On Two-Parameter Families of Symmetric Matrices

Roy Meshulam

Department of Mathematics
Technion
Haifa 32000, Israel

Submitted by Richard A. Brualdi

ABSTRACT

Let \( f \) be a homogeneous polynomial mapping of degree \( d \) from \( \mathbb{C}^2 \) into the space of complex \( n \times n \) symmetric matrices. It is shown that if \( m \circ f(x) = k \) for all \( 0 \neq x \in \mathbb{C}^2 \), then either \( k \) is even or \( d \) is even.

1. INTRODUCTION

Let \( M_{m \times n}(\mathbb{C}) \) denote the space of complex \( m \times n \) matrices. A mapping \( f : \mathbb{C}^l \rightarrow M_{m \times n}(\mathbb{C}) \) is a \( k \)-mapping if \( \text{rank} f(x) = k \) for all \( 0 \neq x \in \mathbb{C}^l \). The mapping \( f = (f_{ij})_{i=1}^{m} j=1^{n} \) is \( d \)-homogeneous if each of its components \( f_{ij} \) is a homogeneous polynomial of degree \( d \).

Let \( S_n(\mathbb{C}) \) denote the space of complex \( n \times n \) symmetric matrices. Here we show:

**Theorem 1.1.** If \( f : \mathbb{C}^2 \rightarrow S_n(\mathbb{C}) \) is a \( d \)-homogeneous \( k \)-mapping, then either \( d \) is even or \( k \) is even.

The following corollary was proved by R. Westwick (Theorem 3, in [1]), and independently by A. Malek [5]. R. Loewy had proved the corollary for spaces of real symmetric matrices [5].

**Corollary 1.2.** Let \( k \) be odd. If \( W \) is a linear subspace of \( S_n(\mathbb{C}) \) such that \( \text{rank} A = k \) for all \( 0 \neq A \in W \), then \( \text{dim} W \leq 1 \).

Theorem 1.1 is proved in Section 2 using a theorem of Grothendieck on holomorphic vector bundles over the complex projective line.
The following examples show that if either $d$ or $k$ is even and $n$ is sufficiently large, then there exist $d$-homogeneous $k$-mappings $f : \mathbb{C}^2 \to S_n(\mathbb{C})$:

(1) $k = 2k'$. For $n \geq k + 1$ define $f : \mathbb{C}^2 \to S_n(\mathbb{C})$ by $f_{i,i+k'}(s, t) = f_{i,i+k'+1}(s, t) = s^d$ for $1 \leq i \leq k'$, and $f_{ij}(s, t) = 0$ otherwise.

(2) $d = 2d'$. For $n \geq 2k$ define $f : \mathbb{C}^2 \to S_n(\mathbb{C})$ by $f_{2i-1,2i-1}(s, t) = s^d$, $f_{2i,2i}(s, t) = t^d$, and $f_{2i-1,2i}(s, t) = (st)^d$ for $1 \leq i \leq k$, and $f_{ij}(s, t) = 0$ otherwise.

2. $k$-MAPPINGS AND VECTOR BUNDLES

Let $\mathbb{C}P^{t-1}$ denote the $t - 1$-dimensional complex projective space, regarded as a quotient space $\mathbb{C}^t - \{0\} / \sim$, where $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{C}^*$. Let $[x] \in \mathbb{C}P^{t-1}$ denote the image of $x \in \mathbb{C}^t$ under the quotient map.

Let $\mathcal{O}(1)$ denote the hyperplane bundle over $\mathbb{C}P^{t-1}$, and let $\mathcal{O}(-1) = \text{Hom}(\mathcal{O}(1), \mathbb{C})$ denote the tautological line bundle over $\mathbb{C}P^{t-1}$. For $l \geq 1$ let $\mathcal{O}(l) = \mathcal{O}(1)^{\otimes l}$ and $\mathcal{O}(-l) = \mathcal{O}(-1)^{\otimes l}$ (see [2 p. 145]).

The total space of $\mathcal{O}(-l)$ is given by

$$E(\mathcal{O}(-l)) = \{(xL, v_1 \otimes \cdots \otimes v_l) : [x] \in \mathbb{C}P^{t-1}, v_1, \ldots, v_l \in \text{Span}\{x\}\}.$$ 

For a matrix $A \in M_{m \times n}(\mathbb{C})$ let Row $A$ (Col $A$) denote the space spanned by the rows (columns) of $A$.

Let $f : \mathbb{C}^t \to M_{m \times n}(\mathbb{C})$ be a $d$-homogeneous $k$-mapping. We associate with $f$ the holomorphic $k$-plane bundle $\xi(f)$ whose total space is

$$E(\xi(f)) = \{(xL, v) : [x] \in \mathbb{C}P^{t-1}, v \in \text{Row}f(x)\}.$$ 

Let $f^T : \mathbb{C}^t \to M_{n \times m}(\mathbb{C})$ denote the transpose of $f : f^T(x) = f(x)^T$. The total space of $\xi(f^T)$ is $\{(xL, u) : [x] \in \mathbb{C}P^{t-1}, u \in \text{Col}f(x)\}$.

**Proposition 2.1.** Let $f : \mathbb{C}^t \to M_{m \times n}(\mathbb{C})$ be a $d$-homogeneous $k$-mapping, then

$$\xi(f^T) \cong \text{Hom}(\xi(f), \mathbb{C}) \otimes \mathcal{O}(-d).$$

**Proof.** In the following we regard $f(x)$ both as an $m \times n$ matrix and as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$.

The fiber of $\xi(f)$ over $[x] \in \mathbb{C}P^{t-1}$ is $[x] \times \text{Row}f(x)$. Its dual may be identified with the quotient space $[x] \times \mathbb{C}^n / \ker f(x)$ as follows: For a column vector $z = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ let $\bar{z} = z + \ker f(x) \in \mathbb{C}^n / \ker f(x)$. For $v = (v, \ldots, v_n) \in$
Row $f(x)$ let $([x], z)$ act on $([x], v)$ by $v \cdot z = \sum_{i=1}^{n} v_{i}z_{i}$. Thus the total space of the dual bundle $\text{Hom}(\xi(f), C)$ is

$$\{([x], z) : [x] \in \mathbb{CP}^{n-1}, z \in C^{n}/\ker f(x)\},$$

and the total space of $\text{Hom}(\xi(f), C) \otimes \mathcal{O}(-d)$ is

$$\{([x], z \otimes \lambda_{1}x \otimes \cdots \otimes \lambda_{d}x) : [x] \in \mathbb{CP}^{n-1}, z \in C^{n}/\ker f(x), \lambda_{1}, \ldots, \lambda_{d} \in C\}.$$

We now define a vector bundle morphism

$$\varphi : \text{Hom}(\xi(f), C) \otimes \mathcal{O}(-d) \to \xi(f^{T})$$

by

$$\varphi([x], z \otimes \lambda_{1}x \otimes \cdots \otimes \lambda_{d}x) = ([x], \lambda_{1} \cdots \lambda_{d}f(x)z).$$

The $d$-homogeneity of $f$ implies that $\varphi$ is a well-defined isomorphism.

We shall need the following result of Grothendieck [3]:

**Theorem 2.2 (Grothendieck).** For any holomorphic $k$-plane bundle $\xi$ over $\mathbb{CP}^{1}$ there exists a unique sequence of integers $a_{1} \geq \cdots \geq a_{k}$ such that $\xi \cong \bigoplus_{i=1}^{k} \mathcal{O}(a_{i})$.

**Proof of Theorem 1.1.** Let $f : C^{2} \to S_{n}(C)$ be a $d$-homogeneous $k$-mapping, and consider $\xi(f)$, the holomorphic $k$-plane bundle associated with $f$. By Grothendieck's theorem $\xi(f) \cong \bigoplus_{i=1}^{k} \mathcal{O}(a_{i})$ for some $a_{1} \geq \cdots \geq a_{k}$; hence by Proposition 2.1,

$$\bigoplus_{i=1}^{k} \mathcal{O}(a_{i}) \cong \xi(f) = \xi(f^{T}) \cong \text{Hom}(\xi(f), C) \otimes \mathcal{O}(-d) \cong \bigoplus_{i=1}^{k} \mathcal{O}(-a_{i} - d).$$

It follows that $a_{i} = -a_{k+1-i} - d$ for all $1 \leq i \leq k$. Thus if $k$ is odd, then $d = -2a_{k+1/2}$ is even.

**Remarks**

1. It is shown in Hartshorne [4, p. 384, Exercise 2.6] that Grothendieck's theorem generalizes to any algebraically closed field (I am grateful to A. Yekutieli for this information). It follows that Theorem 1.1 is valid over any algebraically closed field.

2. For other applications of vector bundles to spaces of matrices of fixed rank see [7, 8, 6].
REFERENCES

5 R. Loewy, Private communication.

Received 5 February 1993; final manuscript accepted 4 May 1993