Sum Complexes - a New Family of Hypertrees

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Abstract

A k-dimensional hypertree X is a k-dimensional complex on n vertices with a full $(k-1)$-dimensional skeleton and $\binom{n-1}{k}$ facets such that $H_k(X;\mathbb{Q}) = 0$. Here we introduce the following family of simplicial complexes. Let n, k be integers with $k+1$ and n relatively prime, and let A be a $(k+1)$-element subset of the cyclic group $\mathbb{Z}_n$. The *sum complex* X_A is the pure k-dimensional complex on the vertex set $\mathbb{Z}_n$ whose facets are $\sigma \subset \mathbb{Z}_n$ such that $|\sigma| = k+1$ and $\sum_{x\in\sigma} x \in A$. It is shown that if n is prime then the complex X_A is a k-hypertree for every choice of A. On the other hand, for n prime X_A is k-collapsible iff A is an arithmetic progression in $\mathbb{Z}_n$.

1 Introduction

What is the high-dimensional analogue of a tree? Several approaches to this question can be found in the literature. Here we follow the lead of Kalai [1]. We start with some standard notations. All simplicial complexes we consider X have n vertices, and we always identify the vertex set of X with the cyclic group $\mathbb{Z}_n$. The number of i-dimensional faces of X is denoted by $f_i(X)$. We denote by Δ_{n-1} the $(n-1)$-simplex on the vertex set $\mathbb{Z}_n$ and by $\Delta_{n-1}^{(i)}$ the i-dimensional skeleton of Δ_{n-1}. A *k-hypertree* is a simplicial complex $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ such that $f_k(X) = \binom{n-1}{k}$ and with a vanishing

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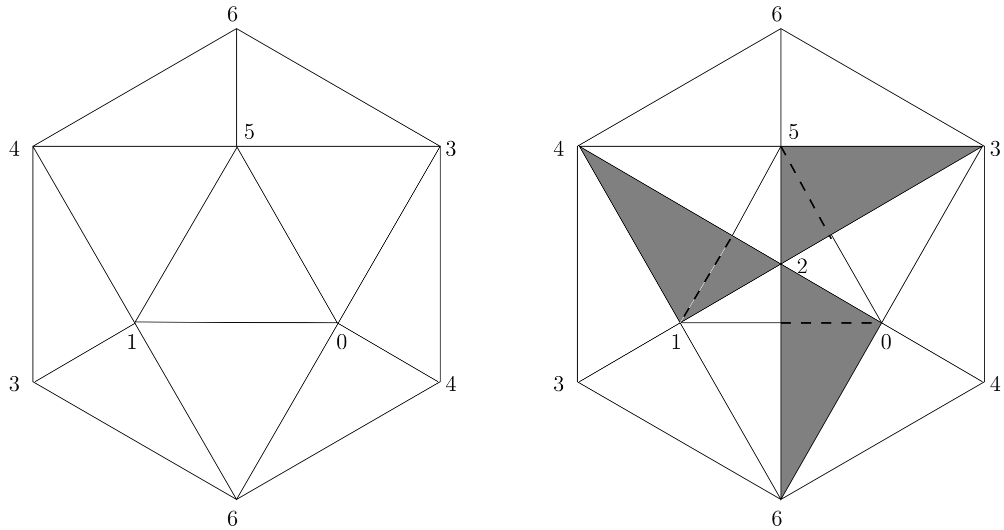


(a) The 6-point triangulation of $\mathbb{RP}^2$ (b) X_A for $A = \{0, 1, 3\} \subset \mathbb{Z}_7$

Figure 1

k-th rational homology $H_k(X; \mathbb{Q}) = 0$. Throughout the paper we assume that $k + 1$ is coprime to n. For $a \in \mathbb{Z}_n$, let X_a be the following collection of subsets of $\mathbb{Z}_n$:

$$X_a = \{\sigma \subset \mathbb{Z}_n : |\sigma| = k + 1 \ , \ \sum_{x \in \sigma} x = a\} \ .$$

For a subset $A \subset \mathbb{Z}_n$ of cardinality $k + 1$, define the *Sum Complex* X_A by

$$X_A = \Delta_{n-1}^{(k-1)} \cup (\cup_{a \in A} X_a) \ .$$

Example: Let $n = 7$, $k = 2$ and $A = \{0, 1, 3\} \subset \mathbb{Z}_7$. The 2-dimensional complex X_A (figure 1b) is obtained from the standard 6-point triangulation of the real projective plane $\mathbb{RP}^2$ on the vertices $\{0, 1, 3, 4, 5, 6\}$ (figure 1a) by replacing the face $\{0, 1, 5\}$ with the three faces $\{0, 1, 2\}$, $\{0, 2, 5\}$, $\{1, 2, 5\}$, and adding the faces $\{2, 3, 5\}$, $\{0, 2, 6\}$ and $\{1, 2, 4\}$. X_A is clearly homotopy equivalent to $\mathbb{RP}^2$.

In this paper we are concerned with topological and combinatorial properties of X_A. Let $\mathbb{F}$ be a field and let $h_i(X_A; \mathbb{F}) = \dim_{\mathbb{F}} H_i(X_A; \mathbb{F})$. Since $X_A \supset \Delta_{n-1}^{(k-1)}$ it follows that $h_0(X_A; \mathbb{F}) = 1$ and $h_i(X_A; \mathbb{F}) = 0$ for $1 \leq i \leq k - 2$. Since $k + 1$ is coprime to n, it follows that for any $y \in \mathbb{Z}_n$, the number of $\sigma \subset \mathbb{Z}_n$ of cardinality $k+1$ that satisfy $\sum_{x\in\sigma} x = y$ is $\frac{1}{n}\binom{n}{k+1}$. Therefore $f_k(X_A) = \frac{k+1}{n}\binom{n}{k+1} = \binom{n-1}{k}$. The Euler-Poincaré relation $\sum_{i\geq 0}(-1)^i f_i(X_A) = \sum_{i\geq 0}(-1)^i h_i(X_A;\mathbb{F})$ then implies that $h_{k-1}(X_A;\mathbb{F}) = h_k(X_A;\mathbb{F})$. In the sequel we assume that the characteristic of $\mathbb{F}$ does not divide n.

Let ω be a fixed primitive n-th root of unity in the algebraic closure $\overline{\mathbb{F}}$. For $x \in \mathbb{Z}_n$ let $e(x) = \omega^x$. The $n \times n$ Fourier matrix M over $\overline{\mathbb{F}}$ is given by $M(u,v) = e(-uv)$ for $u, v \in \mathbb{Z}_n$. For a subset $B \subset \mathbb{Z}_n$ of cardinality $k+1$ let $M_{A,B}$ denote the $(k+1)\times(k+1)$ submatrix of M determined by A and B. Let $\mathcal{B}_{n,k}$ denote the family of all $(k+1)$-element subsets of $\mathbb{Z}_n$ that contain 0.

Theorem 1.1.

$$h_{k-1}(X_A;\mathbb{F}) = h_k(X_A;\mathbb{F}) = \frac{1}{k+1}\sum_{B\in\mathcal{B}_{n,k}} \dim\ker M_{A,B} \ . \tag{1}$$

The Fourier transform matrix $M = (M_{uv})$ of $\mathbb{Z}_n$ over $\overline{\mathbb{Q}} = \mathbb{C}$ is given by $M_{uv} = \exp(-2\pi i uv/n)$. A classical result of Chebotarëv (see e.g. [3]) asserts that if n is prime then any square submatrix of M is nonsingular. Theorem 1.1 therefore implies

Corollary 1.2. *If n is prime then X_A is a k-hypertree.*

If A is an arithmetic progression in $\mathbb{Z}_n$ then $M_{A,B}$ is a Vandermonde matrix for all $B \in \mathcal{B}_{n,k}$. Hence, by Theorem 1.1, X_A is $\mathbb{F}$-acyclic for any $\mathbb{F}$ whose characteristic is coprime to n. More is in fact true. Let σ be a face of dimension at most $k-1$ of a simplicial complex X which is contained in a *unique* maximal face τ of X, and let $[\sigma,\tau] = \{\eta \ : \ \sigma \subset \eta \subset \tau\}$. The operation $X \to Y = X - [\sigma,\tau]$ is called an *elementary k-collapse.* X is *k-collapsible* if there exists a sequence of elementary k-collapses

$$X = X_1 \to X_2 \to \cdots \to X_m = \{\emptyset\} \ .$$

Note that if $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ is k-collapsible and $f_k(X) = \binom{n-1}{k}$, then X is $\mathbb{Z}$-acyclic.

Theorem 1.3. *Let n be a prime and let A be a subset of $\mathbb{Z}_n$ of cardinality $k+1$. Then X_A is k-collapsible iff A is an arithmetic progression in $\mathbb{Z}_n$.*

Theorems 1.1 and 1.3 are proved in Sections 2 and 3. In Section 4 we compute the homology of X_A for $A = \{0,1,3\}$. We conclude in Section 5 with some remarks concerning possible extensions and open problems.

2 Homology of X_A

We first recall some topological terminology (see e.g. [2]). Let X be a finite simplicial complex on the vertex set V. For a set S and a field $\mathbb{K}$, let $\mathcal{L}(S,\mathbb{K})$ denote the $\mathbb{K}$-linear space of all $\mathbb{K}$-valued functions on S. The space $C^m(X;\mathbb{K})$ of $\mathbb{K}$-valued m-cochains of X consists of all functions $\phi \in \mathcal{L}(V^{m+1},\mathbb{K})$ such that $\phi(v_0,\ldots,v_m) = \mathrm{sgn}(\pi)\phi(v_{\pi(0)},\ldots,v_{\pi(m)})$ for any permutation π on $\{0,\ldots,m\}$, and such that $\phi(v_0,\ldots,v_m)=0$ if $\{v_0,\ldots,v_m\}$ is not an m-dimensional simplex of X. (In particular, $\phi(v_0,\ldots,v_m)=0$ if $v_i = v_j$ for some $i \neq j$.) The coboundary operator $d_m : C^m(X;\mathbb{K}) \to C^{m+1}(X;\mathbb{K})$ is given by

$$d_m\phi(v_0,\ldots,v_{m+1}) = \sum_{i=0}^{m+1}(-1)^i\phi(v_0,\ldots,\hat{v}_i,\ldots,v_{m+1}) \quad .$$

Let $Z^m(X;\mathbb{K}) = \ker d_m$ denote the space of m-cocycles of X over $\mathbb{K}$ and let $B^m(X;\mathbb{K}) = \mathrm{Im}\ d_{m-1}$ denote the space of m-coboundaries of X over $\mathbb{K}$. The m-dimensional cohomology space of X with coefficients in $\mathbb{K}$ is

$$H^m(X;\mathbb{K}) = \frac{Z^m(X;\mathbb{K})}{B^m(X;\mathbb{K})} \quad .$$

Let $h^m(X,\mathbb{K}) = \dim_{\mathbb{K}} H^m(X;\mathbb{K})$. Then $h^m(X,\mathbb{K}) = h^m(X,\mathbb{F}) = h_m(X;\mathbb{F})$ for any algebraic extension $\mathbb{K}$ of $\mathbb{F}$. In order to establish Theorem 1.1 we may therefore assume that $\mathbb{F}$ already contains a primitive n-th root of unity ω.

The Fourier transform of a function $\phi \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ is the function $\mathcal{F}(\phi) = \widehat{\phi} \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ given by

$$\widehat{\phi}(u_1,\ldots,u_k) = \sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} \phi(x_1,\ldots,x_k)e(-\sum_{j=1}^{k} u_j x_j) \quad .$$

The Fourier transform is an automorphism of $\mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$.

The proof of Theorem 1.1 involves computing the image of $H^{k-1}(X;\mathbb{F})$ under the Fourier transform. We first consider the Fourier image of the $(k-1)$-coboundaries.

Claim 2.1.

$$\mathcal{F}(B^{k-1}(X_A;\mathbb{F})) = \{g \in C^{k-1}(X_A;\mathbb{F}) : \mathit{support}(g) \subset \mathbb{Z}_n^k - (\mathbb{Z}_n - \{0\})^k\} \quad .$$

Proof: Let $\psi \in C^{k-2}(X_A;\mathbb{F})$. Then

$$\widehat{d_{k-2}\psi}(u_1,\ldots,u_k) = \sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} d_{k-2}\psi(x_1,\ldots,x_k)e(-\sum_{j=1}^{k} u_jx_j) =$$

$$\sum_{(x_1,\ldots,x_k)\in\mathbb{Z}_n^k} \big(\sum_{i=1}^{k}(-1)^{i+1}\psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)\big)e(-\sum_{j=1}^{k} u_jx_j) =$$

$$\sum_{i=1}^{k}(-1)^{i+1}\sum_{x_i} e(-u_ix_i) \sum_{x_1,\ldots,\hat{x}_i,\ldots,x_k} \psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)e(-\sum_{j\neq i} u_jx_j) =$$

$$n\sum_{i=1}^{k}(-1)^{i+1}\delta(0,u_i) \sum_{x_1,\ldots,\hat{x}_i,\ldots,x_k} \psi(x_1,\ldots,\hat{x}_i,\ldots,x_k)e(-\sum_{j\neq i} u_jx_j)$$

where $\delta(0,u_i)=1$ if $u_i=0$ and is zero otherwise. Therefore

$$\mathcal{F}(B^{k-1}(X_A;\mathbb{F})) \subset \{g \in C^{k-1}(X_A;\mathbb{F}) : \text{support}(g) \subset \mathbb{Z}_n^k - (\mathbb{Z}_n - \{0\})^k\} \ .$$

Equality follows since both spaces have dimension $\binom{n-1}{k-1}$ over $\mathbb{F}$.

$\square$

We next study the Fourier image of the $(k-1)$-cocycles of X_A. Fix a $\phi \in C^{k-1}(X_A;\mathbb{F})$. For $a \in \mathbb{Z}_n$ define a function $f_a \in \mathcal{L}(\mathbb{Z}_n^k;\mathbb{F})$ by

$$f_a(x_1,\ldots,x_k) = d_{k-1}\phi\big(a - \sum_{i=1}^{k} x_i, x_1,\ldots,x_k\big) =$$

$$\phi(x_1,\ldots,x_k) + \sum_{i=1}^{k}(-1)^i\phi\big(a - \sum_{j=1}^{k} x_j, x_1,\ldots,\hat{x}_i,\ldots,x_k\big) \ .$$

Let T be the automorphism of $\mathbb{Z}_n^k$ given by

$$T(u_1,\ldots,u_k) = (u_2-u_1,\ldots,u_k-u_1,-u_1) \ .$$

Then $T^{k+1}=I$ and for $1 \le i \le k$

$$T^i(u_1,\ldots,u_k) = (u_{i+1}-u_i,\ldots,u_k-u_i,-u_i,u_1-u_i,\ldots,u_{i-1}-u_i).$$

Claim 2.2. *Let* $u = (u_1, \ldots, u_k) \in \mathbb{Z}_n^k$. *Then*

$$\widehat{f_a}(u) = \widehat{\phi}(u) + \sum_{i=1}^{k} (-1)^{ki} e(-u_i a) \widehat{\phi}(T^i u) \quad . \tag{2}$$

Proof: For $1 \leq i \leq k$ let $\psi_i \in \mathcal{L}(\mathbb{Z}_n^k, \mathbb{F})$ be given by

$$\psi_i(x_1, \ldots, x_k) = \phi\big(a - \sum_{j=1}^{k} x_j, x_1, \ldots, \hat{x}_i, \ldots, x_k\big) \quad .$$

Then

$$\widehat{\psi}_i(u) = \sum_{(x_1,\ldots,x_k) \in \mathbb{Z}_n^k} \phi\big(a - \sum_{j=1}^{k} x_j, x_1, \ldots, \hat{x}_i, \ldots, x_k\big) e(-\sum_{j=1}^{k} u_j x_j) \quad .$$

Substituting

$$y_j = \begin{cases} a - \sum_{\ell=1}^{k} x_\ell & j = 1 \\ x_{j-1} & 2 \leq j \leq i \\ x_j & i+1 \leq j \leq k \end{cases}$$

it follows that

$$\sum_{j=1}^{k} u_j x_j = (a - y_1) u_i + \sum_{j=2}^{i} (u_{j-1} - u_i) y_j + \sum_{j=i+1}^{k} (u_j - u_i) y_j \ .$$

Therefore

$$\widehat{\psi}_i(u) = e(-u_i a) \sum_{y=(y_1,\ldots,y_k) \in \mathbb{Z}_n^k} \phi(y) e(u_i y_1 - \sum_{j=2}^{i} (u_{j-1} - u_i) y_j - \sum_{j=i+1}^{k} (u_j - u_i) y_j) =$$

$$e(-u_i a) \widehat{\phi}(-u_i, u_1 - u_i, \ldots, u_{i-1} - u_i, u_{i+1} - u_i, \ldots, u_k - u_i) =$$
$$e(-u_i a) (-1)^{i(k-i)} \widehat{\phi}(T^i u) \ . \tag{3}$$

Now (2) follows from (3) since $f_a = \phi + \sum_{i=1}^{k} (-1)^i \psi_i$.

□

For $u \in \mathbb{Z}_n^k$ let $E_u = \{T^i u : 0 \leq i \leq k\}$ and let

$$L_u = \bigcap_{a \in A} \{g \in \mathcal{L}(E_u, \mathbb{F}) : g(u) + \sum_{i=1}^{k} (-1)^{ki} e(-u_i a) g(T^i u) = 0\}. \qquad (4)$$

Let $\phi \in Z^{k-1}(X_A; \mathbb{F})$. Then for all $a \in A$ and $(x_1, \ldots, x_k) \in \mathbb{Z}_n^k$

$$f_a(x_1, \ldots, x_k) = d_{k-1}\phi\big(a - \sum_{i=1}^{k} x_i, x_1, \ldots, x_k\big) = 0 \ .$$

Eqn. (2) then implies that for all $a \in A$ and $u \in \mathbb{Z}_n^k$

$$\widehat{\phi}(u) + \sum_{i=1}^{k} (-1)^{ki} e(-u_i a) \widehat{\phi}(T^i u) = 0 \ .$$

Writing $\widehat{\phi}_{|E_u}$ for the restriction of $\widehat{\phi}$ to E_u we obtain

Corollary 2.3. *Let $\phi \in C^{k-1}(X_A; \mathbb{F})$. Then $\phi \in Z^{k-1}(X_A; \mathbb{F})$ iff $\widehat{\phi}_{|E_u} \in L_u$ for all $u \in \mathbb{Z}_n^k$.*

□

Let the symmetric group S_k act on $\mathbb{Z}_n^k$ by

$$\sigma\big((u_1, \ldots, u_k)\big) = (u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(k)})$$

and let $G_{n,k}$ denote the subgroup of $\mathrm{Aut}(\mathbb{Z}_n^k)$ generated by T and S_k. The subset

$$D_{n,k} = \{(u_1, \ldots, u_k) \in (\mathbb{Z}_n - \{0\})^k : u_i \neq u_j \text{ for } i \neq j\}$$

is clearly invariant under $G_{n,k}$.

Claim 2.4.
(i) Let $\sigma \in S_k$ and $1 \leq i \leq k$. Then $\eta = T^i \sigma T^{-\sigma^{-1}(i)} \in S_k$ and $\mathrm{sgn}(\eta) = (-1)^{k(i+\sigma^{-1}(i))}\mathrm{sgn}(\sigma)$.
(ii) Any element of $G_{n,k}$ can be written uniquely as σT^i where $\sigma \in S_k$ and $0 \leq i \leq k$. $G_{n,k}$ acts freely on $D_{n,k}$.
(iii) $L_u = L_{T^j u}$ for all $u \in D_{n,k}$ and $0 \leq j \leq k$.

Proof: (i) For $1 \le \ell \le k$ let $\tau_\ell \in S_k$ be given by

$$\tau_\ell(i) = \begin{cases} k-\ell+1+i & 1 \le i \le \ell-1 \\ k-\ell+1 & i=\ell \\ i-\ell & \ell+1 \le i \le k \ . \end{cases}$$

It can be checked that

$$\eta = T^i \sigma T^{-\sigma^{-1}(i)} = \tau_{k-i+1}^{-1} \sigma \tau_{k-\sigma^{-1}(i)+1} \ .$$

Noting that $\mathrm{sgn}(\tau_\ell) = (-1)^{k\ell+1}$ it thus follows that

$$\mathrm{sgn}(\eta) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau_{k-i+1})\mathrm{sgn}(\tau_{k-\sigma^{-1}(i)+1}) = (-1)^{k(i+\sigma^{-1}(i))}\mathrm{sgn}(\sigma) \quad .$$

(ii) It follows from (i) that

$$G_{n,k} = \{\sigma T^i : \sigma \in S_k \ , \ 0 \le i \le k\} \ .$$

Let $u = (u_1, \dots, u_k) \in D_{n,k}$ and let $v = (v_1, \dots, v_k) = \sigma T^i u$. If $i \neq 0$ then

$$\sum_{j=1}^{k} v_j = \sum_{j=1}^{k} u_j - (k+1)u_i \neq \sum_{j=1}^{k} u_j$$

and therefore $\sigma T^i u \neq u$. It follows that $G_{n,k}$ acts freely on $D_{n,k}$ and that the representation of an element of $G_{n,k}$ as σT^i is unique.
(iii) Let $g \in L_u$ and $a \in A$. Then

$$g(T^j u) + \sum_{i=1}^{k} (-1)^{ik} e(-(T^j u)_i a) g(T^{i+j} u) =$$

$$g(T^j u) + \sum_{i=1}^{k-j} (-1)^{ik} e(-(u_{i+j} - u_j)a) g(T^{i+j} u) +$$

$$(-1)^{(k-j+1)k} e(u_j a) g(u) + \sum_{i=k-j+2}^{k} (-1)^{ik} e(-(u_{i-k+j-1} - u_j)a) g(T^{i+j} u) =$$

$$(-1)^{jk} e(u_j a)\Big(g(u) + \sum_{i=1}^{k} (-1)^{ik} e(-u_i a) g(T^i u)\Big) = 0. \qquad (5)$$

Hence $g \in L_{T^j u}$.

□

Proof of Theorem 1.1: Let $R \subset D_{n,k}$ be a fixed set of representatives of the orbits of $G_{n,k}$ on $D_{n,k}$. Then $|R| = \frac{|D_{n,k}|}{|G_{n,k}|} = \frac{1}{k+1}\binom{n-1}{k}$. Consider the mapping

$$\Theta : Z^{k-1}(X_A;\mathbb{F}) \to \bigoplus_{u\in R} L_u$$

given by

$$\Theta(\phi) = \left(\widehat{\phi}_{|E_u} \ : \ u \in R\right) \ .$$

Claim 2.5.

$$\ker\Theta = B^{k-1}(X_A;\mathbb{F}) \ .$$

Proof:

$$\ker\Theta = \{\phi \in Z^{k-1}(X_A;\mathbb{F}) : \widehat{\phi}_{|E_u} = 0 \text{ for all } u \in R\} =$$

$$\{\phi \in Z^{k-1}(X_A;\mathbb{F}) : \widehat{\phi}(u) = 0 \text{ for all } u \in D_{n,k}\} = B^{k-1}(X_A;\mathbb{F})$$

by Claim 2.1.

□

Claim 2.6. Θ *is surjective.*

Proof: Let $(g_u : u \in R) \in \bigoplus_{u\in R} L_u$. Define $g \in C^{k-1}(X_A;\mathbb{F})$ by

$$g(v) = \begin{cases} 0 & v \notin D_{n,k} \\ \text{sgn}(\sigma) g_u(T^j u) & v = \sigma T^j u \text{ where } u \in R. \end{cases}$$

Clearly $\Theta(\mathcal{F}^{-1}(g)) = (g_u : u \in R)$. To show that $\mathcal{F}^{-1}(g) \in Z^{k-1}(X_A;\mathbb{F})$ it suffices by Corollary 2.3 to check that $g \in L_v$ for all $v \in \mathbb{Z}_n^k$. If $v \notin D_{n,k}$ then $g_{|E_v} = 0$. Suppose then that $v = \sigma T^j u \in D_{n,k}$ where $u \in R$ and $0 \le j \le k$. Combining Claim 2.4(i) and Eq. (5) it follows that

$$g(v) + \sum_{i=1}^{k} (-1)^{ik} e(-v_i a) g(T^i v) =$$

$$g(\sigma T^j u) + \sum_{i=1}^{k} (-1)^{ik} e(-(\sigma T^j u)_i a) g(T^i \sigma T^j u) =$$

$$\mathrm{sgn}(\sigma)g_u(T^ju)+\sum_{i=1}^{k}(-1)^{ik}e(-(T^ju)_{\sigma^{-1}(i)}a)(-1)^{k(i+\sigma^{-1}(i))}\mathrm{sgn}(\sigma)g_u(T^{\sigma^{-1}(i)+j}u) =$$

$$\mathrm{sgn}(\sigma)\Big(g_u(T^ju)+\sum_{i=1}^{k}(-1)^{ik}e(-(T^ju)_i a)g_u(T^{i+j}u)\Big) =$$

$$= (-1)^{jk}\mathrm{sgn}(\sigma)e(u_j a)\Big(g_u(u)+\sum_{i=1}^{k}(-1)^{ik}e(-u_i a)g_u(T^iu)\Big)= 0.$$

□

Claims 2.5 and 2.6 imply that

$$H^{k-1}(X_A,\mathbb{F}) \cong \bigoplus_{u\in R} L_u \ . \tag{6}$$

For $u = (u_1,\dots,u_k) \in D_{n,k}$ let $B_u = \{0,u_1,\dots,u_k\}$. Then $\dim L_u = \dim\ker M_{A,B_u}$. Combining (6) with Claim 2.4(iii) it thus follows that

$$h^{k-1}(X_A;\mathbb{F}) = \sum_{u\in R}\dim L_u =$$

$$\frac{1}{k+1}\sum_{u\in R}\sum_{j=0}^{k}\dim L_{T^ju} = \frac{1}{k+1}\sum_{B\in\mathcal{B}_{n,k}}\dim\ker M_{A,B} \ .$$

□

3 When is X_A collapsible?

In this section we prove Theorem 1.3, so that in this section n is prime. We find it convenient to maintain the vertices in a face sorted according to the order induced from $\mathbb{N}$, and also refer to subsets of $\mathbb{F}_n$ as sorted vectors and not only as sets.

3.1 Equivalence

Let $\phi : \mathbb{F}_n \to \mathbb{F}_n$ be the linear map $\phi(x) = \alpha x + \beta$. It is clear that the image of X_a under ϕ is X_t where $t = \alpha a + (k+1)\beta$. We say that the complexes $X_{a_0,\ldots,a_k}$ and $X_{b_0,\ldots,b_k}$ are *equivalent* iff there exist a permutation π on $\{b_0, \ldots, b_k\}$ and α, β s.t. $\pi(b_i) = \alpha a_i + (k+1)\beta$ for every $0 \leq i \leq k$. Equivalent complexes are clearly isomorphic.

It is an easy observation that $a_0, \ldots, a_k$ is an arithmetic progression iff $X_{a_0,\ldots,a_k}$ is equivalent to the complex $X_{0,\ldots,k}$. We show that $X = X_{0,\ldots,k}$ is collapsible whence $X_{a_0,\ldots,a_k}$ is collapsible for $a_0, \ldots, a_k$ an arithmetic progression.

3.2 Proof of sufficiency

To show that X is collapsible we introduce an order $\prec_R$ by which we remove the k-faces from X. We need first some preliminary definitions. With every k-face $u \in X$ we associate a vector $h(u)$ of dimension $\lceil \frac{k}{2} \rceil$. The i-th coordinate in h counts how many integers in the interval $[u_i, u_{k-i}]$ do not belong to $\{u_i, \ldots, u_{k-i}\}$. Namely, the i-th coordinate of $h(u)$ is:

$$h_i(u) := u_{k-i} - u_i - (k - 2i)$$

Clearly $h_i(u)$ is non-increasing in i. For every two k-faces $u, v \in X$ we say that $u \prec_L v$ if $h(u)$ is lexicographically smaller than $h(v)$. When $h(u) = h(v)$ we say that $u \equiv_L v$. It should be clear that h is invariant under set reversal i.e. $x \to n - x$. It is also invariant under shifts that "do not overflow" in the obvious sense, but we will not be using this fact. If $u \not\equiv_L v$ for some $u, v \in X$, we denote by $\delta_L(u, v)$ the first index for which $h(u)$ and $h(v)$ differ. Thus if $u \prec_L v$ and $\delta_L(u, v) = i$ then $h_j(u) = h_j(v)$ for all $j < i$ and $h_i(u) < h_i(v)$.
For $i, j \in \mathbb{F}_n$ it is convenient to define $\rho(i, j)$ as $i - j$ if $i > j$ and as $j - i$ otherwise. This is extended as usual to: $\rho(i, A) = \min\{\rho(i, a) \mid a \in A\}$ and $\rho(A, B) = \min\{\rho(a, b) \mid a \in A, b \in B\}$.
If $u \in X_i$ and $v \in X_j$ we say that $u \prec_I v$ if i is closer than j to $\{0, k\}$, i.e., if $\rho(i, \{0, k\}) < \rho(j, \{0, k\})$. We say that $u \equiv_I v$ when $\rho(i, \{0, k\}) = \rho(j, \{0, k\})$, namely, $i = j$ or $i = k - j$. Letting $i' = \rho(i, \{0, k\})$, it is clear that $u \prec_I v$ iff $i' < j'$. If $u \not\equiv_I v$, we denote by $\delta_I(u, v) = \min\{i', j'\} = \rho(\{i, j\}, \{0, k\})$.

We are now ready to define the relation $\prec_R$. This is done in terms of the relations $\prec_L$ and $\prec_I$. To begin, $u \equiv_R v$ iff $u \equiv_L v$ and $u \equiv_I v$. If $u \preceq_L v$ and $u \preceq_I v$ and at least one inequality is proper, then $u \prec_R v$. Finally,

when $u \prec_L v$ and $u \succ_I v$, the order $\prec_R$ is determined according to the smaller of $\delta_I(u,v), \delta_L(u,v)$. Namely, if $\delta_I(u,v) < \delta_L(u,v)$ then $u \succ_R v$ and if $\delta_I(u,v) \geq \delta_L(u,v)$ then $u \prec_R v$.
To sum up, for $u, v \in X$:

1. If $u \equiv_I v$ and $u \equiv_L v$ then $u \equiv_R v$.

2. If $u \equiv_I v$ and $u \prec_L v$ then $u \prec_R v$.

3. If $u \prec_I v$, then $u \prec_R v$ unless

   (a) $u \succ_L v$ and

   (b) $\delta_L(u,v) \leq \delta_I(u,v)$

   In which case $u \succ_R v$.

To clarify this definitions a little bit more, we present an example from the complex $X_{0,1,2,3}$ over $\mathbb{F}_7$. Let $u = \{0,1,2,5\}$, $v = \{1,2,5,6\}$. The set u has two missing integers between 0 and 5 and no missing integers between 1 and 2, hence $h(u) = h(\{0,1,2,5\}) = (2,0)$. Similarly $h(v) = h(\{1,2,5,6\}) = (2,2)$. Also, $u \prec_L v$ because $(2,0)$ is lexicographically smaller than $(2,2)$. Furthermore, $\delta_L(u,v) = 1$ because the first coordinate the vectors differ is the second coordinate (and we start indexing coordinates from zero). Now $u \in X_1$ since $0+1+2+5 \equiv 1 \mod 7$. Similarly $v \in X_0$. We next calculate that $1' = 1 = \rho(1, \{0,7\})$ and $0' = 0$. Hence $v \prec_I u$ because $0' < 1'$, and $\delta_I(u,v) = \min\{0,1\} = 0$. To recap, $u \prec_L v$ and $v \prec_I u$, so we turn to compare $0 = \delta_I(u,v) < \delta_L(u,v) = 1$, it follows that in this case the order R is determined by I, hence $\{0,1,2,5\} = u \succ_R v = \{1,2,5,6\}$. A full description of the order R on $X_{0,1,2,3}$ over $\mathbb{F}_7$ is shown in Figure 2 and Figure 3:

A few words are in order about Figure 2. The rows are sorted by the lexicographic order of $h(\cdot)$. The columns on the right include all facets of X sorted by value of i'. Note that for each value of h and each i' there are two facets that attain this pair of values. The leftmost column gives the value of $\delta_L(x,y)$ for every two consecutive lines in the table.

We now turn to show that X can indeed be collapsed in the order $\prec_R$. That is, for every $x \in X$ it is possible to apply an elementary collapse step to x if all the $\prec_R$-predecessors of x have already been collapsed. In order to show this, we need to point out an free$(k-1)$-face that is contained in x. What we will show is that for $x \in X_a$, the face $\hat{x} := x \setminus \{x_a\}$ is free. (Note

Figure 2: $X_{0,1,2,3}$ parameters over $\mathbb{F}_7$

$\delta_L(x,y)$	$h(x)$	$i'=0$	$i'=1$
	$(0,0)$	$\{2,3,4,5\},\{1,2,3,4\}$	
0			
	$(1,0)$	$\{0,1,2,4\},\{2,4,5,6\}$	$\{2,3,4,6\},\{0,2,3,4\}$
1			
	$(1,1)$		$\{0,1,3,4\},\{2,3,5,6\}$
0			
	$(2,0)$	$\{1,3,4,6\},\{0,2,3,5\}$	$\{0,1,2,5\},\{1,4,5,6\}$
1			
	$(2,1)$		$\{1,3,5,6\},\{0,1,3,5\}$
1			
	$(2,2)$	$\{1,2,5,6\},\{0,1,4,5\}$	
0			
	$(3,0)$		$\{0,4,5,6\},\{0,1,2,6\}$
1			
	$(3,1)$	$\{0,3,5,6\},\{0,1,3,6\}$	

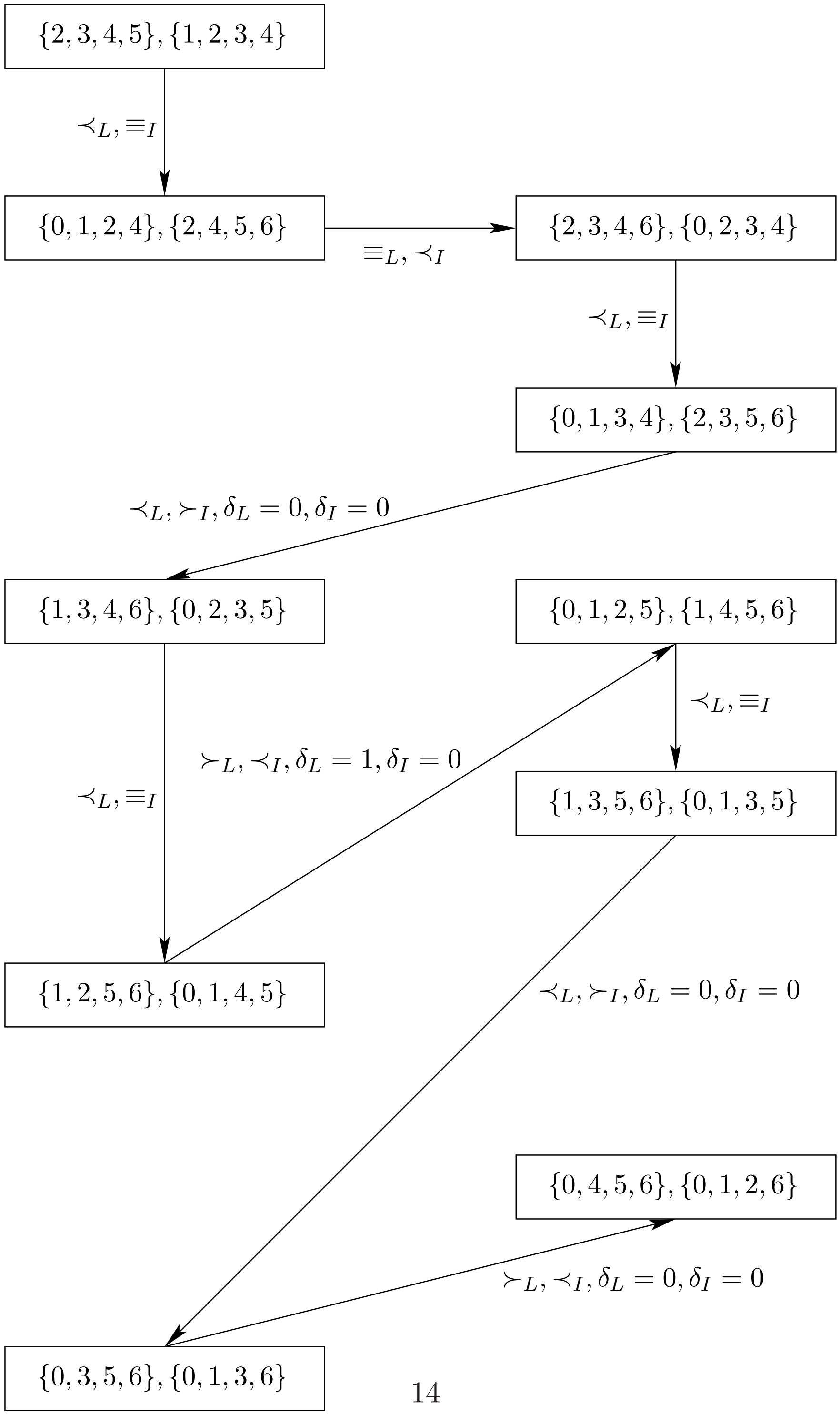
that since $x \in X = X_{0,\dots,k}$, there indeed must exist some $a \in \{0,\dots,k\}$ s.t. $x \in X_a$). It may be helpful to mention that a plays a double role here. It is an index in the vector x as well as the sum of the elements of x. Being free means that all the k-faces containing $\hat{x}$, precede x in the order $\prec_R$. A k-face that contains $\hat{x}$ has the form $y^{(b)} := \hat{x} \cup \{x_a + (b-a)\}$ with $0 \le b \le k$ and $b \ne a$. Clearly, $y^{(b)}$ is a k-face in X iff $x_a + (b-a) \notin \hat{x}$. Also, in this case $y^{(b)} \in X_b$, as we assume below.

The proof that $y^{(b)} \prec_R x$ has two cases:

1. We first consider the case where $y^{(b)} \succeq_I x$. Since $y^{(b)} \in X_b$ and $x \in X_a$, the meaning of $y^{(b)} \succeq_I x$ is that $b' \ge a'$. Therefore $\delta_I(y^{(b)}, x)$ which is the smaller of a' and b' equals a'. This means that b lies between a and $k-a$ (whether a or $k-a$ is bigger is immaterial here).

   - Consequently, $x_a + (b-a)$ is in the interval $[x_{a'}, x_{k-a'}]$. It follows that the first and last $a'-1$ elements of x and $y^{(b)}$ are identical. In particular, $h_i(y^{(b)}) = h_i(x)$ for $i < a'$.
   - We recall that $y^{(b)}$ is created by removing x_a from x and replacing it by the term $x_a + (b-a)$. Thus the interval $[y^{(b)}_{a'}, y^{(b)}_{k-a'}]$ is shorter than $[x_{a'}, x_{k-a'}]$. It follows that the first coordinate where $h(y^{(b)})$

Figure 3: The order of collapse determined by $\prec_R$



and $h(x)$ differ is the a'-th coordinate, where $h_{a'}(y^{(b)}) < h_{a'}(x)$. Consequently, $y^{(b)} \prec_L x$ and $a' = \delta_L(y^{(b)}, x)$.

- If $y^{(b)} \equiv_I x$ then we are done, because we already know that $y^{(b)} \prec_L x$. By definition of $\prec_R$ this yields the desired conclusion $y^{(b)} \prec_R x$.
- If $y^{(b)} \succ_I x$ then from the previous points we conclude that $a' = \delta_L(y^{(b)}, x) = \delta_I(y^{(b)}, x)$. To sum up, $y^{(b)} \prec_L x$ and $\delta_L(y^{(b)}, x) = \delta_I(y^{(b)}, x)$, which yields by definition, $y^{(b)} \prec_R x$, as claimed.

2. Now consider the case $y^{(b)} \prec_I x$. This means that $b' < a'$. Therefore $b' = \delta_I(y^{(b)}, x)$. Consequently b does not lie between a and $k - a$.

   - It follows that $x_a + (b - a) \in [x_{b'}, x_{k-b'}]$. Consequently, the first and last $b' + 1$ elements of x and $y^{(b)}$ are identical. In particular, $h_i(y^{(b)}) = h_i(x)$ for $i \leq b'$. Thus $\delta_L(y^{(b)}, x) > b'$.
   - If $y^{(b)} \preceq_L x$ then $y^{(b)} \prec_R x$ and we are done.
   - If $y^{(b)} \succ_L x$ then from the previous points we conclude that $b' = \delta_I(y^{(b)}, x) < \delta_L(y^{(b)}, x)$. Hence $y^{(b)} \succ_I x$ and $\delta_I(y^{(b)}, x) < \delta_L(y^{(b)}, x)$. Again, by definition, $y^{(b)} \prec_R x$, as claimed.

This completes the proof that $X_{0,\ldots,k}$ is collapsible and hence that $X = X_{a_0,\ldots,a_k}$ is collapsible whenever $a_0, \ldots, a_k$ is an arithmetic progression.

3.3 Proof of necessity

We now turn to show that if $a_0, \ldots, a_k$ is not arithmetic, then $X_{a_0,\ldots,a_k}$ is not collapsible. In fact we show that in this case exactly $k + 1$ elementary collapse steps can be carried out.
For $X \subseteq \mathbb{F}_n$ we denote as usual by $X + a$ the a-shift of X, namely, the set $\{x + a | x \in X\}$. We start with the following simple observation.

Observation 3.1. *Let n be a prime. A subset $X \subsetneq \mathbb{F}_n$ is an arithmetic progression iff there is an element l for which $|(X + l) \setminus X| = 1$.*

When is the $(k - 1)$-face $x_1, \ldots, x_k$ free a free face? This is the case iff, for each $k \geq i \geq 1$ the element $x_i + \sum_{j=1}^{k} x_j$ belongs to the set $\{a_0, \ldots, a_k\}$. If $x_i + \sum x_j = a_l$ it means that $x_1, \ldots, x_k$ cannot be extended to a k-face in X_{a_l}. This translates into a linear system of equations in $x_1, \ldots, x_k$ whose matrix has 2's along the main diagonal and 1's elsewhere. Such a matrix is nonsingular, so the solution is unique. Also, all the k terms $x_i + \sum x_j$ are distinct, so the only choice we have in constructing this linear system is which of the $k+1$ elements in $\{a_0, \ldots, a_k\}$ to omit. There are $k+1$ such choices which yields $k+1$ distinct collapse steps that can be carried out.

We now explicitly describe the $k+1$ collapse steps that can be carried out. Each of these collapsible faces has the form $x^{(t)} := \{a_0 + l_t, \ldots, a_k + l_t\} \in X_{a_t}$ for some $l_0, \ldots l_k$

The condition $x^{(t)} \in X_{a_t}$ determines l_t via $l_t = \frac{a_t - \sum_{i=0}^{k} a_i}{k+1}$. We claim that the face $y := x^{(t)} \setminus \{a_t + l_t\}$ is free. The sum of y's elements is $-l_t$, so that for every $i \neq t$ we need to add the term $\{a_i + l_t\}$ to y in order to attain the sum a_i. This is, however, impossible since $\{a_i + l_t\}$ is a member of y.

We turn to show that after these first $k+1$ collapse steps are carried out, there remain no free $(k-1)$-faces in X. In order for a $(k-1)$-face y to be free following the above collapses, y has to be contained in exactly one of these $k+1$ collapsed faces. Since $\{a_0, \ldots, a_k\}$ is not an arithmetic progression, by Observation 3.1, the intersection of any two of the $x^{(t)}$ contains at most $k-1$ elements. In particular there is no $(k-1)$-face that they both contain. Thus we have to consider only $(k-1)$-faces y which are contained in one of the $x^{(t)}$ and exactly one more k-face.

It follows that y must be of the form $x^{(t)} \setminus \{a_j + l_t\}$ for some j and t. The sum of y's elements is $a_t - a_j - l_t$. If y is contained as well in a k-face $z \in X_{a_i}$, then necessarily $z_i = z = y \cup \{a_i - a_t + a_j + l_t\}$. We are assuming that y becomes free with the collapse of $x^{(t)}$, so there must be exactly one index i for which z_i is a legal k-face different from $x^{(t)}$. It follows that $x^{(t)}$ and $x^{(t)} + (a_j - a_t)$ must have k elements in common. Again by Observation 3.1 this means that the elements in $x^{(t)}$ form an arithmetic progression, a contradiction. The proof of Theorem 1.3 is now complete.

4 Example: Homology of $X_{\{0,1,3\}}$

For a prime p and an integer n indivisible by p, let $U_{p,n}$ be the group of n-th roots of unity in $\overline{\mathbb{F}_p}$.

Proposition 4.1. *Let $k = 2$, $A = \{0, 1, 3\}$. Let p be a prime and suppose n*

is coprime to $3p$*. Then*

$$h_1(X_A;\mathbb{F}_p) = \frac{1}{3}\big|\big\{\{u,v\} \subset U_{p,n} - \{1\} : u \neq v \text{ and } 1+u+v=0\big\}\big| \ .$$

Proof: Let $B = \{0,k,\ell\}$ with $0 < k < \ell < n$ and let $u = \omega^{-k}, v = \omega^{-\ell}$. Then

$$\det M_{A,B} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & u & v \\ 1 & u^3 & v^3 \end{bmatrix} =$$

$$uv^3 - vu^3 + u^3 - u - v^3 + v = (u-1)(v-1)(v-u)(u+v+1) \ .$$

It follows that

$$\text{rk } M_{A,B} = \begin{cases} 2 & 1+u+v=0 \\ 3 & \text{otherwise.} \end{cases}$$

Thus the Proposition follows directly from Theorem 1.

□

Corollary 4.2. *Let* $k=2$*,* $A = \{0,1,3\}$*. Let* p *be a prime and suppose* $n = p^m - 1$ *is coprime to 3. Then*

$$h_1(X_A;\mathbb{F}_p) = \begin{cases} \frac{n-1}{6} & p=2 \\ \frac{n-2}{6} & p=3 \\ \frac{n-4}{6} & p>3. \end{cases}$$

Proof: Clearly $\mathbb{F}_{p^m}^* = U_{p,n}$. Therefore, by Proposition 4.1

$$h_1(X_A;\mathbb{F}_p) = \frac{1}{6}\big|\big\{u \in \mathbb{F}_{p^m}^* - \{1\} : -(1+u) \notin \{0,1,u\}\big\}\big|.$$

The Corollary now follows since

$$\big\{u \in \mathbb{F}_{p^m}^* - \{1\} : -(1+u) \notin \{0,1,u\}\big\} = \begin{cases} \mathbb{F}_{2^m}^* - \{1\} & p=2 \\ \mathbb{F}_{3^m}^* - \{\pm 1\} & p=3 \\ \mathbb{F}_{p^m}^* - \{\pm 1, -2, -\frac{1}{2}\} & p>3. \end{cases}$$

□

5 Concluding Remarks

Theorem 1.1 provides an explicit description of the homology of the sum complex X_A over fields of characteristic coprime to n. In particular, it follows via Chebotarëv's Theorem that if n is prime then X_A is $\mathbb{Q}$-acyclic, i.e. X_A is a k-hypertree. When A is an arithmetic progression, X_A was shown to be k-collapsible, and in particular $\mathbb{Z}$-acyclic. One natural question is whether there exist other A's for which X_A is $\mathbb{Z}$-acyclic. Kalai's k-dimensional Cayley's formula [1] suggests that most k-hypertrees are not $\mathbb{Z}$-acyclic. Likewise we conjecture that X_A is not $\mathbb{Z}$-acyclic for most $(k+1)$-subsets $A \subset \mathbb{Z}_n$. One possible approach to the question of $\mathbb{F}_p$-acyclicity of X_A for primes $p \nmid n$ is via the following reduction. Let $S_{\mathbb{F}}(A)$ be the $\mathbb{F}$-linear space of polynomials in $\mathbb{F}[x]$ spanned by the monomials $\{x^a : a \in A\}$. Theorem 1.1 then implies that X_A is $\mathbb{F}_p$-acyclic iff $\deg \gcd(f(x), x^n - 1) \leq k$ for all $0 \neq f(x) \in S_{\overline{\mathbb{F}_p}}(A)$.

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