), \dots$, whereas in our case the same point x generates infinitely many orbits. One of the most difficult and interesting problems here is to describe the attractors of the new dynamical systems i.e. subsets in the space that can be reached by orbits (sometimes very specific ones, see proofs of Theorem 5, Theorem 6 and Theorem 18) beginning at a fixed point x .

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References

- [Ch] C. Chester, *Techniques in Partial differential equations*, Mc.Graw - Hill Book Company, New York, 1972.
- [KCG] M.Kuczma, B.Choczewski and R.Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its applications 32, Cambridge University Press, Cambridge, 1990.
- [GGV] I.Gelfand, M.Graev and N.Vilenkin, *Integral Geometry and Representation Theory*, Academic Press, New York, 1966.
- [P1] B.Paneah, *On Solvability of Functional Equations Relating to Dynamical System with Two Generators*, Functional Analysis and Its Applications, v.37, No.1, pp.46-60, 2003.
- [P2] B.Paneah, *Dynamical Approach to Some Problems in Integral Geometry*, Trans.Amer. Math.Soc., v.356, No.7, pp.2757-2780, 2003.
- [P3] B.Paneah, *On a Problem in Integral Geometry connected to the Dirichlet Problem for Hyperbolic Equations*, International Mathematical Research Notes (IMRN), (1997), No.5, 213 - 222.
- [P4] B.Paneah, *Dynamic Methods in the General Theory of Cauchy Type Functional Equations*, Contemporary Mathematics, v.364, pp.205 - 223, 2004.
- [P5] B.Paneah, *Noncommutative Dynamical Systems with Two Generators and Their Applications in Analysis*, Discret and Continuous Dynamical Systems, v.9, No.6, pp.1411-1420, 2003.

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