Let $A = \{a_1, \ldots, a_t\} \subset \{0, 1\}^n$, $F_1, \ldots, F_t \subset [n] = \{1, \ldots, n\}$. The ordered family of pairs $((a_1, F_1), \ldots, (a_t, F_t))$ is incompatible if for any $1 \leq k < l \leq t$ there exists a $j \in F_k$ such that $a_{kj} \neq a_{lj}$.

It is shown that for any incompatible family $((a_1, F_1), \ldots, (a_t, F_t))$, there exists a $1-1$ mapping $\phi : [t] \rightarrow 2^n$ such that $\phi(i) \subset F_i$ and $\phi(i)$ is shattered by $A$.

1 On the trace of incompatible vectors

Let $a = (a_1, \ldots, a_n)$ be a vector in $\{0, 1\}^n$, and $S$ a subset of $[n] = \{1, \ldots, n\}$. The trace of $a$ on $S$ is $a|_S = (a_j : j \in S) \in \{0, 1\}^S$.

A subset $S \subset [n] = \{1, \ldots, n\}$ is shattered by $A \subset \{0, 1\}^n$ if for each $\epsilon = (\epsilon_j : j \in S) \in \{0, 1\}^S$ there exists an $a \in A$ such that $a|_S = \epsilon$.

Let $S(A)$ denote the family of all $S \subset [n]$ which are shattered by $A$.

The following basic result was proved by Sauer [5] and Perles and Shelah [6].

**Theorem 1.1** ([5],[6]) If $|A| > \sum_{i=0}^d \binom{n}{i}$ then there exists an $S \in S(A)$ such that $|S| > d$.

Different proofs and extensions of Theorem 1.1 were given by Alon [1], Frankl [2], Frankl and Pach [3] and others. See the recent survey by Füredi and Pach [4].

We first give a simple proof of the following extension of Theorem 1.1 which is also implicit in all previous proofs.
Theorem 1.2 \(|A| \leq |\mathcal{S}(A)| \) for any \(A \subset \{0,1\}^n\)

Proof: Let \(U\) denote the \(\mathbb{Z}_2\)-linear space of multilinear polynomials in \(\mathbb{Z}_2[x_1, \ldots, x_n]\). With each \(a = (a_1, \ldots, a_n) \in A\) we associate the polynomial \(f_a(x) = \prod_{j=1}^n (x_j + a_j + 1) \in U\). For each non-shattered subset \(T \in 2^n - \mathcal{S}(A)\) we choose a vector \(b_T = (b_{T,j} : j \in T) \in \{0,1\}^T\) such that \(b_T \neq a|_T\) for all \(a \in A\). Let \(g_T(x) = \prod_{j \in T} (x_j + b_j + 1) \in U\). Note that for \(a, a' \in A\), \(f_a(a') = \delta(a, a')\), and \(g_T(a) = 0\) for all \(T \not\in \mathcal{S}(A)\).

Claim 1.3 The family \(\{f_a(x) : a \in A\} \cup \{g_T(x) : T \not\in \mathcal{S}(A)\}\) is linearly independent in \(U\).

Proof: Suppose
\[\sum_{a \in A} \alpha_a f_a(x) + \sum_{T \not\in \mathcal{S}(A)} \beta_T g_T(x) = 0.\] (1)

Substituting \(a' \in A\) in Eq. (1) we obtain \(\alpha_{a'} = 0\). It thus remains to show that \(\{g_T(x) : T \not\in \mathcal{S}(A)\}\) is linearly independent. This follows from the fact that the unique highest degree monomials in the expansions of the \(g_T\)'s are all different.

\[\square\]

Claim 1.3 implies \(|A| + (2^n - |\mathcal{S}(A)|) \leq \dim U = 2^n\), hence \(|A| \leq |\mathcal{S}(A)|\).

\[\square\]

Next we consider the following extension of Theorem 1.2. Let \(A = \{a_1, \ldots, a_t\} \subset \{0,1\}^n\), where \(a_i = (a_{i1}, \ldots, a_{in})\), and let \(F_1, \ldots, F_t \subset [n]\). The ordered family of pairs \(((a_1, F_1), \ldots, (a_t, F_t))\) is incompatible if for any \(1 \leq k < l \leq t\) there exists a \(j \in F_k\) such that \(a_{kj} \neq a_{lj}\).

Theorem 1.4 For any incompatible family \(((a_1, F_1), \ldots, (a_t, F_t))\), there exists a \(1-1\) mapping \(\phi : [t] \to 2^{[n]}\) such that \(\phi(i) \subset F_i\) and \(\phi(i)\) is shattered by \(A\).
Proof: With each pair \((a_i, F_i)\) we associate the polynomial 
\[ f_i(x) = \prod_{j \in F_i} (x_j + a_{ij} + 1) \in U. \]
Let \(V = \{g(x) \in U : g(a_i) = 0 \text{ for all } 1 \leq i \leq t\}\), and let \(W = U/V\). For \(f \in U\) let \(\overline{f} \in W\) denote the image of \(f\) under the quotient map.

Claim 1.5 \(\overline{f_1}, \ldots, \overline{f_t}\) are linearly independent in \(W\).

Proof: Suppose \(\sum_{k=1}^t \lambda_k f_k(x) \in V\). The incompatibility condition implies that \(f_k(a_l) = 0\) whenever \(k < l\). It follows that for all \(1 \leq l \leq t\)
\[ 0 = \sum_{k=1}^t \lambda_k f_k(a_l) = \lambda_l + \sum_{k=l+1}^t \lambda_k f_k(a_l) \]
and so \(\lambda_1 = \cdots = \lambda_t = 0\).

Claim 1.6 For any \(F \subset [n]\)
\[ \prod_{j \in F} x_j \in \text{Span}\{ \prod_{j \in S} x_j : S \in 2^F \cap \mathcal{S}(A) \}. \]

Proof: We apply induction on \(|F|\). If \(F \in \mathcal{S}(A)\) then we are done. Otherwise there exists an \(\epsilon \in \{0, 1\}^F\) such that \(\epsilon \neq a_i F\) for all \(1 \leq i \leq t\).
It follows that \(g(x) = \prod_{j \in F} (x_j + \epsilon_j + 1)\) satisfies \(g(a_i) = 0\) for all \(1 \leq i \leq t\) and so \(g(x) \in V\). Therefore
\[ \prod_{j \in F} x_j = \prod_{j \in F} x_j - g(x) \in \text{Span}\{ \prod_{j \in S'} x_j : S' \subsetneq F \} \]
and the Claim follows from the induction hypothesis.

Claim 1.6 implies that for each \(1 \leq i \leq t\) we may expand
\[ \overline{f_i}(x) = \sum_{S \in 2^F \cap \mathcal{S}(A)} \mu_{i,S} \prod_{j \in S} x_j. \]
Consider the \(t \times 2^n\) matrix \(M\) indexed by \([t] \times 2^{[n]}\) and given by \(M(i, S) = \mu_{i,S}\) if \(S \in 2^F \cap \mathcal{S}(A)\), and zero otherwise.
Claim 1.5 implies that \( \text{rank}_{\mathbb{Z}_2} M = t \), so in particular there exists a 1–1 mapping \( \phi : [t] \to 2^{[n]} \) such that \( M(i, \phi(i)) \neq 0 \) for all \( 1 \leq i \leq t \). It follows that \( \phi(i) \subset F_i \) and that \( \phi(i) \) is shattered by \( A \).

\[ \square \]

For a vector \( a \in \{0,1\}^n \) let \( \text{Supp}\ a = \{1 \leq i \leq n : a_i = 1\} \).

Let \( A \subset \{0,1\}^n \) and let \( a_1, \ldots, a_t \) be an ordering of \( A \) such that

\[ |\text{Supp}\ a_k| \geq |\text{Supp}\ a_l| \text{ for all } k \leq l. \]

The ordered family of pairs \((a_1, \text{Supp} a_1), \ldots, (a_t, \text{Supp} a_t)\) is clearly incompatible, hence Theorem 1.4 implies the following result implicit in Frankl and Pach [3]:

**Corollary 1.7 ([3])** For any \( A \subset \{0,1\}^n \) there exists a 1–1 mapping \( \phi : A \to \mathcal{S}(A) \) such that \( \phi(a) \subset \text{Supp} a \) for all \( a \in A \).

\[ \square \]

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**References**


