GERMS AND MULTIGERMS OF LEGENDRIAN CURVES

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ABSTRACT. The first main theorem states that in the real analytic category any two diffeomorphic multigerms of Legendrian curves in a fixed contact 3-space are contactomorphic. The same holds in the C^∞ category beyond a certain case of infinite codimension. One of corollaries (and the most difficult part) of this theorem is the equivalence of the following properties of a Legendrian curve multigerm γ: (i) γ does not determine an orientation of $\mathbb{R}^3$ in the sense that there are contact structures defining different orientations with respect to which γ is Legendrian; (ii) the image of γ admits an orientation-reversing symmetry. The second main theorem states the equivalence of (i), (ii), (iii), and (iv), where (iii) is the property that γ is planar, i.e. its image is contained in a non-singular surface, and (iv) is the property that γ is quasi-homogeneous. The paper also contains application of these results to contact classification of germs and multigerms of Legendrian curves.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Multigerms of Legendrian curves. Reduction theorem.** By a *multigerm* of a curve in $\mathbb{R}^3$ we mean the collection $\gamma = (\gamma_1, ..., \gamma_m)$ of germs at $0 \in \mathbb{R}$ of maps $(\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ defined up to permutation of the components γ_i. A component γ_i might be non-singular ($\gamma_i'(0) \neq 0$) or singular ($\gamma_i'(0) = 0$). The image of a multigerm is the union of the images of its components. Two multigerms of curves in $\mathbb{R}^3$ are called *diffeomorphic* if their images can be brought one to the other by a local diffeomorphism $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$.

Given a contact 1-form α on $\mathbb{R}^3$ we denote by (α) the corresponding contact structure - the field of kernels of α. A local contactomorphism of the contact space $(\mathbb{R}^3, (\alpha))$ is a local diffeomorphism $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ preserving the contact structure, i.e. $\Phi^*\alpha = Q\alpha$, where Q is a function germ, $Q(0) \neq 0$. Two multigerms of curves in a contact space $(\mathbb{R}^3, (\alpha))$ are called *contactomorphic* if their images can be brought one to the other by a local contactomorphism of this contact space.

A multigerm $\gamma = (\gamma_1, \ldots, \gamma_m)$ in a contact space $(\mathbb{R}^3, (\alpha))$ is called *integral* or *Legendrian* if $\gamma_i^*\alpha \equiv 0, \ i = 1, \ldots m$.

Theorem 1.1. *In the real analytic category any two diffeomorphic Legendrian multigerms in a fixed contact 3-space are contactomorphic. The same holds in the C^∞ category provided that the multigerms do not belong to a certain set E_∞ of infinite codimension defined below.*

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Definition of the set E_∞. By E_∞ in Theorem 1.1 we denote the set of C^∞ multigerms of curves in $\mathbb{R}^3$ containing either two non-singular components with infinite order of tangency [1] or a singular component whose Taylor series can be reduced by a non-degenerate change of coordinates to a series of the form $(f(t), 0, 0)$.

Remarks.

1. Theorem 1.1 does not hold for Legendrian multigerms of curves in $\mathbb{R}^{2n+1}$ if $n \geq 2$. In this case there are invariants with respect to the group of contactmorphisms beyond those with respect to the group of diffeomorphisms, see [1] and [16].

2. Theorem 1.1 also holds in the holomorphic category (one has to replace $\mathbb{R}^3$ by $\mathbb{C}^3$). In the holomorphic category for the case of multigerms with one component it was proved by G. Ishikawa in [11] using a general theory of Legendrian maps developed in [12]. The holomorphic category is much simpler than the real analytic category because, as we will explain below, the main difficulty in proving Theorem 1.1 is related to orientation of $\mathbb{R}^3$ defined by contact structures. (Recall that any contact structure (α) on $\mathbb{R}^3$ defines an orientation of $\mathbb{R}^3$ since the sign of the volume form $\alpha \wedge d\alpha$ remains the same when multiplying α by a non-vanishing function). The proof of Theorem 1.1 in the holomorphic category does not require the most difficult part of the proof in the real analytic category, see section 3.5.

The starting point for Theorem 1.1 is a result on structural stability of germs of singular Legendrian submanifolds of a contact manifold (of any odd dimension) proved by A. Givental' in [10]. This result implies Theorem 1.1 *provided* that one of the Legendrian multigerms is fixed and the other can be brought to the first one by a local diffeomorphism ϕ with the linear approximation sufficiently close to id.

The possibility to refuse from the assumption that the linear approximation of ϕ is close to id is, from the first look, surprising. In fact, take an orientation-reversing diffeomorphism Ψ and a Legendrian multigerm γ. Assume that the multigerm $\tilde{\gamma} = \Psi \circ \gamma$ is also Legendrian with respect to the same contact structure. Since any contactomorphism preserves orientation of $\mathbb{R}^3$ then the existence of a contactomorphism Φ bringing the image of $\tilde{\gamma}$ to the image of γ implies that the image of γ admits an orientation-reversing symmetry (the composition $\Phi \circ \Psi$). On the other hand, there are Legendrian multigerms (even with one component) whose image does not admit an orientation-reversing symmetry, see examples in Section 2.

1.2. **Legendrian multigerms and orientation of the space.** The given above counterexample is wrong because the assumption that the image of a Legendrian multigerm γ does not admit an orientation-reversing symmetry cannot hold simultaneously with the assumption that $\Psi \circ \gamma$ is Legendrian with respect to the same contact structure for some orientation-reversing diffeomorphism Ψ. In order to explain this in simpler terms we need the following notation and definition that will be used throughout the paper.

[1]The tangency of two non-singular components γ_1, γ_2 means that the vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ are proportional. In this case in suitable local coordinates the images of γ_1 and γ_2 are described by the equations $y = z = 0$ and $y = g_1(x), z = g_2(x)$, where $g_1(x), g_2(x) = o(x)$. The order of tangency is infinite if the function germs $g_1(x)$ and $g_2(x)$ have zero Taylor series.

Notation. By $Leg(\mathbb{R}^3)$ we denote the set of multigerms γ of curves in $\mathbb{R}^3$ that are Legendrian with respect to some contact structure (α) (depending on γ).

Definition. Let $\gamma \in Leg(\mathbb{R}^3)$. We will say that γ determines orientation of $\mathbb{R}^3$ if all contact structures with respect to which γ is Legendrian define the same orientation.

The construction given in the end of Section 1.1 is impossible because one has the following statement:

(*) *Assume that a multigerm $\gamma \in Leg(\mathbb{R}^3)$ does not determine orientation of $\mathbb{R}^3$. Then in the real analytic category the image of γ admits an orientation-reversing symmetry. If $\gamma \notin E_\infty$ then the same holds in the C^∞ category.*

This statement is a corollary of Theorem 1.1. On the other hand, (*) is the most difficult part of this Theorem, modulo well-developed techniques.

1.3. **Four equivalent properties of Legendrian multigerms.** Theorem 1.2 below states the equivalence of four properties of any multigerm $\gamma \in Leg(\mathbb{R}^3)$ - the two properties given in (*), the planarity, and the quasi-homogeneity.

Definition. A multigerm γ of a curve in $\mathbb{R}^3$ is planar if its image belongs to a *non-singular* surface (2-dimensional submanifold) of $\mathbb{R}^3$.

Definition. A multigerm of a curve in $\mathbb{R}^3$ is called quasi-homogeneous if there exist $\lambda_1, \lambda_2, \lambda_3 > 0$ (weights) such that γ is RL-equivalent [2] to a multigerm whose components have the form

$$x = at^{r\lambda_1}, \;\; y = bt^{r\lambda_2}, \;\; z = ct^{r\lambda_3}, \quad a, b, c, r \in \mathbb{R},$$

where the local coordinate system (x, y, z) and the weights $\lambda_1, \lambda_2, \lambda_3 > 0$ are the same for all components and the numbers a, b, c, r depend on a component.

Theorem 1.2. *In the real analytic category (respectively the C^∞ category) the following statements are equivalent for any multigerm $\gamma \in Leg(\mathbb{R}^3)$ (respectively for any multigerm $\gamma \in Leg(\mathbb{R}^3)$ beyond the set E_∞):*
(i) γ does not determine orientation of $\mathbb{R}^3$;
(ii) the image of γ admits an orientation-reversing symmetry;
(iii) γ is planar;
(iv) γ is quasi-homogeneous.

The implications $(iii) \implies (ii) \implies (i)$ are trivial, but the other implications are not. Note that each of the implications $(ii) \implies (iii)$, $(iii) \implies (ii)$, $(iii) \implies (iv)$ and $(iv) \implies (iii)$ is violated by certain multigerms (even with one component) of space curves. By Theorem 1.2 such multigerms do not belong to $Leg(\mathbb{R}^3)$, i.e. they are not Legendrian with respect to any contact structure on $\mathbb{R}^3$.

[2] Two multigerms $\gamma = (\gamma_1, ..., \gamma_m)$ and $\tilde{\gamma} = (\tilde{\gamma}_1, ..., \tilde{\gamma}_m)$ are called RL-equivalent if they can be brought one to the other by a local diffeomorphism $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ (change of coordinates) and local diffeomorphisms $\phi_i : (\mathbb{R}, 0) \to (\mathbb{R}, 0), i = 1, ..., m$ (independent reparameterization of the components), i.e up to permutation of the components one has $\tilde{\gamma}_i = \Phi \circ \gamma_i \circ \phi_i, \; i = 1, ..., m$.

Theorem 1.3. (Corollary of Theorem 1.2). *In the real analytic category (respectively the C^∞ category) any multigerm $\gamma \in Leg(\mathbb{R}^3)$ (respectively any multigerm $\gamma \in Leg(\mathbb{R}^3)$ beyond the set E_∞) satisfies one of the following two conditions:*

(a) γ is planar and quasi-homogeneous;

(b) γ is neither planar nor quasi-homogeneous.

It is clear that any planar quasi-homogeneous multigerm belongs to $Leg(\mathbb{R}^3)$. Within multigerms that are neither planar nor quasi-homogeneous there are multigerms both in and beyond the class $Leg(\mathbb{R}^3)$, see Example 2.5 and Sections 6-8.

1.4. **Plan for the paper.** In Section 2 we give several simple examples illustrating Theorems 1.2, 1.3 and the notion of the determinacy of orientation by a Legendrian multigerm. Theorem 1.1 is proved in Section 3 using the well known techniques (the homotopy method) and the most difficult implication $(i) \implies (ii)$ in Theorem 1.2. The proof of Theorem 1.2 is rather long - it is contained in Section 4.

In sections 5-9 we present applications of Theorems 1.1 and 1.2 to contact classification of Legendrian multigerms in a fixed contact space. [3]

The simplest application holds for multigerms of classes $\mathbf{A}_i$, section 5. The image of multigerms of the class $\mathbf{A}_i$ is diffeomorphic to the singular submanifold of $\mathbb{R}^3$ given by the equation $y^2 = x^{i+1} = z = 0$. The class $\mathbf{A}_2 \cup \mathbf{A}_4 \cup \cdots$ consists of singular curves with non-zero 2-jet whose Taylor series in not RL-equivalent to $(t^2, 0, 0)$. The class $\mathbf{A}_1 \cup \mathbf{A}_3 \cup \cdots$ consists of multigerms with two components; the components are non-singular curves with a finite order of tangency at the origin. It is easy to see that $\mathbf{A}_i \cap Leg(\mathbb{R}^3) = \mathbf{A}_i$, i.e. any multigerm of the class $\mathbf{A}_i$ is Legendrian with respect to some contact structure. This makes application of Theorem 1.1 very simple - by this theorem all Legendrian multigerms of the class $\mathbf{A}_i$ in a fixed contact space are contactomorphic.

Section 6 is devoted to contact classification of $\mathbf{E}$ singularities - germs of singular curves with zero 2-jet, non-zero 3-jet, and the Taylor series not RL-equivalent to $(t^3, 0, 0)$. The classification of such singularities with respect to diffeomorphisms is obtained in [9], see also [2]. It is described, in the notations in [2], by normal forms $E_{6k,p,i}$ and $E_{6k+2,p,i}$, where $k \geq 1, 0 \leq p \leq k, 0 \leq i \leq min(k-1, p)$. We prove that $E_{6k,p,i}, E_{6k+2,p,i} \in Leg(\mathbb{R}^3)$ if and only if $p = i + 1$. Within the class $\mathbf{E} \cap Leg(\mathbb{R}^3)$ there are planar quasi-homogeneous multigerms (the case $p = k, i = k-1$) and multigerms that are neither planar nor quasi-homogeneous (the case $p = i+1 < k$). Theorem 1.1 implies that in a fixed contact space all multigerms diffeomorphic to $E_{6k,i+1,i}$ (respectively $E_{6k+2,i+1,i}$) are contactomorphic.

In Section 7 we obtain few more classification results for germs of Legendrian curves (multigerms with one component); this allows us to determine and classify all *contact-simple* germs of Legendrian curves. [4] We prove that a germ γ of an Legendrian curve is contact-simple within Legendrian germs (i.e. any Legendrian germ

[3] Theorems 1.1 and 1.2 also can be applied for classification of germs of Legendrian curves in a 4-dimensional Engel manifold and its generalization - an n-dimensional "monster" manifold constructed in [14] using expressed in modern terms E. Cartan prolongation procedure. We hope that this application also leads to a number of results relating singularities of plane and space curves with singularities of Cartan-Goursat flags, see [14]. We also hope that there are applications in the theory of implicit ODE's and in the theory of Legendrian knots.

[4] Results of Section 7 were obtained in the author's work [17], by a different method, using certain pre-normal form for Legendrian curves.

sufficiently close to γ is contactomorphic to one of a finite number of fixed Legendrian germs) if and only if γ is simple as a space curve, with respect to the group of diffeomorphisms (i.e. any space curve germ sufficiently close to γ is diffeomorphic to one of a finite number of fixed space curve germs). Therefore our theorem on determination of contact-simple germs of Legendrian curves is the same as the theorem in [9] on determination of simple germs of space curves. Nevertheless, the classification of contact-simple germs of Legendrian curves differs from the classification of simple space curves in [9] because there are many simple singularities of space curves that are not Legendrian with respect to any contact structure, in particular quasi-homogeneous non-planar curves and planar non-quasi-homogeneous curves which do not belong to $Leg(\mathbb{R}^3)$ by Theorem 1.3. On the other hand, we prove that there is a 1-1 correspondence between contact-simple singularities of Legendrian curves in $\mathbb{R}^3$ and simple singularities of curves in $\mathbb{R}^2$, determined and classified in [8]. This correspondence is not canonical - it holds in a local coordinate system in which the contact structure is described by the 1-form $dz - ydx$ and requires a (non-canonical) projection $(x, y, z) \to (x, y)$.

In section 8 we classify, with respect to contactomorphisms, Legendrian multigerms (γ_1, γ_2) such that γ_1 is a curve with a cusp singularity (a curve diffeomorphic to $(t^2, t^{2k+1}, 0))$ and γ_2 is a non-singular curve. We denote this class by $(\mathbf{A}_{2k}, l)$. We use the classification of multigerms of this class with respect to the group of diffeomorphism obtained in [13]. At first we consider the case $k = 1$. Using Theorem 1.2 we prove that a multigerm in $(\mathbf{A}_2, l)$ is Legendrian with respect to some contact structure if and only if it is is planar and quasi-homogeneous (the class $(\mathbf{A}_2, l)$ has singularities violating these properties). Theorem 1.1 implies that in a fixed contact space any Legendrian multigerm $(\gamma_1, \gamma_2) \in (\mathbf{A}_2, l)$ is contactomorphic to one of two normal forms without parameters. These two normal forms correspond to the cases that the non-singular component γ_2 is not tangent (respectively tangent) to the limit tangent line $l(\gamma_1)$ to the image of the component γ_1.

The contact classification of multigerms of the class $(\mathbf{A}_{2k}, l)$ is a bit more involved if $k \geq 2$. In this case $(\mathbf{A}_{2k}, l)$, but for $k \geq 2$ the class $(\mathbf{A}_{2k}, l) \cap Leg(\mathbb{R}^3)$ contains certain singularities that are neither planar nor quasi-homogeneous. The class $(\mathbf{A}_{2k}, l) \cap Leg(\mathbb{R}^3)$ can be completely described using Theorem 1.2. Using Theorem 1.1 we prove that in a fixed contact space any multigerm $(\gamma_1, \gamma_2) \in (\mathbf{A}_{2k}, l)$ is contactomorphic to one and only one of $2k$ normal forms without parameters.

In Section 9 we use Theorems 1.1 - 1.3 in order to obtain a contact classification of Legendrian multigerms with three non-singular components.

In the Appendix we discuss an alternative way of contact classification of Legendrian multigerms in the fixed contact space $(\mathbb{R}^3, (dz - ydx))$, using the projections $\pi_{x,y} : (x, y, z) \to (x, y)$ and $\pi_{x,z} : (x, y, z) \to (x, z)$. Let $\gamma, \tilde{\gamma}$ be Legendrian multigerms. We prove that γ and $\tilde{\gamma}$ are contactomorphic if one of the following holds: (a) the projections $\pi_{x,z}\gamma$ and $\pi_{x,z}\tilde{\gamma}$ are diffeomorphic (via any local diffeomorphism) provided that these projections do not contain components with the zero Taylor series; (b) the projections $\pi_{x,y}\tilde{\gamma}$ are diffeomorphic via a local diffeomorphism preserving the volume form $dx \wedge dy$ up to multiplication by a *number*.

In view of these sufficient conditions and results in [17] and Section 10 we give example showing that in general the contact classification of germs and multigerms of Legendrian curves in $\mathbb{R}^3$ cannot be reduced to a classification of plane curves. Namely, we give examples illustrating the following statements:

1. If two Legendrian multigerms γ and $\tilde{\gamma}$ in the contact space $(\mathbb{R}^3, (dz - ydx))$ are contactomorphic then (a) the projections $\pi_{x,z}\gamma$ and $\pi_{x,z}\tilde{\gamma}$ do not need to be diffeomorphic; (b) the projections $\pi_{x,y}\gamma$ and $\pi_{x,y}\tilde{\gamma}$ do not need to be diffeomorphic;
2. If two Legendrian multigerms γ and $\tilde{\gamma}$ in the contact space $(\mathbb{R}^3, (dz - ydx))$ have diffeomorphic projections $\pi_{x,y}\gamma$ and $\pi_{x,y}\tilde{\gamma}$ then γ and $\tilde{\gamma}$ do not need to be diffeomorphic (and consequently contactomorphic).

Examples illustrating these statements for the projection $\pi_{x,z}$ are very simple, they hold already for **E** singularities. In the case of the projection $\pi_{x,y}$ the singularities in the examples are deeply degenerate. This is in correspondence with results in [17] and Section 10.

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2. Examples.

The following examples illustrate Theorems 1.2 and 1.3 and the notion of determinacy of orientation of the space by a Legendrain multigerm.

Example 2.1. Consider a planar quasi-homogeneous multigerm of the form

$$\gamma = (\gamma_1, ..., \gamma_m), \quad \gamma_i : \ x = a_i t^q, \ y = b_i t^p, \ z = 0,$$

where $a_i, b_i \in \mathbb{R}$. The multigerm γ is Legendrain with respect to any contact structure of the form $(dz + \lambda_1 ydx + \lambda_2 xdy)$ with λ_1, λ_2 satisfying the relation $q\lambda_1 + p\lambda_2 = 0$. Within such contact structures there are those that define different orientations of $\mathbb{R}^3$, for example $(dz + pydx - qxdy)$ and $(dz - pydx + qxdy)$.

Example 2.2. Consider the multigerm $\gamma = (\gamma_1, \gamma_2)$ with two components

$$\gamma_1 : \ x = t^2, \ y = t^5, \ z = 0, \qquad \gamma_2 : \ x = t, \ y = t^2, \ z = t^3.$$

It is Legendrain with respect to the contact structure $dz + \lambda_1 ydx + \lambda_2 xdy$ with $\lambda_1 = -15, \lambda_2 = 6$. It is easy to check that γ is not planar. By Theorem 1.2 the multigerm γ determines orientation of $\mathbb{R}^3$, the image of γ does not admit an orientation-reversing symmetry, and γ is not quasi-homogeneous.

Example 2.3. Let us show, using Theorems 1.2 and 1.3, that the multigerm

$$\gamma = (\gamma_1, \gamma_2), \quad \gamma_1 : \ x = t^2, \ y = t^5, \ z = 0, \quad \gamma_2 : \ x = t, \ y = t^3, \ z = at^2$$

is not Legendrain with respect to any contact structure on $\mathbb{R}^3$ for any $a \in \mathbb{R}$.

If $a \neq 0$ then γ admits an orientation-reversing symmetry, for example $(x, y, z) \to (x, -y + \frac{2xz}{a}, z)$. On the other hand, if $a \neq 0$ then it can be easily showed that γ is not planar. Therefore $\gamma \notin Leg(\mathbb{R}^3)$ by Theorem 1.2. In the case $a = 0$ $\gamma \notin Leg(\mathbb{R}^3)$ by a different part of Theorem 1.2, namely, by Theorem 1.3: in this case γ is planar, but one can prove that it is not quasi-homogeneous.

Example 2.4. Consider the following germ of a singular curve:

$$\psi : x = t^3,\ y = t^7 + at^8,\ z = bt^{11},\quad a, b \in \mathbb{R}.$$

This germ has the following properties:

(a) It is easy to check that if $b \neq 0$ then ψ is not a planar curve (for any $a \in \mathbb{R}$);

(b) If $a \neq 0$ and $b \neq 0$ then $\psi \in Leg(\mathbb{R}^3)$. For example, ψ is Legendrain with respect to the contact structure $(dz + \lambda_1 ydx + \lambda_2 xdy)$, where (λ_1, λ_2) is the solution of the linear system $3\lambda_1 + 7\lambda_2 = 0,\ 3\lambda_1 + 8a\lambda_2 + 11b = 0$ ($\lambda_1 == \frac{77b}{3a}, \lambda_2 = -\frac{11b}{a}$;)

(c) If $a = 0$ then the image of ψ admits an orientation-reversing symmetry, for example $(x, y, z) \to (-x, -y, -z)$;

(d) If $a \neq 0$ then it is easy to prove that the germ ψ is not quasi-homogeneous.

These properties and Theorems 1.2 and 1.3 imply the following statements:

(i) If $a \neq 0$ and $b \neq 0$ then the curve ψ belongs to $Leg(\mathbb{R}^3)$ and defines orientation of $\mathbb{R}^3$. The image of the curve ψ does not admit an orientation-reversing symmetry;

(ii) If $a = 0, b \neq 0$ or $a \neq 0, b = 0$ then $\psi \notin Leg(\mathbb{R}^3)$ (if $a = 0$ and $b \neq 0$ then the curve ψ admits an orientation-reversing symmetry, but it is not planar; if $a \neq 0$ and $b = 0$ then ψ is planar, but not quasi-homogeneous).

Example 2.5. Many singularities of space curves are described by the normal form

$$\psi :\ x = t^q,\ y = t^p \pm t^s,\ z = t^r,\quad q < p < s < r \tag{2.1}$$

where the integers q, p, s, r satisfy the following assumptions:

(a) p is not divisible over q; (b) $s \notin \{\alpha_1 q + \alpha_2 p\}$, α_1, α_2 are integers ≥ 0;

(c) $s + q$ is not divisible over p; (d) $r \notin \{\alpha_1 q + \alpha_2 p\}$, α_1, α_2 are integers ≥ 0.

The explanation of (a) - (d) is as follows, see [8], [9]. If one of the assumptions (a)-(c) is violated then the germ ψ is ψ is quasi-homogeneous - it is RL-equivalent to (t^q, t^p, t^r) or (t^q, t^s, t^r). One can prove that under assumptions (a) - (c) the germ ψ is not quasi-homogeneous. If (d) is violated then the r-jet of ψ can be simplified to $(t^q, t^p \pm t^s, 0)$. If $q = 3$ then under assumptions (a)-(d) ψ is not planar, see [9]. This is not so if $q \geq 4$. For example the germ $(t^4, t^6 + t^7, t^{9+2k})$ satisfies (a)-(d) for any $k \geq 0$, but if $k \geq 4$ then it is RL-equivalent to $(t^4, t^6 + t^7, 0)$, see [9].)

Proposition 2.1. *Let ψ be a germ of form (2.1) with q, p, s, r satisfying (a)-(d). Then $\psi \in Leg(\mathbb{R}^3)$ if and only if $r = s + q$.*

Proof. Assume that ψ is Legendrain with respect to a contact 1-form $\alpha = Adx + Bdy + Cdz$. Assumptions (a) and (b) easily imply that $A(0) = B(0) = 0$. Consequently $C(0) \neq 0$ and there is no loss of generality to assume that $C \equiv 1$. Then the functions A and B satisfy the relation

$$F(t) = A(t^q, t^p \pm t^s, t^r) \cdot q \cdot t^{q-1} + B(t^q, t^p \pm t^s, t^r) \cdot (pt^{p-1} \pm st^{s-1}) + r \cdot t^{r-1} \equiv 0.$$

Let $f_1 t + f_2 t^2 + \cdots$ be the Taylor series of $F(t)$. Using (a)-(d) it is easy to obtain

$$f_{p+q-1} = q \cdot \frac{\partial A}{\partial y}(0) + p \cdot \frac{\partial B}{\partial x}(0) = 0;\quad f_{s+q-1} = \pm \left(q \cdot \frac{\partial A}{\partial y}(0) + s \cdot \frac{\partial B}{\partial x}(0) \right) = \delta,$$

where $\delta = 0$ if $r \neq q + s$ and $\delta = r$ if $r = q + s$. It follows that if $r \neq q + s$ then $(\partial A/\partial y)(0) = (\partial B/\partial x)(0) = 0$. In this case $(\alpha \wedge d\alpha)(0) = 0$, i.e. the 1-form α is not contact. The contradiction means that if $r \neq q + s$ then $\psi \notin Leg(\mathbb{R}^3)$. If $r = q + s$ then $\psi \in Leg(\mathbb{R}^3)$ because ψ is Legendrain with respect to the contact

1-form $dz + \lambda_1 ydx + \lambda_2 xdy$ with (λ_1, λ_2) being the solution of the linear system $q\lambda_1 + p\lambda_2 = 0,\ q\lambda_1 + s\lambda_2 = \pm r$. □

Remind that under conditions (a)-(d) the germ (2.1) is not quasi-homogeneous. Therefore if $r = s + q$ then by Theorems 1.2, 1.3 and 2.1 the germ (2.1) is not planar and determines orientation of $\mathbb{R}^3$. The latter can also be seen from the proof of Proposition 2.1: if ψ is Legendrain with respect to a contact 1-form $\alpha = dz + Adx + Bdy$ then $(\alpha \wedge d\alpha)(0) = (\lambda_2 - \lambda_1)dx \wedge dy \wedge dz$, where λ_1 and λ_2 are uniquely determined by the numbers q, p, s, r.

3. Proof of Theorem 1.1 (using Theorem 1.2)

In this section we prove Theorems 1.1 using the implication $(i) \implies (ii)$ in Theorem 1.2. We work in a fixed category which is either real analytic or C^∞. At the end of the section we also prove Theorem 1.1 in the holomorphic category.

3.1. Reformulation of Theorem 1.1.

At first we reformulate Theorem 1.1.

Theorem 3.1. (equivalent to Theorem 1.1). *Let γ be a multigerm of a curve in $\mathbb{R}^3$. In the C^∞ category assume that $\gamma \notin E_\infty$. If γ is Legendrain with respect to contact structures (α) and (β) then there exists a local diffeomorphism preserving the image of γ and sending (β) to (α).*

Theorem 1.1 implies Theorem 3.1 by the Darboux theorem on the diffeomorphness of all local contact structures. Let us show that Theorem 3.1 implies Theorem 1.1. Let γ and $\tilde\gamma$ be Legendrain multigerms with respect to a contact structure (α) in Theorem 1.1, and let Φ be a local diffeomorphism sending the image of $\tilde\gamma$ to the image of γ. Let $\hat\gamma = \Phi \circ \tilde\gamma$. The multigerms γ and $\hat\gamma$ have the same image and consequently in the real analytic category $\hat\gamma$ is Legendrian with respect to (α). The same holds in the C^∞ category due to the assumption $\gamma, \tilde\gamma \notin E_\infty$. [5] On the other hand $\hat\gamma$ is Legendrain with respect to the contact structure $(\beta) = (\Phi^*\alpha)$. By Theorem 3.1 there exists a local diffeomporphism Ψ sending (β) to (α) preserving the image of $\hat\gamma$. Then the composition $\Psi \circ \Phi$ is a local contactomorphism of the contact space $(\mathbb{R}^3, (\alpha))$ sending the image of $\tilde\gamma$ to the image of γ.

3.2. The case that the contact structures (α) and (β) are transversal.

The transversality of (α) and (β) means that $(\alpha \wedge \beta)(0) \neq 0$. This is an easy case in Theorem 3.1. In this case the vector field Y defined by the relation $Y \rfloor \Omega = \alpha \wedge \beta$, where Ω is a non-degenerate volume form on $\mathbb{R}^3$, does not vanish at 0 and the image of the multigerm γ belongs to the phase curve l of Y containing 0. This phase curve l is a non-singular 1-dimensional submanifold of $\mathbb{R}^3$ and it is Legendrain with respect to the contact structures (α) and (β). By Darboux-Givental' theorem (see [4]) there exists a local diffeomorphism sending (β) to (α) and preserving l *pointwise*. Consequently, this diffeomorphism preserves the image of γ (also pointwise).

[5] in order to conclude that $\hat\gamma^*\alpha \equiv 0$ it suffices to assume that the multigems γ and $\hat\gamma$ have no components with the zero Taylor series and consequently any component is immersed at any point $t \neq 0$ sufficiently close to 0.

3.3. **The case that the contact structures (α) and (β) are tangent. Reduction to the case that they define the same orientation.** We have reduced Theorem 3.1 to the case that the 1-forms α and β are tangent at 0: $(\alpha \wedge \beta)(0) = 0$. Since α and β are non-vanishing 1-forms we may assume that $\alpha(0) = \beta(0)$.

Now we use the implicatioin $(i) \implies (ii)$ in Theorem 1.2. It allows to assume, without loss of generality, that the contact structures (α) and (β) in Theorem 3.1 define the same orientation of $\mathbb{R}^3$. In fact, assuming that we have proved Theorem 3.1 for this case, the proof for the case that (α) and (β) define different orientation is as follows. By the statement $(i) \implies (ii)$ in Theorem 1.2 the image of γ admits an orientation-reversing symmetry Φ. It follows that γ is Legendrain with respect the contact structures $(\Phi^*\alpha)$ and (β). These contact structures define the same orientation of $\mathbb{R}^3$ and therefore there exists a local diffeomorphism Ψ sending (β) to $(\phi^*\alpha)$ preserving the image of γ. The composition $\Phi^{-1} \circ \Psi$ preserves the image of γ and sends (β) to (α).

We have reduced Theorem 3.1 to the following proposition.

Proposition 3.2. *Let γ be a multigerm of a curve in $\mathbb{R}^3$. In the C^∞ category assume that $\gamma \notin E_\infty$. Assume that γ is Legendrain with respect to germs of contact 1-forms α and β satisfying the following assumptions:*

(a) $\alpha(0) = \beta(0) = 0$;

(b) the orientations on $T_0\mathbb{R}^3$ defined by α and β are the same.

Then there exists a local diffeomorphism Φ which preserves the image of γ and brings (β) to (α).

3.4. **Proof of Proposition 3.2.** The assumptions (a) and (b) allow to prove this proposition by the homotopy method, see [6] and [18]. Let $\alpha_s = \alpha + s(\beta - \alpha)$. We will prove that that there exists a family of local diffeomorphism Φ_s, $s \in [0,1]$ preserving the image of the multigerm γ and such that $\Phi_s^*(\alpha_s) = \alpha_0$ for any $s \in [0,1]$.

Lemma 3.3. *Under assumptions (a) and (b) the germ $\alpha_s = \alpha + s(\beta - \alpha)$ is contact for all $s \in [0,1]$, i.e. $(\alpha_s \wedge d\alpha_s)(0) \neq 0$.*

Proof. Fix a non-degenerate volume form Ω on $T_0\mathbb{R}^3$ and consider the function $F(s)$ such that $(\alpha_s \wedge d\alpha_s)(0) = F(s)\Omega$. Condition (a) implies that $F(s)$ is a linear function: $F(s) = k_1 s + k_2$ for some $k_1, k_2 \in \mathbb{R}$. Condition (b) implies that $F(0)$ and $F(1)$ are non-zero numbers of the same sign. Therefore $F(s) \neq 0$ for any $s \in [0,1]$. □

Lemma 3.3 allows to define a family of germs of vector fields Y_s by the relation

$$Y_s \rfloor (\alpha_s \wedge d\alpha_s) = \alpha \wedge \beta, \quad s \in [0,1]. \tag{3.1}$$

By condition (a) one has $Y_s(0) = 0$. This allows to define a family Φ_s, $s \in [0,1]$, of local diffeomorphisms by the system of ODEs

$$\frac{d\Phi_s}{ds} = Y_s(\Phi_s), \quad \Phi_0 = id. \tag{3.2}$$

We will show that Φ_s is the required family of local diffeomorphisms: Φ_s preserves the image of the multigerm γ and sends the contact structure (α_s) to the (α_0).

Take a point $p \neq 0$ close to $0 \in \mathbb{R}^3$ and belonging to the image of γ. Let γ_i be a component of γ passing through p: $p = \gamma_i(t_0), t_0 \neq 0$. In the real analytic category $\gamma_i'(t_0) \neq 0$. The same holds in the C^∞ category due to the assumption $\gamma \notin E_\infty$.

Therefore in a small enough neighborhood of p the image of the the component γ_i is a non-singular 1-dimensional submanifold $l_p \subset \mathbb{R}^3$. Since γ is Legendrain with respect to 1-forms α and β then l_p is tangent to the 2-spaces $ker\alpha(p)$ and $ker\beta(p)$. If $(\alpha\wedge\beta)(p)\neq 0$ then (3.1) implies that $Y_s(p)\in ker\alpha(p)\cap ker\beta(p)$. if $(\alpha\wedge\beta)(p)=0$ then by (3.1) one has $Y_s(p)=0$. Consequently either $Y_s(p)=0$ or the vector $Y_s(p)$ is tangent to the image of γ_i. It follows that the family of local diffeomorphisms defined by (3.2) preserves the image of γ_i for any $s\in[0,1]$. Therefore Φ_s preserves the image of the multigerm γ.

Now we will prove that $\Phi_s^*(\alpha_s)=\alpha_0$, i.e that there exists a family H_s of function germs such that $H_s(0)\neq 0$ and $H_s\cdot\Phi^*\alpha_s=\alpha_0$. Note that we have proved above that $Y_s \rfloor \alpha = Y_s \rfloor \beta = 0$ for any $s\in[0,1]$. Consequently $Y_s \rfloor \alpha_s = 0$. Let L_{Y_s} be the Lie derivative along the vector field Y_s. Then $L_{Y_s}\alpha_s = Y_s \rfloor d\alpha_s$. This relations, (3.1), and the relation $\beta = \frac{d\alpha s}{ds}$ imply

$$\left(L_{Y_s}\alpha_s + \frac{d\alpha_s}{ds}\right)\wedge\alpha_s = 0.$$

Since $\alpha_s(0)\neq 0$ for any $s\in[0,1]$ it follows that there exists a family h_s of function germs, $s\in[0,1]$, such that

$$L_{Y_s}\alpha_s + h_s\alpha_s + \frac{d\alpha_s}{ds} = 0. \tag{3.3}$$

Define a family H_s of function germs by the ODE

$$\frac{dH_s}{ds} = h_s(\Phi_s)\cdot H_s, \quad H_0\equiv 1. \tag{3.4}$$

Consider the function $G(s)=H_s\cdot\Phi_s^*\alpha_s$ whose values are 1-forms. Differentiating this function using (3.2) and (3.4) we obtain

$$G'(s) = H_s\cdot\Phi_s^*\left[L_{Y_s}\alpha_s + h_s\alpha_s + \frac{d\alpha_s}{ds}\right].$$

By (3.3) $G'(s)\equiv 0$. Since $G(0)=\alpha_0$ by the initial conditions in (3.2) and (3.4) then $G(s)\equiv\alpha_0$, i.e $H_s\cdot\Phi_s^*\alpha_s\equiv\alpha_0$. It remains to note that $H_s(0)\neq 0$ because the ODE (3.4) is linear.

3.5. **Proof of Theorem 1.1 in the holomorphic category.** A little modification of the given arguments gives a complete proof of Theorem 1.1 in the holomorphic category. In the holomorphic category Theorem 1.1 also can be reformulated as Theorem 3.1 and the reduction to the case $\alpha(0)=\beta(0)$ is the same. The rest of the proof is the same if the path $\alpha_s=\alpha+s(\beta-\alpha), s\in[0,1]$ consists of contact 1-forms. If this is not so then the path α_s should be replaced by another path of the form $\tilde\alpha_s=\alpha+f(s)(\beta-\alpha)$, where $f(s):[0,1]\to\mathbb{C}$ is a smooth function such that $f(0)=0, f(1)=1$ and $(\alpha_s\wedge d\alpha_s)(0)\neq 0$ for all $s\in[0,1]$. It is clear that such a *complex-valued* function $f(s)$ exists. The family of vector fields Y_s defined by (3.1) should be replaced by the family $\tilde Y_s$ such that

$$\tilde Y_s \rfloor (\tilde\alpha_s\wedge d\tilde\alpha_s) = \tilde\alpha_s\wedge\frac{d\tilde\alpha_s}{ds} = f'(s)\cdot\alpha\wedge\beta.$$

The given above arguments with minor modifications imply that the family of local diffeomorphisms Φ_s defined by (3.2) with Y_s replaced by $\tilde Y_s$ preserves the image of γ and brings the contact structures $(\tilde\alpha_s)$ to $(\tilde\alpha_0)=(\alpha)$.

4. Proof of Theorem 1.2

4.1. **Outline of the proof.** As we mentioned in Section 1, the implications $(iii) \implies (ii) \implies (i)$ in Theorem 1.2 are trivial. In order to prove Theorem 1.2, we have to prove the implications $(iv) \implies (i)$ and $(i) \implies (iii), (iv)$.

The implication $(iv) \implies (i)$ is proved in Section 4.2. The implications $(i) \implies (iii), (iv)$ can be joined into the following statement.

Theorem 4.1. *Let γ be a multigerm of a curve in $\mathbb{R}^3$. In the C^∞ category assume that $\gamma \notin E_\infty$. If γ is Legendrain with respect to contact structures (α) and (β) defining different orientations of $\mathbb{R}^3$ then γ is a planar and quasi-homogeneous.*

In what follows we will prove this theorem under the following assumption:

(A) the image of γ is not contained in the union $l_1 \cup l_2$ of two non-singular non-tangent 1-dimensional submanifolds of $\mathbb{R}^3$.

Let us show that there is no loss of generality to assume **(A)** - if **(A)** is violated then the conclusion Theorem 4.1 is trivial. In fact, take local coordinates in which l_1 and l_2 are the x- and y-axes. The image of any component of γ is contained either in l_1 or in l_2, therefore in the real analytic category each of the components has, up to reparameterization, the form $(at^p, 0, 0)$ or $(0, bt^q, 0)$. In the C^∞ category the assumption $\gamma \notin E_\infty$ implies that each of the component of γ has, up to reparameterization, the form $(t, 0, 0)$ or $(0, t, 0)$. In any case the multigerm γ is planar and quasi-homogeneous with weights $(1, 1, 1)$.

Introduce the vector field $Y_{\alpha,\beta}$ defined by the relation

$$Y_{\alpha,\beta} \rfloor \Omega = \alpha \wedge \beta, \tag{4.1}$$

where Ω is a non-degenerate volume form on $\mathbb{R}^3$. We will say that $Y_{\alpha,\beta}$ is a vector field associated to the contact structures (α) and (β). It is defined up to multiplication by a non-vanishing functions since the 1-forms α and β and the volume form Ω are defined up to multiplication by a non-vanishing function.

Definition. A germ Y of a vector field on $\mathbb{R}^3$ is tangent to a germ $\psi : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ of a curve in $\mathbb{R}^3$ if the vectors $\dot{\psi}(t)$ and $Y(\psi(t))$ are linearly dependent for any t close to 0. The vector field Y is tangent to a multigerm of a curve in $\mathbb{R}^3$ if it is tangent to each of the components of the multigerm.

The integrability of the multigerm γ with respect to the contact structures (α) and (β) implies that the vector field $Y_{\alpha,\beta}$ is tangent to γ. In fact, take a point $p \in \mathbb{R}^3$ near the origin and let γ_i be a component of γ whose image contains p: $\gamma_i(t_0) = p$. If the contact structures (α) and (β) are transversal at p then the vectors $\gamma'(t_0)$ and $Y_{\alpha,\beta}(p)$ are contained in the same line $ker\alpha(p) \cap ker\beta(p)$. If the contact planes $ker\alpha(p)$ and $ker\beta(p)$ coincide then by (4.1) $Y_{\alpha,\beta}(p) = 0$.

Assumption **(A)** implies that the contact forms α and β in Theorem 4.1 are tangent at 0: $(\alpha \wedge \beta)(0) = 0$, see Section 3.2. Consequently $Y_{\alpha,\beta}(0) = 0$. The proof of Theorem 4.1 is based on the analysis of the singularity of the vector field $Y_{\alpha,\beta}$.

Proposition 4.2. *Let α and β be the 1-forms in Theorem 4.1 and let $(\alpha \wedge \beta)(0) = 0$. Within the case* **(A)** *the vector field $Y_{\alpha,\beta}$ has the following properties:*

(i) the germ of set of singular points of $Y_{\alpha,\beta}$ is a non-singular 1-dimensional submanifold of $\mathbb{R}^3$;

(ii) the eigenvalues of the vector field $Y_{\alpha,\beta}$ at the origin are $\lambda_1, \lambda_2, 0$, where λ_1, λ_2 are non-zero real numbers of the same sign.

We prove Proposition 4.2 in a separate section 5. This proposition is the most difficult part of the proof of Theorem 4.1.

Properties (i) and (ii) in Proposition 4.2 allow to bring $Y_{\alpha,\beta}$, by a change of coordinates, to the following normal form.

Proposition 4.3. (real analytic and C^∞ categories). *Any germ of a vector field satisfying (i) and (ii) in Proposition 4.2 is diffeomorphic to a vector field*

$$\dot{x} = f_1(x,y,z), \ \dot{y} = f_2(x,y,z), \ \dot{z} = 0, \tag{4.2}$$

$$f_1(0,0,z) = f_2(0,0,z) \equiv 0. \tag{4.3}$$

This proposition is proved in Section 4.3. Propositions 4.2 and 4.3 allow to fix local coordinates in which the vector field $Y_{\alpha,\beta}$ has form (4.2)-(4.3).

Proposition 4.4. *Let γ be a multigerm of a curve in $\mathbb{R}^3$ which is tangent to a vector field of the form (4.2)-(4.3). If γ satisfies* **(A)** *and in the C^∞ category $\gamma \notin E_\infty$ then the image of γ is contained in the plane $z = 0$.*

Proposition 4.4 is proved in Section 4.4. Propositions 4.2, 4.3 and 4.4 imply that the multigerm γ in Theorem 4.1 is planar. In order to prove that γ is quasi-homogeneous we consider the restriction Y of the vector field $Y_{\alpha,\beta}$ to the plane $z = 0$. It is a vector field on $\mathbb{R}^2(x,y)$ with non-zero real eigenvalues of the same sign at the singular point $0 \in \mathbb{R}^2$. The multigerm γ lives in the plane $\mathbb{R}^2(x,y)$; it is tangent to Y. The quasi-homogeneity of γ in Theorem 4.1 follows from Propositions 4.2, 4.3 and 4.4 and the following statement proved in Section 4.5.

Proposition 4.5. *Let γ be a multigerm of a curve in $\mathbb{R}^2$. In the C^∞ category assume that γ contains no components with the zero Taylor series. If γ is tangent to a vector field Y on $\mathbb{R}^2$ with non-zero real eigenvalues of the same sign at the singular point $0 \in \mathbb{R}^2$ then the multigerm γ is quasi-homogeneous.*

4.2. **Proof of the implication** $(iv) \implies (i)$**.** Let γ_i be any component of a quasi-homogeneous multigerms of a curve in $\mathbb{R}^3$ with weights $\lambda_1, \lambda_2, \lambda_3 > 0$ in local coordinates (x,y,z). Consider the 1-forms

$$\theta_1 = \lambda_1 x dy - \lambda_2 y dx, \ \theta_2 = \lambda_1 x dz - \lambda_3 z dx, \ \theta_3 = \lambda_2 y dz - \lambda_3 z dy.$$

The quasi-homegeneity of γ impleis that $\gamma_i^* \theta_1 = \gamma_i^* \theta_2 = \gamma_i \theta_3 \equiv 0$. Let α be the germ a contact 1-form with respect to which γ is Legendrain. Then $\gamma_i^*(\alpha + r\theta_j) \equiv 0, j = 1,2,3$, for any $r \in \mathbb{R}$. Since the weights λ_i are positive and $\alpha(0) \neq 0$ then there exists at least one $j \in \{1,2,3\}$ such that $(\alpha \wedge d\theta_j)(0) \neq 0$. We may assume that $(\alpha \wedge d\theta_1)(0) \neq 0$. Denote $\alpha_r = \alpha + r\theta_1$. Since $\theta_1(0) = 0$ then

$$(\alpha_r \wedge d\alpha_r)(0) = (\alpha \wedge d\alpha)(0) + r(\alpha \wedge d\theta_1)(0), \quad r \in \mathbb{R}.$$

For sufficiently big r the 1-forms α_r and α_{-r} are contact and define different orientations of $\mathbb{R}^3$. The multigerm γ is Legendrain with respect to these contact 1-forms, and consequently γ does not determine orientation of $\mathbb{R}^3$.

4.3. **Proof of Proposition 4.3.** Take a local coordinate system in which the vector field $Y_{\alpha,\beta}$ vanishes at points of the z-axes and consequently has the form

$$f_1(x,y,z)\frac{\partial}{\partial x} + f_2(x,y,z)\frac{\partial}{\partial y} + f_3(x,y,z)\frac{\partial}{\partial z}$$

with f_1, f_2 satisfying (4.3) and $f_3(x,y,z)$ such that $f_3(0,0,z) \equiv 0$. The eigenvalues at a singular point $(0,0,z)$ are $\lambda_1(z), \lambda_2(z), 0$, where $\lambda_1(0)$ and $\lambda_2(0)$ are real numbers of the same sign. The absence of resonant relations $0 = k_1\lambda_1(0) + k_2\lambda_2(0)$ with integers $k_1, k_2 \geq 0, (k_1,k_2) \neq (0,0)$ implies that $f_3(x,y,z)$ can be reduced to 0 by a formal change of coordinates, see [5]. The condition that $\lambda_1(0)$ and $\lambda_2(0)$ are real numbers of the same sign implies that normal form (3.2)-(3.3) also holds in the C^∞ and real analytic categories, see [5] and [19].

4.4. **Proof of Proposition 4.4.** Let

$$\gamma_i : \ x = a(t), \ y = b(t), \ z = c(t)$$

be any component of γ. The tangency of $Y_{\alpha,\beta}$ to γ implies the relation

$$c'(t) \cdot f_1(a(t), b(t), c(t)) = c'(t) \cdot f_2(a(t), b(t), c(t)) \equiv 0. \tag{4.4}$$

Let $\hat{a}, \hat{b}, \hat{c}$ be the Taylor series of $a(t), b(t), c(t)$. The vector field $Y_{\alpha,\beta}$ has two non-zero eigenvalues and consequently

$$det\left(\frac{\partial(f_1,f_2)}{\partial(x,y)}\right)(0) \neq 0. \tag{4.5}$$

Relations (4.3), (4.4) and (4.5) imply that either $\hat{c} = 0$ or $\hat{a} = \hat{b} = 0$. In the real analytic category we obtain the alternative

(AL) $c(t) \equiv 0$ or $a(t) = b(t) \equiv 0$.

The same alternative holds in the C^∞ category due to the assumption $\gamma \notin E_\infty$. In fact, if one of the series $\hat{a}, \hat{b}$ is not zero then by (4.3), (4.4), (4.5) one has $c(t) \equiv 0$. On the other hand, if $\hat{a} = \hat{b} = 0$ then the assumption $\gamma \notin E_\infty$ implies that $c'(0) \neq 0$ and (4.3),(4.4), (4.5) imply that $a(t) = b(t) \equiv 0$.

Now we will show that the case $a(t) = b(t) \equiv 0$ is impossible for any component of the multigerm γ satisfying **(A)**. Take one of contact forms with respect to which γ is Legendrain, for example the 1-form α, and write it in local coordinates of normal form (4.2) – (4.3): $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$. Assume that γ contains a component $x = 0, y = 0, z = c(t) \not\equiv 0$. Then in the real analytic category $\alpha_3(0) = 0$. The same holds in the C^∞ category because the assumption $\gamma \notin E_\infty$ implies $c'(0) \neq 0$. Consequently $(\alpha_1(0), \alpha_2(0)) \neq (0,0)$. Consider the vector field $\alpha_1(x,y,0)\frac{\partial}{\partial y} - \alpha_2(x,y,0)\frac{\partial}{\partial x}$. It is a non-singular vector field in the (x,y)-plane. Denote by l its phase curve containing the origin. Then l is a non-singular 1-dimensional submanifold of $\mathbb{R}^3$. By (AL) any component of the multigerm γ belongs either to the z-axes or to the (x,y)-plane and it follows that the image of γ belongs to the union of l and the z-axes. This contradicts to assumption **(A)**.

4.5. **Proof of Proposition 4.5.** We will use local normal forms for vector fields on the plane. It is well-known that the germ of a vector field Y satisfying the assumptions of Proposition 4.5 can be brought to one of the following normal forms

$$\dot{x} = \lambda_1 x, \quad \dot{y} = \lambda_2 y; \tag{4.6}$$

$$\dot{x} = \lambda x, \quad \dot{y} = N\lambda y + \kappa x^N, \quad N \geq 1, \ \lambda \neq 0, \kappa \neq 0. \tag{4.7}$$

This norml form holds in either the rela analytic or the C^∞ category, see [5]. Proposition 4.5 is a direct corollary of the following lemma.

Lemma 4.6. *Let $\gamma : x = a(t), y = b(t)$ be the germ of a plane curve. In the C^∞ category assume that at least one of the germs $a(t), b(t)$ has non-zero Taylor series.*
(i) If γ is tangent to a vector field of form (4.6) then, up to a reparameterization $t \to \phi(t), \phi'(0) \neq 0$, it the form

$$a(t) = a \cdot t^{\lambda_1 r,}, \quad b(t) = b \cdot t^{\lambda_2 r}, \quad r, a, b \in \mathbb{R}. \tag{4.8}$$

(ii) If γ is tangent to a vector field of form (4.7) then $a(t) \equiv 0$.

The proof of this lemma in the C^∞ category requires the following known statement (its proof and the proof of much more general statements of the same type can be found, for example, in [Be]):

Lemma 4.7. *Let $Q(t, u)$ be a C^∞ function-germ such that $Q(t, 0) \equiv 0$. Let $f(t)$ be a C^∞ function germ satisfying the equation $t^r \cdot f'(t) = Q(t, f(t))$ for some $r \geq 0$. If $f(t)$ has the zero Taylor series then $f(t) \equiv 0$.*

Proof of Lemma 4.6, (i). We may assume that $a(t)$ has non-zero Taylor series. Then there is no loss of generality to assume that $a(t) = at^p, a \neq 0$. The tangency of the curve $(at^p, b(t))$ to a vector field of form (4.6) means the relation

$$\lambda_2 p b(t) = \lambda_1 t b'(t). \tag{4.9}$$

This equation can be easily solved in formal power series. If $\lambda_2 p / \lambda_1$ is not an integer number then the only formal solution of (4.9) is $b(t) \equiv 0$. If $\lambda_2 p / \lambda_1$ is an integer number i then any formal solution of equation (4.9) has the form $b(t) = bt^i, b \in \mathbb{R}$. Therefore any analytic solution of (4.9) and, by Lemma 4.7, any C^∞ solution of (4.9) has the form $b(t) = bt^{\lambda_2 p/\lambda_1}$, where $b = 0$ if $\lambda_2 p / \lambda_1$ is not an integer number and $b \in \mathbb{R}$ if $\lambda_2 p / \lambda_1$ is an integer. In any case we obtain (4.8) with $r = p/\lambda_1$.

Proof of Lemma 4.6, (ii). At first let us show that the function germ $a(t)$ has the zero Taylor series. Assume, to get contradiction, that this is not so. Then there is no loss of generality to assume that $a(t) = at^p, a \neq 0$. The tangency of the curve $(at^p, b(t))$ to a vector field of form (4.7) means the relation

$$F(t) = \lambda t b'(t) - pN\lambda b(t) - p\kappa a^N t^{pN} \equiv 0.$$

Let $\sum f_i t^i$ be the Taylor series of $F(t)$. Then F_{Np} does not depend on $b(t)$ and is equal to $F_{Np} = -p\kappa a^n \neq 0$. Contradiction.

We have proved that the function germ $a(t)$ has the zero Taylor series. Consequently in the real analytic category $a(t) \equiv 0$. In order to prove Lemma 4.6 in the C^∞ category we have to prove that $a(t) \equiv 0$ under the assumption that $a(t)$ has the zero Taylor series and the Taylor series of $b(t)$ is not zero. In this case there is no loss of generality to assume that $b(t) = bt^p, b \neq 0$. The tangency of the curve $(a(t), bt^p)$ to a vector field of form (4.7) means the relation

$$a'(t) \cdot \left(N\lambda b t^p + \kappa a^N(t)\right) = \lambda p b t^{p-1} a(t). \tag{4.10}$$

Since the function germ $a(t)$ has the zero Taylor series then $a(t) = t^p \tilde{a}(t)$ for some C^∞ function germ $\tilde{a}(t)$. Equation (4.10) for $a(t)$ is equivalent to a certain equation for $\tilde{a}(t)$ of the form $t^p \tilde{a}'(t) = Q(t, \tilde{a}(t))$, where Q is a C^∞ function germ of two variables. By Lemma 4.7 $\tilde{a}(t) \equiv 0$. Consequently $a(t) \equiv 0$.

5. Proof of Proposition 4.2

5.1. **Outline of the proof.** The starting point is the following lemma.

Lemma 5.1. *Let (α) and (β) be tangent at the origin contact structures defining different orientations of $\mathbb{R}^3$. Then the vector field $Y_{\alpha,\beta}$ has the following properties:*

(a) the sum of its eigenvalues at the origin is not zero;

(b) the ideal generated by the three coefficients of $Y_{\alpha,\beta}$ in some coordinate system (this ideal does not depend on the choice of the coordinate system) is generated by two function germs (and consequently one of the eigenvalues of $Y_{\alpha,\beta}$ is zero).

Lemma 5.1 is proved in Section 5.2. Note that the multigerm γ is not involved in Lemma 5.1. In Sect. 5.3 we will show that Proposition 4.2 follows from Lemma 5.1 and the following property of any Legendrain multigerm satisfying **(A)**.

Proposition 5.2. *Let $\gamma = (\gamma_1, ..., \gamma_m) \in Leg(\mathbb{R}^3)$ be a multigerm satisfying assumption* **(A)**. *In the C^∞ category we assume that $\gamma \notin E_\infty$. There exists a local coordinate system (x, y, z) and a positive number r such that the following holds. If $\theta = \theta_1 dx + \theta_2 dy + \theta_3 dz$ is a 1-form such that $\gamma_i^* \theta \equiv 0, i = 1, .., m$ then*

$$\theta_1(0) = \theta_2(0) = 0, \ \frac{\partial \theta_1}{\partial x}(0) = 0, \ \frac{\partial \theta_1}{\partial y}(0) + r\frac{\partial \theta_2}{\partial x}(0) = 0. \tag{5.1}$$

Proposition 5.2 is proved in Sections 5.4 - 5.6 for the case that the multigerm γ satisfies, except **(A)**, at least one of the following assumptions:

(B1) the multigerm γ contains three non-singular components such that no two of them are tangent at 0.

(B2) the multigerm γ contains two tangent at 0 non-singular components; the order of tangency is finite.

(B3) the multigerm γ contains a singular component whose Taylor series is not RL-equivalent to a series of the form $(at^r, 0, 0), r \geq 2$.

One can easily check that in the C^∞ category the validity of at least one of the assumptions **(B1) - (B3)** follows from **(A)** and the condition $\gamma \notin E_\infty$. In the real analytic category there are multigerms γ satisfying **(A)** and violating each of the assumptions **(B1)**,**(B2)**,**(B3)**, but in Section 5.7 we reduce Theorem 5.2 to the case that at least one of these assumptions holds.

5.2. **Proof of Lemma 5.1.** We may assume that $\alpha(0) = \beta(0)$. Take local coordinates in which $\alpha = A_1 dx + A_2 dy + dz$, $\beta = B_1 dx + B_2 dy + dz$, where $A_i(0) = B_i(0) = 0$. Calculate the vector field $Y_{\alpha,\beta}$ using the volume form $\Omega = dx \wedge dy \wedge dz$:

$$Y_{\alpha,\beta} = (A_2 - B_2)\frac{\partial}{\partial x} + (B_1 - A_1)\frac{\partial}{\partial y} + (A_1 B_2 - A_2 B_1)\frac{\partial}{\partial z}$$

Statement (i) follows from the relation $A_2 B_1 - A_1 B_2 = (A_2 - B_2)\cdot A_1 + (B_1 - A_1)\cdot A_2$. The sum of the eigenvalues of $Y_{\alpha,\beta}$ is equal to

$$trace\, Y_{\alpha,\beta} = \frac{\partial(A_2 - B_2)}{\partial x}(0) + \frac{\partial(B_1 - A_1)}{\partial y}(0) = (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})(0) - (\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y})(0).$$

The assumption that the contact structures (α) and (β) define different orientations means that one of the numbers $(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})(0)$, $(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y})(0)$ is positive and the other is negative. Consequently $trace\, Y_{\alpha,\beta} \neq 0$.

5.3. **Lemma 5.1 and Proposition 5.2 imply Proposition 4.2.** Let γ, α, β be as in Proposition 4.2. Take local coordinates as in Proposition 5.2. Let

$$\alpha = A_1 dx + A_2 dy + A_3 dz, \quad \beta = B_1 dx + B_2 dy + B_3 dz.$$

Then

$$A_1(0) = A_2(0) = B_1(0) = B_2(0) = 0, \quad \frac{\partial A_1}{\partial x}(0) = \frac{\partial B_1}{\partial x}(0) = 0, \tag{5.2}$$

$$\frac{\partial A_1}{\partial y}(0) + r\frac{\partial A_2}{\partial x}(0) = 0, \quad \frac{\partial B_1}{\partial y}(0) + r\frac{\partial B_2}{\partial x}(0) = 0. \tag{5.3}$$

It follows that $A_3(0) \neq 0, B_3(0) \neq 0$ and therefore there is no loss of generality to assume that $A_3 = B_3 \equiv 1$. The vector field $Y_{\alpha,\beta}$ has form (5.1), up to multiplication by a non-vanishing function. Relations (5.2) imply that the matrix of linear approximation of $Y_{\alpha,\beta}$ is a triangular matrix with diagonal entries (eigenvalues)

$$\lambda_1 = \frac{\partial(A_2 - B_2)}{\partial x}(0), \quad \lambda_2 = \frac{\partial(B_1 - A_1)}{\partial y}(0), \quad \lambda_3 = 0.$$

Relations (5.3) imply that $\lambda_2 = r\lambda_1$. The case $\lambda_1 = \lambda_2 = 0$ is impossible by the statement (a) of Lemma 5.1. Therefore the vector field $Y_{\alpha,\beta}$ has two non-zero real eigenvalues of the same sign. It remains to prove that the set of singular points of $Y_{\alpha,\beta}$ is a non-singular 1-dimensional submanifold of $\mathbb{R}^3$. This easily follows from the statement (b) in Lemma 5.1 and the fact that $Y_{\alpha,\beta}$ has two non-zero eigenvalues.

5.4. **Proof of Proposition 5.2 under assumption (B1).** Let $\gamma_1, \gamma_2, \gamma_3$ be the components satisfying **(B1)**. The condition $\gamma \in Leg(\mathbb{R}^3)$ implies that the vectors $\gamma_1'(0), \gamma_2'(0), \gamma_3'(0)$ are linearly dependent. In fact, if these vectors are linearly independent and α is a 1-form such that $\gamma_1^*\alpha = \gamma_2^*\alpha = \gamma_3^*\alpha \equiv 0$ then $\alpha(0) = 0$, i.e. α is not contact. Thus the vectors $\gamma_1'(0), \gamma_2'(0), \gamma_3'(0)$ are linearly dependent, but any two of them are linearly independent. Therefore in suitable local coordinates

$$\gamma_1'(0) = \frac{\partial}{\partial x}, \ \gamma_2'(0) = \frac{\partial}{\partial y}, \ \gamma_3'(0) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Let $\theta = \theta_1 dx + \theta_2 dy + \theta_3 dz$ be a 1-form such that $\gamma_i^*\theta \equiv 0, \ i = 1, 2, 3$. Then one has the relations

$$\theta_1(0) = \theta_2(0) = 0, \ \frac{\partial \theta_1}{\partial x}(0) = \frac{\partial \theta_2}{\partial y}(0) = 0, \ \frac{\partial \theta_1}{\partial y}(0) + \frac{\partial \theta_2}{\partial x}(0) = 0.$$

Therefore Proposition 5.2 holds with the chosen local coordinates and with $r = 1$.

5.5. **Proof of Proposition 5.2 under assumption (B2).** Let γ_1, γ_2 be the components satisfying **(B2)**. In suitable coordinates their images can be described by equations [6]

$$y = z = 0, \quad y - x^{N+1} = z = 0, \quad N \geq 1. \tag{5.4}$$

Let $\theta = \theta_1 dx + \theta_2 dy + \theta_3 dz$ be a 1-form such that $\gamma_1^*\theta = \gamma_2^*\theta \equiv 0$, where γ_1, γ_2 have form (5.4). Then one has the relations

$$\theta_1(x, 0, 0) \equiv 0, \quad \theta_1(x, x^{N+1}, 0) + (N+1)x^N\theta_2(x, x^{N+1}, 0) \equiv 0.$$

[6] Normal form (5.4) can be easily obtained as follows. The images of γ_1 and γ_2 can be described by equations $y = z = 0$ and $y = f_1(x), z = f_2(x)$, where $f_i(x) = x^{N+1}g_i(x)$, $(g_1(0), g_2(0)) \neq (0,0)$. Here N is the order of tangency between γ_1 and γ_2. Applying a change of coordinates of the form $(x, y, z) \to (\phi(x), \pm y, \pm z)$ and possibly $(x, y, z) \to (x, z, y)$ we can reduce $f_1(x)$ to x^{N+1} and $f_2(x)$ to $x^{N+1}\tilde{f}_2(x)$. To get (5.4) it remains to change z by $z - y\tilde{f}_2(x)$.

Considering the coefficients at x^i, $i = 0, 1, N, N+1$ in the Taylor series of the left hand side of this relation we obtain

$$\theta_1(0) = \theta_2(0) = \frac{\partial \theta_1}{\partial x}(0) = 0, \quad \frac{\partial \theta_1}{\partial y}(0) = (N+1)\frac{\partial \theta_2}{\partial x}(0).$$

Therefore Proposition 5.2 holds in the chosen coordinates $r = N + 1$.

5.6. **Proof of Proposition 5.2 under assumption (B3).** Let γ_1 be a singular component satisfying **(B3)**. Then γ_1 is RL-equivalent to a curve of the from

$$\psi: \ x = t^p, \ y = t^q + f_1(t), \ z = f_2(t), \tag{5.5}$$

$$2 \leq q < p, \ q \neq 0 \ mod \ p, \quad f_1(t) = o(t^q), \ f_2(t) = o(t^q). \tag{5.6}$$

We will assume that the functions $f_1(t)$ and $f_2(t)$ satisfy the following conditions

$$f_1^{(2p)}(0) = f_2^{(2p)}(0) = f_2^{(p+q)}(0) = 0. \tag{5.7}$$

These conditions can be reached by a change of coordinates of the form $y \to y + a_1x^2$, $z \to z + a_2x^2 + a_3xy$ with suitable a_1, a_2, a_3. Let $\theta = \theta_1(x,y,z)dx + \theta_2(x,y,z)dy + \theta_3(x,y,z)dz$ be a 1-form such that $\psi^*\theta \equiv 0$. Then one has the relation

$$pt^{p-1}\theta_1(\hat{\psi}(t)) + (q \cdot t^{q-1} + f_1'(t))\theta_2(\hat{\psi}(t)) + f_2'(t)\theta(\hat{\psi}(t)) \equiv 0.$$

Using (5.6) and (5.7) we see that the coefficients at $t^{p-1}, t^{q-1}, t^{2p-1}$ and t^{p+q-1} in the Taylor series of the left hand side of this relation are equal to $p \cdot \theta_1(0)$, $q \cdot \theta_2(0)$, $p \cdot \frac{\partial \theta_1}{\partial x}(0)$ and $p \cdot \frac{\partial \theta_1}{\partial y}(0) + q \cdot \frac{\partial \theta_2}{\partial x}(0)$ respectively. Therefore

$$\theta_1(0) = \theta_2(0) = \frac{\partial \theta_1}{\partial x}(0), \quad p \cdot \frac{\partial \theta_1}{\partial y}(0) + q \cdot \frac{\partial \theta_2}{\partial x}(0) = 0,$$

and Proposition 5.2 holds with the chosen coordinate system and with $r = q/p$.

5.7. **Real analytic category. Reduction of Proposition 5.2 to the case that one of the assumptions (B1) - (B3) holds.** Let $\gamma = (\gamma_1, ..., \gamma_m)$. Construct a multigerm $\tilde{\gamma} = (\tilde{\gamma}_1, ..., \tilde{\gamma}_m)$ as follows. If γ_i is a non-singular component then $\tilde{\gamma}_i = \gamma_i$. If γ_i is a singular component whose image is not contained in in a non-singular 1-dimensional submanifold of $\mathbb{R}^3$ then also $\tilde{\gamma}_i = \gamma_i$. In the remaining case that γ_i is a singular component whose image is contained in a non-singular 1-dimensional submanifold l_i of $\mathbb{R}^3$ we define $\tilde{\gamma}_i$ to be a non-singular curve whose image is l_i. Let θ be any 1-form. It is clear that in the real analytic category the relation $\gamma_i^*\theta \equiv 0$ implies the relation $\tilde{\gamma}_i^*\theta = 0$. Therefore in order to prove Proposition 5.2 for the multigerm γ it suffices to prove it for the multigerm $\tilde{\gamma}$.

Since the image of γ is contained in the image of $\tilde{\gamma}$ then the multigerm $\tilde{\gamma}$ also satisfies **A**. Let us show that $\tilde{\gamma}$ also satisfies at least one of the assumptions **(B1) - (B3)**. The construction of $\tilde{\gamma}$ implies that it satisfies one of the following conditions:

(a) $\tilde{\gamma}$ contains non-singular components only;

(b) $\tilde{\gamma}$ contains a singular component whose image is not contained in a non-singular 1-dimensional submanifold of $\mathbb{R}^3$.

In case (a) the image of $\tilde{\gamma}$ is the union $l_1 \cup \cdots \cup l_s$ of s different non-singular 1-dimensional submanifolds of $\mathbb{R}^3$ (it is not excluded that $s < m$). By assumption **A** $s \geq 3$. In the real analytic category the order of tangency between two different non-singular components of a multigerm is always finite. It follows that in case (a) either **(B1)** or **(B2)** holds. In case (b) one has **(B1)** by the following lemma.

Lemma 5.3. (real analytic category). *Let* $\psi : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ *be the germ of a singular curve whose Taylor series* $\hat{\psi}$ *is RL-equivalent to a series of the form* $(t^r, 0, 0), r \geq 2$. *Then* ψ *is RL-equivalent to the curve* $(t^r, 0, 0)$.

Proof. There is no loss of generality to assume that $\psi : x = t^r, y = b(t), z = c(t)$. Let $\hat{b}(t), \hat{c}(t)$ be the Taylor series of $b(t)$ and $c(t)$. The RL-equivalence of $\hat{\psi}$ to $(t^r, 0, 0)$ implies that there exist formal power series $\hat{B}(x, y, z), \hat{C}(x, y, z)$ such that $\hat{b}(t) = \hat{B}(t^r, 0, 0)$, $\hat{c}(t) = \hat{C}(t^r, 0, 0)$. It follows that $b(t) = g(t^r), c(t) = h(t^r)$, where g and h are *analytic* functions in one variable. A local diffeomorphism $(x, y, z) \to (x, y + g(x), z + h(x))$ brings the curve $(t^r, 0, 0)$ to the curve ψ. □

6. Contact classification of A singularities

The simplest application of results of section 1 is the classification of singular Legendrian submanifolds of a contact 3-space diffeomorphic to submanifolds

$$A_i : \quad \{(x, y, z) : \; y^2 - x^{i+1} = z = 0.\}$$

Submanifolds of $\mathbb{R}^3$ diffeomorphic to A_i correspond to the following singularity classes of multigerms of curves in $\mathbb{R}^3$:

$\mathbf{A}_2 \cup \mathbf{A}_4 \cup \cdots$: singular curves ψ with non-zero 2-jet whose Taylor series is not RL-equivalent to $(t^2, 0, 0)$; $\psi \in \mathbf{A}_{2k}$ if $j^{2k_1}\psi$ is RL-equivalent to $(t^2, 0, 0)$ and $j^{2k+1}\psi$ is not equivalent to $(t^2, 0, 0)$);

$\mathbf{A}_1 \cup \mathbf{A}_3 \cup \cdots$: multigerms $\gamma = (\gamma_1, \gamma_2)$ consisting of two non-singular components with finite order of tangency; $\gamma \in A_{2k+1}$ if the order of tangency is equal to k (the zero order of tangency means that γ_1 and γ_2 are not tangent).

The classes $\mathbf{A}_{2k}$ and $\mathbf{A}_{2k+1}$ are described by the following normal form with respect to the RL-equivalence:

$$\hat{A}_{2k} : \quad (t^2, t^{2k+1}, 0), \quad \hat{A}_{2k+1} : \quad (t, 0, 0), \; (t, t^{k+1}, 0)$$

see [1] an Sect. 5.5. The image of $\hat{A}_{2k}$ is exactly A_{2k}. The image of $\hat{A}_{2k+1}$ is given by the equations $y(y - x^{k+1}) = z = 0$. The function $y(y - x^{k+1})$ is R-equivalent to $y^2 - x^{2k+2}$. Therefore the image of $\hat{A}_{2k+1}$ is diffeomorphic to A_{2k+1}.

Fix a contact structure, for example the standard contact structure $(dz - ydx)$. Consider the following singular curve and a multigerm with two components which are Legendrain with respect to this contact structure:

$$\tilde{A}_{2k} : \qquad x = t^2, \; y = t^{2k+1}, \; z = 2t^{2k+3}/(2k+3);$$

$$\tilde{A}_{2k+1} : \quad \gamma_1 : \; x = t, \; y = z = 0, \qquad \gamma_2 : \; x = t, \; y = t^{k+1}, \; z = t^{k+2}/(k+2).$$

These multigerms can be reduced to $\hat{A}_{2k}$ and $\hat{A}_{2k+1}$ by a change $(x, y, z) \to (x, y, z + rxy)$ with a suitable r. Therefore Theorem 1.1 implies the following result.

Theorem 6.1. (real analytic and C^∞ categories). *Let* $i = 2k$ *or* $i = 2k + 1$.
1. $\mathbf{A}_i \cap Leg(\mathbb{R}^3) = \mathbf{A}_i$.
2. Let $\gamma \in \mathbf{A}_i$ *be a Legendrain multigerm in a fixed contact structure* $(\mathbb{R}^3, (dz - ydx))$. *Then* γ *is contactomorphic to the multigerm* $\tilde{A}_i$.

By Theorem 6.1 all Legendrain multigerms of the class $\mathbf{A}_i$ in a fixed contact space are contactomorphic. Consequently, all singular Legendrian submanifolds diffeomorphic to A_i are contactomorphic. We obtain the following corollary.

Corollary 6.2. (real analytic and C^∞ categories). *Any singular Legendrian submanifold of the contact space* $(\mathbb{R}^3, (dz - ydx))$ *diffeomorphic to* A_i *is contactomorphic to the singular submanifold given by the equations* $y^2 = x^{i+1}$, $z = 2xy/(i+3)$.

7. Contact classification of E singularities

In this section we give a complete contact classification of germs of Legendrain singular curves in $\mathbb{R}^3$ of class **E**. The class **E** consists of germs $\psi : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ satisfying the following conditions: (a) ψ has zero 2-jet and non-zero 3-jet; (b) the Taylor series of ψ is not RL-equivalent to $(t^3, 0, 0)$.

In the previous section we showed that $\mathbf{A} \cap Leg(\mathbb{R}^3) = \mathbf{A}.For\mathbf{E}$ singularities this is not so: within such singularities there are those that are not Legendrian with respect to any contact structure. The class $\mathbf{E} \cap Leg(\mathbb{R}^3)$ is substantially smaller than **E**. For example, the classification of **E** singularities with respect to diffeomorphisms starts with the following four singularities, see [9]:

$$(t^3, t^4, t^5) \leftarrow (t^3, t^4, 0)^* \leftarrow (t^3, t^5, t^7) \leftarrow (t^3, t^5, 0)^*.$$

Here we marked by $*$ the two singularities that belong to $Leg(\mathbb{R}^3)$. The other two singularities do not belong to $Leg(\mathbb{R}^3)$ by Theorem 1.3: they are quasi-homogeneous, but not planar. The ierarchy of **E** singularities continues with

$$\begin{array}{lll} (t^3, t^5, t^7) & & \\ \uparrow & & \\ (t^3, t^7, t^8) \leftarrow & (t^3, t^7 + t^8, t^{11})^* \leftarrow & (t^3, t^7 + t^8, 0) \\ \uparrow & \uparrow & \\ (t^3, t^8, t^{10}) & (t^3, t^7, t^{11}). & \end{array}$$

Within these six singularities only one belongs to $Leg(\mathbb{R}^3)$ $(t^3, t^7 + t^8, t^{11}) \in Leg(\mathbb{R}^3)$ by Proposition 2.1. The other five singularites do not belong to $Leg(\mathbb{R}^3)$ by Theorem 1.3 - each of them is either quasi-homogeneous and not planar, or planar and not quasi-homogeneous.

Now we present a general result on distinguishing the class $\mathbf{E} \cap Leg(\mathbb{R}^3)$ in **E**. The classification of all E singularities with respect to the RL-equivalence is known due to [9], see also [2]. Any singular curve of class E is RL equivalent to one and only one of the following curves, up to reduction $\pm \to +$ given in the remark below (we use the notations in [2]):

$E^\pm_{6k,p,i}$: $x = t^3$, $y = t^{3k+1} \pm t^{3k+2+3i}$, $z = t^{3k+2+3p}$,

$E^\pm_{6k+2,p,i}$: $x = t^3$, $y = t^{3k+2} \pm t^{3k+4+3i}$, $z = t^{3k+4+3p}$,

where $k \ge 1$, $0 \le p \le k$, $0 \le i \le min(k-1, p)$.

Remark. The curves $E^\pm_{6k,p,i}$ (respectively $E^\pm_{6k+2,p,i}$) are diffeomorphic if and only if one of the following holds: (a) i is an even number (respectively odd number); (b) $i = k - 1$; (c) $i = p$.

Proposition 7.1. *Let* $l = 6k$ *or* $l = 6k+2$. $E^\pm_{l,p,i} \in Leg(\mathbb{R}^3)$ *if and only if* $p = i+1$.

Proof. It is easy to check that the **E** singularities have the following properties:

• if $p = k, i = k-1$ then the curves $E^\pm_{6k,p,i}$ and $E^\pm_{6k+2,p,i}$ are planar and quasi-homogeneous (these curves are RL-equivalent to $(t^3, t^{3k+1}, 0)$ and $(t^3, t^{3k+2}, 0)$);

• if $p = k, i \leq k-2$ then the curves $E^{\pm}_{6k,p,i}$ and $E^{\pm}_{6k+2,p,i}$ are planar, but not quasi-homogeneous (these curves are RL-equivalent to $(t^3, t^{3k+1} \pm t^{3k+2+3i}, 0)$ and $(t^3, t^{3k+2} \pm t^{3k+4+3i}, 0)$);

• if $i = p \leq k-1$ then the curves $E^{\pm}_{6k,p,i}$ and $E^{\pm}_{6k+2,p,i}$ are quasi-homogeneous, but not planar (these curves are RL-equivalent to $(t^3, t^{3k+1}, t^{3k+2+3p})$ and $(t^3, t^{3k+2}, t^{3k+4+3p})$);

• in the remaining case $i < p \leq k-1$ the curves $E^{\pm}_{6k,p,i}$ and $E^{\pm}_{6k+2,p,i}$ are neither planar, nor quasi-homogeneous.

In the cases (a) $p = k$ and (b) $i = p$ Proposition 7.1 is a corollary of Theorem 1.3 and the given above properties of the **E** singularities. In the remaining case $i < p \leq k-1$ Proposition 7.1 follows from Proposition 2.1. □

Consider now the following curves of the class **E** that are Legendrain with respect to the contact structure $(dz - ydx)$:

$\tilde{E}^{\pm}_{6k,i+1,i}: \quad x = t^3, \ y = t^{3k+1} \pm t^{3k+2+3i}, \ z = \int_0^t y(s)x'(s)$;

$\tilde{E}^{\pm}_{6k+2,i+1,i}: \quad x = t^3, \ y = t^{3k+2} \pm t^{3k+4+3i}, \ z = \int_0^t y(s)x'(s)$,

where $k \geq 1, \ 0 \leq i \leq k-1$.

Theorem 7.2. (real analytic and C^∞ categories). *Any germ $\psi \in$**E** of a Legendrain curve in the contact space $(\mathbb{R}^3, dz - ydx))$ is contactomorphic to one of the curves $\tilde{E}^{\pm}_{6k,i+1,i}, \tilde{E}^{\pm}_{6k+2,i+1,i}$, where $k \geq 1, 0 \leq i \leq k-1$. Within these curves no two are contactomorphic up to reduction $\pm \to +$ in the given above remark. The curves $\tilde{E}^{\pm}_{6k,k,k-1}, \tilde{E}^{\pm}_{6k+2,k,k-1}$ are planar and quasi-homogeneous. The curves $\tilde{E}^{\pm}_{6k,i+1,i}, \tilde{E}^{\pm}_{6k+2,i+1,i}$ with $i \leq k-2$ are neither planar nor quasi-homogeneous.*

Theorem 7.2 follows from Proposition 7.1 and Theorem 1.1: it suffices to observe that the curves $\tilde{E}^{\pm}_{6k,i+1,i}$ and $\tilde{E}^{\pm}_{6k+2,i+1,i}$ can be reduced to $E^{\pm}_{6k,i+1,i}$ and $E^{\pm}_{6k+2,i+1,i}$ by a change of the form $z \to k_1 z + k_2 xy$ with suitable k_1, k_2.

8. Contact-simple germs of Legendrain curves

Denote by $MG^m(\mathbb{R}^n)$ the space of mulrigerms $(\gamma_1, ..., \gamma_m)$, where $\gamma_i : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ are germs of curves in $\mathbb{R}^n$.

Definition. A multigerm $\gamma \in MG^m(\mathbb{R}^n)$ is called simple in $MG^m(\mathbb{R}^n)$ if there exist a set $N \subset MG^m(\mathbb{R}^n)$ consisting of a finite number of multigerms and a finite k such that any multigerm in $MG^m(\mathbb{R}^n)$ whose k-jet is sufficiently close to the k-jet of γ is diffeomorphic to one of the multigerms of the set N.

The simple multigerms in $MG^m(\mathbb{R}^n)$ were determined and classified in [8] for $m = 1, n = 2$, in [9] for $m = 1, n = 3$, and recemtly in [13] for any m and n.

Notation. Given a contact structure (α) on $\mathbb{R}^3$ denote by $Leg^m(\mathbb{R}^3, (\alpha))$ the subclass of $MG^m(\mathbb{R}^3)$ consisting of multigerms of Legendrain curves in the contact space $(\mathbb{R}^3, (\alpha))$.

Definition. Let (α) be a contact structure on $\mathbb{R}^3$. A multigerm $\gamma \in Leg^m(\mathbb{R}^3, (\alpha))$ is called contact-simple in $Leg^m(\mathbb{R}^3, (\alpha))$ if there exists a set $N \subset Leg^m(\mathbb{R}^3, (\alpha))$ consisting of a finite number of multigerms and a finite k such that any multigerm in $Leg^m(\mathbb{R}^3, (\alpha))$ whose k-jet is sufficiently close to the k-jet of γ is contactomorphic to one of the multigerms of the set N.

In this section we use results of Section 1 in order to determine and classify all RL-contact-simple germs in $Leg^1(\mathbb{R}^3, (\alpha))$. Most of results of the present section were obtained in [17] by a different method.

Theorem 8.1. *Let (α) be any contact structure on $\mathbb{R}^3$. A germ $\psi \in Leg^1(\mathbb{R}^3, (\alpha))$ is contact-simple in $Leg^1(\mathbb{R}^3, (\alpha))$ if and only if it is simple in $MG^1(\mathbb{R}^3)$.*

Conjecture. The same holds when replacing $Leg^1(\mathbb{R}^3, (\alpha))$ and $MG^1(\mathbb{R}^3)$ by $Leg^m(\mathbb{R}^3, (\alpha)$ and $MG^m(\mathbb{R}^3)$, for any $m \geq 1$.

The part "if" of Theorem 8.1 and of the conjecture follow from Theorem 1.1. The part "only if" of Theorem 8.1 can be easily reduced to the following statement.

Proposition 8.2. *Let (α) be any contact structure on $\mathbb{R}^3$. Let $\psi_1, \psi_2 \in Leg^1(\mathbb{R}^3, (\alpha))$ be germs of the form*

$$\psi_1 = (t^4, t^9, 0) + o(t^9), \quad \psi_2 = (t^5, t^6, 0) + o(t^6).$$

The germs ψ_1 and ψ_2 are not contact-simple in $Leg^1(\mathbb{R}^3, (\alpha))$.

Proof. It is easy to check that the condition $\psi_1, \psi_2 \in Int(\mathbb{R}^3)$ implies that these germs have the following form:

$$\psi_1 = (t^4, t^9 + a_1 t^{10} + a_2 t^{11}, 0) + o(t^{11}; \quad \psi_2 = (t^5, t^6 + b_1 t^7 + b_2 t^8 + b_3 t^9, 0) + o(t^9).$$

The assumption $\psi_1, \psi_2 \in Leg^1(\mathbb{R}^3, (\alpha))$ implies that the contact 1-form $\alpha = Adx + Bdy + Cdz$ satisfies the condition $C(0) \neq 0$ (here x, y, z are the first, the second, and the third coordinates.) Using this observation it is easy to show that there exist families $\psi_{1,\epsilon}, \psi_{2,\epsilon} \in Leg^1(\mathbb{R}^3, (\alpha))$ of the form

$$\psi_{1,\epsilon} = (t^4,\ t^9 + a_1 t^{10} + (a_2 + \epsilon) t^{11},\ t^{12} \cdot f_1(\epsilon, t));$$
$$\psi_{2,\epsilon} = (t^5,\ t^6 + b_1 t^7 + b_2 t^8 + (b_3 + \epsilon) t^9,\ t^{10} \cdot f_2(\epsilon, t)),$$

where $f_1(\epsilon, t)$ and $f_2(\epsilon, t)$ are smooth functions. Now Proposition 8.2 is a corollary of the following classification results (see [8] and [9]): if ϵ and $\tilde{\epsilon}$ are small enough then the 11-jets $j^{11}\psi_{1,\epsilon}$ and $j^{11}\psi_{1,\tilde{\epsilon}}$ (respectively the 9-jets $j^9\psi_{2,\epsilon}$ and $j^9\psi_{2,\epsilon}$) are RL-equivalent if and only if $\epsilon = \tilde{\epsilon}$. □

Theorem 8.1 and results in [9] imply the following corollary.

Theorem 8.3. *A germ $\psi \in Leg^1(\mathbb{R}^3, (\alpha))$ is contact-simple in $Leg^1(\mathbb{R}^3, (\alpha))$ if and only if it it satisfies each of the following requirements: (a) the 4-jet of ψ is not zero; (b) the 9-jet of ψ is not RL-equivalent to (t^4, t^9) or $(t^4, 0)$; (c) the Taylor series of ψ is not RL-equivalent to one of the series $(t^2, 0, 0), (t^3, 0, 0), (t^4, t^6, 0)$.*

In order to obtain the classification of the contact-simple curves in $Leg^1(\mathbb{R}^3, (\alpha)$ one has to take out from the list of simple singularities in $MG^1(\mathbb{R}^3)$ all curves which are not Legendrain with respect to any contact structure.

Theorem 8.4. *Let (α) be any contact structure on $\mathbb{R}^3$ and let $\psi \in Leg^1(\mathbb{R}^3, (\alpha))$ be a non-immersed contact-simple germ in $Leg^1(\mathbb{R}^3, (\alpha))$. Then one has the following alternative:*

(a) the germ ψ is planar, quasi-homogeneous and RL-equivalent to one and only one of the germs

$(t^2, t^{2k+1}, 0),\ (t^3, t^{3k+1}, 0),\ (t^3, t^{3k+2}, 0),\ (t^4, t^5, 0),\ (t^4, t^7, 0),\quad k \geq 1;$

(b) the germ ψ is neither planar nor quasihomogeneous and RL-equivalent to one and only one of the germs

$(t^3,\ t^{3k+1} \pm t^{3k+2+3i},\ t^{3k+5+3i}),\quad k \geq 2,\ i \in \{0,...,k-2\},\quad \pm \hookrightarrow +$ *if i is odd;*
$(t^3,\ t^{3k+2} \pm t^{3k+4+3i},\ t^{3k+7+3i}),\quad k \geq 2,\ i \in \{0,...,k-2\},\quad \pm \hookrightarrow +$ *if i is even;*
$(t^4,\ t^6 + t^{2k+5},\ t^{2k+9}),\quad k \geq 1;$
$(t^4,\ t^5 \pm t^7, t^{11}),\ (t^4, t^7 \pm t^9, t^{13}),\ (t^4,\ t^7 \pm t^{13},\ t^{17}).$

Proof. For curve germs with non-zero 3-jet this theorem was already proved in Sections 6, 7. Consider curve germs with the 4-jet $(t^4, 0, 0)$ and satisfying requirements b) - d) of Theorem 8.3. By classification results in [9] any such curve germ ψ satisfies one of the following conditions: a) ψ is planar, but not quasi-homogeneous; b) ψ is quasi-homogeneous, but not planar; c) ψ is planar, quasi-homogeneous and RL-equivalent to one and only one of the curves $(t^4, t^5, 0)$, $(t^4, t^7, 0)$; d). ψ is neither planar, nor quasi-homogeneous and RL-equivalent to one and only one of the following curves:

$$(t^4, t^6 + t^{2k+5}, t^{2k+7}),\ (t^4, t^6 + t^{2k+5}, t^{2k+9}),\ (t^4, t^6 + t^{2k+5}, t^{2k+13}),\ \ k \geq 1;$$
$$(t^4, t^5 \pm t^7, t^6),\ (t^4, t^5 \pm t^7, t^{11}),\ (t^4, t^7, t^9 + t^{10});$$
$$(t^4, t^7 \pm t^9, t^{10}),\ (t^4, t^7 \pm t^9, t^{13}),\ (t^4, t^7 \pm t^9, t^{17}),\ (t^4, t^7 \pm t^{13}, t^{17}).$$

In case c) $\psi \in Leg(\mathbb{R}^3)$. By Theorem 1.3 in cases a) and b) $\psi \notin Leg(\mathbb{R}^3)$. In order to distinguish germs in $Leg(\mathbb{R}^3)$ within case d) we use Proposition 2.1. By this proposition the curves $(t^4, t^6 + t^{2k+5}, t^{2k+9})$, $(t^4, t^5 \pm t^7, t^{11})$, $(t^4, t^7 \pm t^9, t^{13})$, $(t^4, t^7 \pm t^{13}, t^{17})$ belong to $Leg(\mathbb{R}^3)$ and the other displayed curves do not belong to $Leg(\mathbb{R}^3)$. Now Theorem 8.4 follows from Theorem 1.1. □

Now we establish a 1-1 correspondence between the contact classification of contact-simple curves and the RL-classification of simple germs in $MG^1(\mathbb{R}^2)$ (i.e. germs of plane curves.) The latter classification was obtained in [8]. Any simple germ of a plane curve is RL equivalent to one and only one of the following curves:

$$(t^2, t^{2k+1}),\ (t^3, t^{3k+1}),\ (t^3, t^{3k+2}),\ (t^4, t^5),\ (t^4, t^7); \tag{8.1}$$
$$(t^3, t^{3k+1} \pm t^{3k+2+3i}),\ k \geq 2,\ i \in \{0, ..., k-2\},\ \pm \hookrightarrow + \text{ if } i \text{is odd}; \tag{8.2}$$
$$(t^3, t^{3k+2} \pm t^{3k+4+3i}),\ k \geq 2,\ i \in \{0, ..., k-2\},\ \pm \hookrightarrow + \text{ if } i \text{ is even}; \tag{8.3}$$
$$(t^4, t^6 + t^{2k+5}),\ \ (t^4, t^5 \pm t^7),\ (t^4, t^7 \pm t^9),\ (t^4, t^6 \pm t^{13}),\ k \geq 1 \tag{8.4}$$

The correspondence between the normal forms in Theorem 8.4 and normal forms (10.1)-(10.4) is *not* canonical. It requires to fix a local coordinate system in which the contact structure (α) has the Darboux normal form $(dz - ydx)$.

Notations. Given a germ $\psi = (x(t), y(t), z(t)) \in Leg^1(\mathbb{R}^3, (dz - ydx))$ denote by $\pi_{x,y}(\psi)$ the plane curve germ $(x(t), y(t))$. Given a plane curve germ $\mu : (x(t), y(t))$ denote by $\pi^{-1}_{x,y}(\mu)$ the (unique) germ $\psi \in Leg^1(\mathbb{R}^3, (dz - ydx)$ such that $\pi_{x,y}\psi = \mu$.

Theorem 8.5.
1. A germ $\psi \in Leg^1(\mathbb{R}^3, (dz - ydx))$ is contact-simple in $Leg^1(\mathbb{R}^3, (dz - ydx))$ if and only if the plane curve germ $\pi_{x,y}(\psi)$ is simple in $MG^1(\mathbb{R}^3)$.

2. Let $\psi, \tilde{\psi} \in Leg^1(\mathbb{R}^3, (dz - ydx))$ be contact-simple germs in $Leg^1(\mathbb{R}^3, (dz - ydx))$. The germs $\psi, \tilde{\psi}$ are contactomorphic if and only if the plane curve germs $\pi_{x,y}(\psi)$, $\pi_{x,y}(\tilde{\psi})$ are RL-equivalent;

3. If $\psi \in Leg^1(\mathbb{R}^3, (dz - ydx))$ is a contact-simple non-immersed germ in $Leg^1(\mathbb{R}^3, (dz - ydx))$ then one has the following alternative:

(a) ψ is planar, quasi-homogeneous and contactomorphic to one and only one germ of the form $\pi_{x,y}^{-1}(\mu)$, where μ is a germ in the list (8.1).

(b) ψ is neither planar, nor quasi-homogeneous and contactomorphic to one and only one germ of the form $\pi_{x,y}^{-1}(\mu)$, where μ is a germ in the list (8.2)-(8.4).

Proof. Theorem 8.5 is a corollary of Theorem 8.4, Theorem 1.1, classification results for plane curves in [8] and the following two statements following from the classification results for space curves in [9]:

a). any curve given in Theorem 8.4 is RL-equivalent to a curve $\pi_{x,y}^{-1}(\mu)$, where μ is one of the curves in the list (8.1) - (8.4);

b). if μ is a curve in the list (8.1) - (8.4) then the curve $\pi_{x,y}^{-1}(\mu)$ is RL-equivalent to one of the curves given in Theorem 8.4 □

9. Contact classification of multigerms consisting of a cusp and a non-singular curve.

In this section we use results of section 1 in order to classify, in a fixed contact 3-space, multigerms $\gamma = (\gamma_1, \gamma_2)$, where γ_2 is a non-singular curve and γ_1 is a singular curve with the cusp singularity - a curve diffeomorphic to $A_{2k} : (t^2, t^{2k+1}, 0)$.

Notation. The class of such multigerms will be denoted $(\mathbf{A}_{2k},\mathbf{l})$.

Let us start with the case $k = 1$. The RL-classification of multigerms in $(\mathbf{A}_2,\mathbf{l})$ is described by the following normal forms obtained in [13]:

a). $\gamma_1 : (t^2, t^3, 0), \quad \gamma_2 : (0, 0, t)$ b). $\gamma_1 : (t^2, t^3, 0), \quad \gamma_2 : (0, t, 0)$

c). $\gamma_1 : (t^2, t^3, 0), \quad \gamma_2 : (t, 0, t^2)$ d). $\gamma_1 : (t^2, t^3, 0), \quad \gamma_2 : (t, 0, 0)$

In order to distinguish these normal forms denote by $L(\gamma_1) \subset T_0\mathbb{R}^3$ the tangent plane to a non-singular surface containing the image of γ_1 (all such non-singular surfaces have the same tangent plane at 0) and by $l(\gamma_1) \subset L(\gamma_1)$ the 1-dimensional subspace of $L(\gamma_1)$ which is the limit tangent line to γ_1. If γ_1 is given by the normal form $x = t^2, y = t^3, z = 0$ then $L(\gamma_1)$ is spanned by the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and $l(\gamma_1)$ is spanned by the vector $\frac{\partial}{\partial x}$. Normal form a) holds if γ_2 is transversal to the plane $L(\gamma_1)$. If γ_2 is tangent to $L(\gamma_1)$, but not tangent to $l(\gamma_1)$ then one has normal form b). Normal forms c) and d) hold if γ_2 is tangent to $l(\gamma_1)$.

Normal forms a) - d) imply that any multigerm $\gamma \in (\mathbf{A}_2,\mathbf{l})$ is quasi-homogeneous (the weights are $(2, 3, 1)$ in the case of normal forms a), b), d) and $(2, 3, 4)$ in the case of normal form (d)). It is easy to check that multigerms a) and c) are not planar. Therefore $b), d) \in Leg(\mathbb{R}^3)$ and by Theorem 1.3 $a), c) \notin Leg(\mathbb{R}^3)$. . We obtain the following statement.

Proposition 9.1.

1. Any multigerm of the class $(\mathbf{A}_2,\mathbf{l}) \cap Leg(\mathbb{R}^3)$ is planar and quasi-homogeneous.

2. Let $\gamma = (\gamma_1, \gamma_2) \in (\mathbf{A}_2,\mathbf{l}) \cap Leg(\mathbb{R}^3)$. If γ_2 is not tangent to the limit tangent line $l(\gamma_1)$ then γ is RL-equivalent to multigerm b). If γ_2 is tangent to the limit tangent line $l(\gamma_1)$ then γ is RL-equivalent to multigerm d).

Introduce the multigerms

$$\gamma_{NT}: \quad \gamma_1: \ x=t^2, \ y=t^3, \ z=2t^5/5, \qquad \gamma_2: \ x=0, \ y=t, \ z=0;$$
$$\gamma_T: \quad \gamma_1: x=t^2, \ y=t^3, \ z=2t^5/5, \qquad \gamma_2: x=t, \ y=0, \ z=0.$$

These multigerms are Legendrain with respect to the contact structure $(dz - ydx)$. They can be brought to multigerms b) and d) respectively by a change of coordinates $(x,y,z) \to (x,y,z+rxy)$ with a suitable $r \in \mathbb{R}$. Therefore Theorem 1.1 along with Proposition 9.1 imply the following classification result.

Theorem 9.2. (real analytic and C^∞ categories). *Let $\gamma = (\gamma_1, \gamma_2) \in$* $(\mathbf{A}_2,\mathbf{l})$ *be an Legendrain multigerm in a fixed contact space $(\mathbb{R}^3, (dz - ydx))$. Then γ is contactmorphic to one of the multigerms γ_{NT}, γ_T. The normal form γ_{NT} (respectively γ_T) holds if γ_2 is not tangent (respectively tangent) to the limit tangent line $l(\gamma_1)$.*

The contact classification of multigerms of classes $(\mathbf{A}_{2k},\mathbf{l})$, $k \geq 2$ is more involved: if $k \geq 2$ then the set $(\mathbf{A}_{2k},\mathbf{l}) \cap Leg(\mathbb{R}^3)$ contains non-planar multigerms.

Any multigerm $\gamma = (\gamma_1, \gamma_2) \in (\mathbf{A}_{2k},\mathbf{l})$ is RL-equivalent to a multigerm of one of the following forms, see [13]:

A. $\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: \ (0,0,t);$
B. $\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: \ (0,t,0);$
C. $\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: \ (t, t^i, t^d), \ \ 2 \leq i \leq 2k-1, \ d \geq i+1;$
D. $\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: \ (t, 0, t^d), \ d \geq 2.$

Normal form A holds if γ_2 is transversal to $L(\gamma_1)$, the normal form B holds if γ_2 is tangent to $L(\gamma_1)$, but not tangent to $l(\gamma_1)$. If γ_2 is tangent to $l(\gamma_1)$ then one has one of the normal forms C, D. Here $L(\gamma_1)$ and $l(\gamma_1)$ are the plane and the line in $T_0\mathbb{R}^3$ defined in the same way as in the case $k = 1$.

Multigerms A, B and D are quasi-homogeneous (the weights are $(2, 2k+1, 1)$ in cases A and B and $(2, 2k+1, d(2k+1))$ in case D). Multigerm A is not planar. Multigerm B is planar. One can check that multigerm D is planar if and only if $d \geq 2k+1$. In this case and in this case only D is RL-equivalent to the multigerm $\gamma_1: \ (t^2, t^{2k+1}, 0), \ \ \gamma_2: \ (s, 0, 0)$. By Corollary 1.3 one has $A \notin Leg(\mathbb{R}^3)$, $B \in Leg(\mathbb{R}^3)$ and $D \in Leg(\mathbb{R}^3)$ if and only if $d \geq 2k+1$.

Lemma 9.3. *$C \in Leg(\mathbb{R}^3)$ if and only if $d = i+1$.*

This lemma is proved below. Now we have a complete description of the class $(\mathbf{A}_{2k},\mathbf{l}) \cap Leg(\mathbb{R}^3)$.

Proposition 9.4. *A multigerm $\gamma \in$* $(\mathbf{A}_{2k},\mathbf{l})$ *belongs to $Leg(\mathbb{R}^3)$ if and only if γ is RL-equivalent to one of the multigerms*

$$\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: (0,t,0); \tag{9.1}$$
$$\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: (t, t^i, t^{i+1}), \quad 2 \leq i \leq 2k-1; \tag{9.2}$$
$$\gamma_1: \ (t^2, t^{2k+1}, 0), \quad \gamma_2: (t,0,0). \tag{9.3}$$

These normal forms can be joined to the following one-index family:

$$(A_{2k},l))^i: \quad \gamma_1: \ (t^2,t^{2k+1},0), \quad \gamma_2: \ (t,t^i,t^{i+1}), \quad 1 \le i \le 2k$$

(the boundary values of the index i correspond to (9.1) and (9.3): $(A_{2k},l))^1$ is RL-equivalent to (9.1) and $(A_{2k},l))^{2k}$ is RL-equivalent to (9.3)). Introduce the following multigerms:

$$(A_{2k},l))^i_*, \quad 1 \le i \le 2k: \qquad \begin{array}{l} \gamma_1: \ x = t^2, \ y = t^{2k+1}, \ z = 2t^{2k+3}/(2k+3), \\ \gamma_2: \ x = t, \ y = t^i, \ z = t^{i+1}/(i+1). \end{array}$$

These multigerms are Legendrain with respect to the contact structure $(dz - ydx)$.

Theorem 9.5. (real analytic and C^∞ categories). *Let $\gamma = (\gamma_1,\gamma_2) \in$*$(\mathbf{A}_{2k},\mathbf{l})$ *be a Legendrain multigerm in a fixed contact space $(\mathbb{R}^3, (dz - ydx))$. Then γ is contactmorphic to one and only one of the multigerms $(A_{2k},l))^i_*$, $1 \le i \le 2k$.*

Theorem 9.5 is a corollary of Proposition 9.4 and Theorem 1.1 because the multigerms $(A_{2k},l))^i_*$ can be reduced to $(A_{2k},l))^i$ by a change of coordinates $(x,y,z) \to (x,y,r_1z + r_2xy)$ with suitable r_1, r_2.

Proof of Lemma 9.3. If $d = i+1$ then $C \in Leg(\mathbb{R}^3)$ because in this case multigerm C is RL-equivalent to the multigerm $(A_{2k},l)^*_i$ which is Legendrian with respect to the contact structure $(dz - ydx)$.

Now we prove that if $d > i+1$ then $C \notin Leg(\mathbb{R}^3)$. Assume, to get contradiction, that multigerm C is Legendrain with respect to contact structure $(Adx+Bdy+Cdz)$. It is easy to see that in this case $A(0) = B(0) = 0$ and therefore we may assume that $C \equiv 1$. Then the function germs $A(x,y,z)$ and $B(x,y,z)$ satisfy the relations

$$2A(t^2,t^{2k+1},0) + (2k+1)t^{2k-1}B(t^2,t^{2k+1},0) \equiv 0, \tag{9.4}$$

$$A(t,t^i,t^d) + it^{i-1}B(t,t^i,t^d) + dt^{d-1} \equiv 0. \tag{9.5}$$

Relation (9.4) implies

$$2\frac{\partial A}{\partial y}(0) + (2k+1)\frac{\partial B}{\partial x}(0) = 0, \quad \frac{\partial A}{\partial x}(0) = \frac{\partial^2 A}{\partial x^2}(0) = \cdots = \frac{\partial^{2k-1} A}{\partial x^{2k-1}}(0) = 0. \tag{9.6}$$

Calculate the coefficient at t^i in (9.5). Using the second relation in (9.6) and the constrains $2 \le i \le 2k-1$, $d > i+1$ we obtain

$$\frac{\partial A}{\partial y}(0) + i \cdot \frac{\partial B}{\partial x}(0) = 0. \tag{9.7}$$

Relation (9.7) and the first relation in (9.6) imply $\frac{\partial A}{\partial y}(0) = \frac{\partial B}{\partial x}(0) = 0$. In this case the 1-form $\alpha = Adx + Bdy + dz$ is not contact: $(\alpha \wedge d\alpha)(0) = 0$.

10. Contact classification of multigerms consisting of 3 non-singular components

This section is devoted to contact classification of multigerms $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ consisting of 3 non-singular components.

Notation. Given two germs $A, B : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ denote by $ord(A, B)$ the order of tangency between A and B (if A and B are not tangent then $ord(A, B) = 0$).)

We restrict ourselves to the following singularity classes:

$\mathbf{T}_{0,d}$: $ord(\gamma_1, \gamma_2) = 0$, $ord(\gamma_1, \gamma_3) = d \geq 0$ (up to permutation of the components);

$\mathbf{T}_{1,1}$: $ord(\gamma_1, \gamma_2) = ord(\gamma_1, \gamma_3) = ord(\gamma_2, \gamma_3) = 1$.

Proposition 10.1.

1. *Let $\gamma \in \mathbf{T}_{0,d} \cup \mathbf{T}_{1,1}$, $d \geq 0$. Then $\gamma \in Leg(\mathbb{R}^3)$ if and only if γ is planar.*
2. *Any multigerm $\gamma \in \mathbf{T}_{0,d} \cap Leg(\mathbb{R}^3)$, $d \geq 0$ is RL-equivalent to the multigerm*
$$T_{0,d} : \ (t, 0, 0), \ \ (0, t, 0), \ \ (t, t^{d+1}, 0), \ \ d \geq 0.$$
3. *Any multigerm $\gamma \in \mathbf{T}_{1,1} \cap Leg(\mathbb{R}^3)$ is RL-equivalent to the multigerm*
$$T_{1,1} : \ (t, 0, 0), \ \ (t, t^2, 0), \ \ (t, bt^2, 0), \ \ b \notin \{0, 1\}.$$

The parameter b in $T_{1,1}$ is a modulus. Before proving Proposition 10.1 we present its corollary on Legendrian multigerms $\gamma \in \mathbf{T}_{0,d} \cup \mathbf{T}_{1,1}$ in the contact space $(\mathbb{R}^3, (dz - ydx))$. Consider the following Legendrain multigerms in this contact space:

$$\tilde{T}_{0,d} : \ \gamma_1 : \ x = t, \ y = z = 0, \quad \gamma_2 : x = 0, \ y = t, \ z = 0,$$
$$\gamma_3 : \ x = t, \ y = t^{d+1}, \ z = \tfrac{t^{d+2}}{d+2};$$
$$\tilde{T}_{1,1} : \ \gamma_1 : \ x = t, \ y = z = 0, \quad \gamma_2 : \ x = t, \ y = t^2, \ z = \tfrac{t^3}{3},$$
$$\gamma_3 : \ x = t, \ y = bt^2, \ z = \tfrac{bt^3}{3}, \ \ b \notin \{0, 1\}.$$

These multigerms can be reduced to $T_{0,d}$ and $T_{1,1}$ by a change of coordinates $z \to z + kxy$ with a suitable k. Therefore Proposition 10.1 and Theorem 1.1 imply the following classification result.

Theorem 10.2. (real analytic and C^∞ categories). *Let γ be an Legendrain multigerm of the class $\mathbf{T}_{0,d}$ (respectively $\mathbf{T}_{1,1}$) in the contact space $(\mathbb{R}^3, (dz - ydx))$. Then γ is contactomorphic to the multigerm $\tilde{T}_{0,d}$ (respectively $\tilde{T}_{1,1}$).*

Proof of Proposition 10.1. Consider at first the class $\mathbf{T}_{0,d}$. Any multigerm $\gamma \in \mathbf{T}_{0,d}$ is RL-equivalent to a multigerm of the form

$$(t, 0, 0), \ (0, t, 0), \ (t, t^{d+1}b(t), \ t^{d+1}c(t)), \quad (b(0), c(0)) \neq (0, 0). \tag{10.1}$$

Assume that $c(0) = 0$. Then $b(0) \neq 0$. A change of coordinates $(x, y, z) \to (x, ky + xyf_1(x)z, z + xyf_2(x))$ and a reparameterization $t \to k^{-1}t$ of the second component with suitable $f_1(x), f_2(x)$ and $k \neq 0$ reduces $b(t)$ to 1 and $c(t)$ to 0. We obtain the multigerm $T_{0,d} \in Leg(\mathbb{R}^3)$.

Consider now the case $c(0) \neq 0$. It is easy to prove that in this case multigerm (10.1) is not planar. On the other hand, it is quasi-homogeneous: a change of coordinates $(x, y, z) \to (x, y + zf_1(x), zf_2(x))$ with suitable $f_1(x), f_2(x), f_2(0) \neq 0$, reduces $b(r)$ to 0 and $c(r)$ to 1. By Theorem 1.3 (10.1) $\notin Leg(\mathbb{R}^3)$.

Now we will prove Proposition 10.1 for the class $\mathbf{T}_{1,1}$. Any multigerm $\gamma \in \mathbf{T}_{1,1}$ is RL-equivalent to a multigerm of the form

$$(t,0,0),\ (t,t^2,0),\ (t,t^2b(t),\ t^2c(t)),\quad (b(0),c(0)) \notin (0,0)\cup(1,0). \tag{10.2}$$

The condition $(b(0),c(0)) \neq (0,0)$ (respectively $(b(0),c(0)) \neq (1,0)$) means that the order of tangency between the first and the third component (respectively between the second and the third component) is equal to 1.

Lemma 10.3. *If $c(0) \neq 0$ then (10.2) is RL-equivalent to the multigerm*

$$(t,0,0),\ (t,t^2,0),\ (t,0,t^2). \tag{10.3}$$

If $c(0) = 0$ then (10.2) is RL-equivalent either to the multigerm $T_{1,1}$ or to the multigerm of the form

$$(t,0,0),\ (t,t^2,0),\ (t,bt^2,t^3),\quad b \notin \{0,1\}. \tag{10.4}$$

It is easy to prove that multigerms (10.3) and (10.4) are quasi-homogeneous and not planar. Therefore Proposition 10.1 follows from Lemma 10.3 and Theorem 1.3.

Proof of Lemma 10.3. If $c(0) \neq 0$ in (10.2) then $b(t)$ can be reduced to 0 and $c(t)$ to 1 by a change of coordinates of the form $(x,y,z) \to (x, y + zf_1(x), zf_2(x))$ with suitable $f_1(x), f_2(x), f_2(0) \neq 0$.

Assume now that in (10.2) one has $c(0) = 0$. Then $b(0) \notin \{0,1\}$. Consider the plane curve multigerm $(t,0),(t,t^2),(t,t^2b(t))$. One can prove that under the assumption $b(0) \notin \{0,1\}$ it is RL-equivalent to $((t,0),(t,t^2),(t,bt^2))$, where $b = b(0)$ is a modulus. Therefore (10.2) is RL-equivalent to a multigerm of the form

$$(t,0,0),\ (t,t^2,0),\ (t,bt^2,\ t^3\tilde{c}(t)),\quad b \notin \{0,1\}.$$

A change of coordinates $(x,y,z) \to (x,y,z + y(y - x^{i+1})f(x))$ with a suitable $f(x)$ reduces $\tilde{c}(t)$ to the constant $\tilde{c} = \tilde{c}(0)$. If $\tilde{c} = 0$ then we have the multigerm $T_{1,1}$. If $\tilde{c} \neq 0$ then $\tilde{c}$ can be reduced to 1 by scaling z and we obtain multigerm (10.4).

11. Appendix. Legendrian multigerms in the contact space $(\mathbb{R}^3,(dz - ydx))$: the projections to the (x,z) and the (x,y)-plane.

The contact classification of Legendrain singularities in Sections 6-10 is similar to the classification of multigerms of plane curves with respect to the whole group of diffeomorphisms. In the present section we we present two sufficient conditions for the contactomorphness of multigerms γ and $\tilde{\gamma}$ in the contact space $(\mathbb{R}^3,(dz-ydx))$ in terms of their projections to the (x,y) and the (x,z)-plane. These conditions are not necessary. We present examples showing that in general the contact classification of Legendrain multigerms cannot be reduced to the classification of multigerms of plane curves, even in the case of one component.

Denote by $\pi_{x,z}$ and $\pi_{x,y}$ the projections $(x,y,z) \to (x,z)$ and $(x,y,z) \to (x,y)$. Any Legendrain multigerm with components $\gamma_i : x = a_i(t), y = b_i(t), z = c_i(t)$ in the contact space $(\mathbb{R}^3,(dz - ydx))$ is uniquely determined by the projection $\pi_{x,y}(\gamma)$ - a multigerm of a plane curve with components $(a_i(t), b_i(t))$. It is also uniquely determined by the projection $\pi_{x,z}\gamma$ - a plane curve multigerm with components $(a_i(t), c_i(t))$ provided that the curves $(a_i(t), b_i(t)$ have non-zero Taylor series.

Notation. Denote by F the set of Legendrian multigerms in the contact space $(dz - ydx)$ containing a component $(a(t), b(t), c(t)$ such that the curve $(a(t), c(t))$ has the zero Taylor series.

Proposition 11.1. (real analytic and C^∞ categories). *Let γ and $\tilde{\gamma}$ be Legendrain multigerms in the contact space $(\mathbb{R}^3, (dz-ydx))$ beyond the set F. If the projections $\pi_{x,z}(\gamma)$ and $\pi_{x,z}(\tilde{\gamma})$ are RL-equivalent then γ and $\tilde{\gamma}$ are contactomorphic.*

Proof. Let $(x, z) \to (X(x, z), Z(x, z)$ be a local diffeomorphism sending $\pi_{x,z}\gamma$ to $\pi_{x,z}\tilde{\gamma}$ up to reparameterization of the components. The condition that γ and $\tilde{\gamma}$ are Legendrain with respect to the 1-form $dz - ydx$ and the assumption $\gamma, \tilde{\gamma} \notin F$ imply that any component of the projections $\pi_{x,z}\gamma$ and $\pi_{x,z}\tilde{\gamma}$ has up to reparameterization the form $(\pm t^r, o(t^r))$ (r depends on the component). It follows that

$$\frac{\partial X}{\partial x}(0) \neq 0, \frac{\partial Z}{\partial z}(0) = 0. \tag{11.1}$$

Consider the map

$$(x, y, z) \to (X(x, z), Y(x, y, z), Z(x, z)), \quad Y(x, y, z) = \frac{\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial z}}{\frac{\partial X}{\partial x} + y\frac{\partial X}{\partial z}}. \tag{11.2}$$

Conditions (11.1) imply that (11.2) is a well-defined local diffeomorphism $(\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$. One can check that (11.2) is a contactomorphism of the contact structure $(dz-ydx)$. Since $\gamma, \tilde{\gamma} \notin F$ then γ and $\tilde{\gamma}$ are uniquely determined by the projections $\pi_{x,z}(\gamma)$ and $\pi_x, z(\tilde{\gamma})$ and it follows that contactomorphism (11.2) sends the image of γ to the image of $\tilde{\gamma}$. [7] □

On the other hand, if two Legendrain multigerms in the contact space $(\mathbb{R}^3, (dz - ydx))$ beyond the set F are contactomorphic then their projection to the (x, z)-plane do not need to be diffeomorphic.

Example 10.1. Consider the following Legendrain germs of singular curves in the contact space $(\mathbb{R}^3, (dz - ydx))$:

$\psi: \; x = t^3, \; y = t^4, \; z = \frac{3t^7}{7}$,

$\tilde{\psi}: \; x = t^3, \; y = t^4 + t^5, \; z = \frac{3t^7}{7} + \frac{3t^8}{8}$.

The projections of the curves ψ and $\tilde{\psi}$ to the (x, z)-plane are diffeomorphic to (t^3, t^7) and $(t^3, t^7 + t^8)$ respectively. These two curves are not diffeomorphic. On the other hand, the curves ψ and $\tilde{\psi}$ are RL-equivalent to the same curve $(t^3, t^4, 0)$ and by Theorem 1.1 they are contactomorphic.

[7] We should explain how the contactomorphism (11.2) is constructed. We use the formal notation $y = dz/dx$ and write $Y(x, y, z) = dZ/dX = \frac{(\partial Z/\partial x)dx + (\partial Z/\partial z)dz}{(\partial Z/\partial x)dx + (\partial Z/\partial z)dz}$. We obtain (11.2) by dividing the nominator and the denominator by dx. To explain that these operations are legal one should consider the 3-dimensional manifold C^3 of contact elements whose points are (p, l), where p is a point of the (x, z)-plane and l is a 1-dimensional subspace of $T_p\mathbb{R}^2(x, z)$, see [3], [14]. The manifold C^3 is a circle bundle over $\mathbb{R}^2(x, z)$ endowed with a natural contact structure (α) defined as follows: a curve $(p(t), l(t))$ in C^3 is tangent to (α) if $\dot{p}(t) \in l(t)$ for all t. Any local diffeomorphism Φ of $\mathbb{R}^2(x, z)$ can be lifted to a contactomorphism of the manifold $(C^3, (\alpha))$ - the image under Ψ_Φ of a point (p, l) is the point $((\Phi(p), \Phi_* l)$. The contactomorphism (11.2) is the germ of the lifting to C^3 of the diffeomorphism $(x, z) \to (X, Z)$ at the point (p_0, l_0), where $p_0 = (0, 0), l_0 = span\{\frac{\partial}{\partial x}\}$.

Now we consider the projection to the (x, y)-plane.

Proposition 11.2. (real analytic and C^∞ categories). *Let γ and $\tilde{\gamma}$ be Legendrain multigerms in the contact space $(\mathbb{R}^3, (dz - ydx))$. If the projections $\pi_{x,y}(\gamma)$ and $\pi_{x,y}(\tilde{\gamma})$ are RL-equivalent via a local diffeomorphism preserving the volume form $dx \wedge dy$ up to multiplication by a number then γ and $\tilde{\gamma}$ are contactomorphic.*

Proof. Let $(x, y) \to (X(x, y), Y(x, y))$ be a local diffeomorphism satisfying the assumption of Proposition 11.2. Then $dX \wedge dY = rdx \wedge dy$ for some $r \neq 0$ and consequently $YdX = rydx + dH(x, y)$ for some function germ $H(x, y)$. It follows that the map $\Psi : (x, y, z) \to (X(x, y), Y(x, y), rz + H(x, y))$ is a contactomorphism of the contact space $(\mathbb{R}^3, dz - ydx))$. Since γ and $\tilde{\gamma}$ are uniquely defined by their projections to the (x, y)-plane then Ψ sends the image of γ to the image of $\tilde{\gamma}$. □

Remark. One can show that a local diffeomorphism of the (x, y)-plane can be lifted to a local contactomorphism of the contact space $(\mathbb{R}^3, dz - ydx))$ if and *only if* it satisfies the assumption of Proposition 11.2.

In view of Proposition 11.2 we conclude this section with the following examples:

(a) there are contactomorphic Legendrain multigerms in the contact space $(\mathbb{R}^3, (dz - ydx))$ whose projections to the (x, y)-plane are not RL-equivalent;

(b) there are non-contactomorphic Legendrain multigerms in the contact space $(\mathbb{R}^3, (dz - ydx))$ whose projections to the (x, y)-plane are RL-equivalent.

Example 10.2. Consider the germs ψ_0, ψ_1, ψ_2 of Legendrain curve in the contact space $(\mathbb{R}^3, (dz - ydx))$ with the following projections to the (x, y)-plane:

$\pi_{x,y}\psi_0 : \quad x = t^4, \; y = t^{13} + t^{14} + t^{15},$

$\pi_{x,y}\psi_1 : \quad x = t^4, \; y = (t^{13} + t^{14} + t^{15}) \cdot (1 + t^4),$

$\pi_{x,y}\psi_2 : \quad x = t^4, \; y = t^{13} + t^{14} + t^{15} + \frac{21t^{17}}{17} + \frac{22t^{18}}{17} + \frac{23t^{19}}{17}.$

Let us show that the following statements hold:

1. The projections $\pi_{x,y}\psi_0$ and $\pi_{x,y}\psi_1$ are RL-equivalent;
2. The germs ψ_0 and ψ_1 are not RL-equivalent;
3. The germs ψ_0 and ψ_2 are RL-equivalent;
4. The projections $\pi_{x,y}\psi_0$ and $\pi_{x,y}\psi_2$ are not RL-equivalent.

Statement 1. is obvious - the diffeomorphism $(x, y) \to (x, y(1+x))$ sends $\pi_{x,y}(\psi_0)$ to $\pi_{x,y}(\psi_1)$. Statement 3. follows from Proposition 11.1: the projections of ψ_0 and ψ_2 to the (x, z)-plane have the form

$$\pi_{x,z}(\psi_0) : \quad x = t^4, \; z = f(t), \qquad \pi_{x,z}(\psi_2) : \quad x = t^4, \; z = (1 + t^4) \cdot f(t),$$

where $f(t) = 4t^{17}/17 + 4t^{18}/18 + 4t^{19}/19$. These projections are RL-equivalent, therefore the germs ψ_0 and ψ_1 are contactomorphic by Proposition 11.1.

The proofs of statements 2. and 4. are based on calculations. One can prove that the germ $\pi_{x,y}\psi_2$ is RL-equivalent to a germ of the form

$$x = t^4, \; y = y = t^{13} + t^{14} + t^{15} + at^{19} + o(t^{19}), \; a \neq 0,$$

and that a plane curve of this form is not RL-equivalent to the plane curve $\pi_{x,y}\psi_0$. One also can prove that the space curves ψ_0 and ψ_1 are RL-equivalent, respectively, to the curves of the form

$$x = t^4,\ y = t^{13} + t^{14} + t^{15},\ z = t^{18} + at^{19} + o(t^{23}),$$

$$x = t^4,\ y = t^{13} + t^{14} + t^{15},\ z = t^{18} + at^{19} + bt^{23} + o(t^{23}),$$

where $a, b \neq 0$ and that these two curves are not RL-equivalent.

The singularities Example 10.2 are much more degenerate than in Example 10.1. This is in correspondence with results in [17] and Section 8. Denote by S the set of Legndrain germs in the contact space $(\mathbb{R}^3, (dz - ydx))$ such that the the contactomorphness of Legendrain germs $\gamma, \tilde{\gamma}$ beyond S is the same property as the diffeomorphness of their projections to the (x, y)-plane. Theorem 8.5 implies that $codimS \geq 6$ in the space of all non-immersed Legendrain germs. Using Proposition 11.2 and results in [15] one can prove that $codimS$ is substantially bigger than 6.

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