ON THE MAXIMAL RANK IN A SUBSPACE OF MATRICES

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LET $M_n(F)$ be the space of $n\times n$ matrices over a field F, and let W be a linear subspace of $M_n(F)$.

Flanders [2] proved that if dim $W>m$ and $|F|\geq r+1$, then W contains a matrix of rank $>r$. He also characterized the subspaces W such that dim $W=m$ and W contains no matrix of rank $>r$.

In this note we prove a lower bound on the maximal rank attained in a subspace of matrices (Theorem 1). We then use this bound to derive Flanders' results (Theorems 2 and 3) without restrictions on F.

Let $[n]$ denote $\{1,\ldots,n\}$, and let $\prec$ be the lexicographic order on $[n]\times[n]$. ($(i,j)\prec(i_1,j_1)$ iff $i<i_1$ or $i=i_1$ and $j<j_1$.)

For $A\in M_n(F)$ denote by $p(A)\in[n]\times[n]$, the location of A's lexicographically first non-zero entry:

$$p(A)=\min\{(i,j)\colon\ A(i,j)\neq 0\}$$

For a collection $\mathscr{A}=\{A_1,\ldots,A_m\}$ of $n\times n$ matrices, construct an $n\times n$ matrix B as follows: $B(k,l)=1$ if $(k,l)=p(A_i)$ for some $1\leq i\leq m$, and $B(k,l)=0$ otherwise.

Denote by $\rho(\mathscr{A})$ the minimal number of lines in B (a *line* is either a row or a column) which cover all 1's in B.

THEOREM 1. *Let* $\mathscr{A}=\{A_1,\ldots,A_m\}\subset M_n(F)$. *Then span* $\mathscr{A}$ *contains a matrix of rank* $\geq\rho(\mathscr{A})$.

Proof. Call a set of entries in a matrix independent, if it contains no two entries on the same line. By König's Theorem ([**4**], Theorem 5.1.4 in [**3**]), the maximal size of an independent set of 1's in a 0–1 matrix, is equal to the minimal number of lines, which cover all 1's in that matrix. Hence if $\rho(\mathscr{A})=r$, then there exist $1\leq i_1,\ldots,i_r\leq m$ such that $\{p(A_{i_j})\colon\ 1\leq j\leq r\}$ is independent.

Let $p(A_{i_j})=(k_j,l_j)$ for $1\leq j\leq r$, then $S=\{k_1,\ldots,k_r\}$ and $T=\{l_1,\ldots,l_r\}$ are both of cardinality r. For $1\leq j\leq r$ define $B_j=A_{i_j}[S\mid T]\in M_r(F)$ (the minor determined by restricting the entries to $S\times T$).

We shall prove the theorem by showing that span $\{B_1,\ldots,B_r\}$ contains a non-singular matrix.

We may assume that $k_1<\cdots<k_r$. Let h be the permutation on $[r]$ for which: $l_{h(1)}<\cdots<l_{h(r)}$. Denote the jth row of B_i by b_j.

Clearly B_j's first $(j-1)$ rows are zero, $b_j(k)=0$ for $1\leq k<h^{-1}(j)$, and $b_j(h^{-1}(j))\neq 0$. Let C be the $r\times r$ matrix, whose rows are $b_1,\ldots,b_r$. C is non-singular, because by the preceding remarks, permuting C's rows according to h, we obtain an upper triangular matrix, with non-zeros on the diagonal. Let $D_j=B_jC^{-1}$ for $1\leq j\leq r$. It is easy to check that the following holds:

$$\text{For all } 1\leq j\leq r\text{: } D_j\text{'s first } j-1 \text{ rows are zero and } D_j\text{'s } j\text{th row is the } j\text{th unit vector.} \tag{1}$$

Claim 1. If $D_1,\ldots,D_r$ satisfy (1), *then there exists a* 0–1 *combination of $D_1,\ldots,D_r$ which is non-singular.*

Proof. We use induction on r. The case $r=1$ is trivial. Assume $r>1$. For $1\leq j\leq r-1$ let $D'_j=D_j([r-1]\mid[r-1])\in M_{r-1}(F)$. $D'_1,\ldots,D'_{r-1}$ satisfy (1) for $r-1$, and so, by induction there exist $x_1,\ldots,x_{r-1}\in\{0,1\}$ such that $\sum_{j=1}^{r-1}x_jD'_j$ is non-singular.

Now, since $D_r(i,j)=0$ for all $(i,j)\neq(r,r)$, and $D_r(r,r)=1$, we obtain by expanding the bottom row:

$$\det\left(\sum_{j=1}^{r-1}x_jD_j+D_r\right)=\det\left(\sum_{j=1}^{r-1}x_jD_j\right)+\det\left(\sum_{j=1}^{r-1}x_jD'_j\right) \tag{2}$$

But $\det(\sum_{j=1}^{r-1}x_jD'_j)\neq 0$, so (2) implies that $\sum_{j=1}^{r-1}x_jD_j$ and $\sum_{j=1}^{r-1}x_jD_j+D_r$ cannot both be singular. ■

We return to the proof of the theorem. By the claim $\sum_{j=1}^{r}x_jD_j$ is non-singular for some x_j's, and therefore $\sum_{j=1}^{r}x_jB_j=(\sum_{j=1}^{r}x_jD_j)C$ is also non-singular. This implies that rank $(\sum_{j=1}^{r}x_jA_{i_j})\geq r$. ■

The next result had been proved by Flanders [**2**], for $|F|\geq r+1$:

THEOREM 2. *If W is a subspace of $M_n(F)$, and* $\dim W>rn$, *then W contains a matrix of rank $>r$.* ■

Proof. Choose a basis $\mathcal{A}=\{A_1,\ldots,A_t\}$ of W. By performing a gaussian elimination on $\{A_1,\ldots,A_t\}$ (regarding them as n^2 dimensional vectors), we may assume that $p(A_1),\ldots,p(A_t)$ are all distinct. Since a line in a matrix covers n entries, we cannot cover $p(A_1),\ldots,p(A_t)$ by less than t/n lines. Therefore $\rho(\mathcal{A})\geq t/n>r$, which by Theorem 1 implies that $W=\operatorname{span}\mathcal{A}$ contains a matrix of rank $>r$. ■

Next we discuss a certain extremal case of Theorem 2.

Let F^n be the space of n-tuples over F. Denote by $x\otimes y\in M_n(F)$, the Kronecker product of $x,y\in F^n$. For $A,B\subset F^n$, let $A\otimes B=\operatorname{span}\{x\otimes y\colon x\in A, y\in B\}$.

The following result had been proved by Flanders [**2**], under the assumptions $|F|\geq r+1$ and char$(F)\neq 2$. Atkinson and Lloyd [**1**] had obtained it assuming only $|F|\geq r+1$.

THEOREM 3. *Suppose $W \subset M_n(F)$ is a subspace of dimension m, such that for all $A \in W$, rank $(A) \leq r$. Then either $W = E \otimes F^n$ or $W = F^n \otimes E$, for some r dimensional subspace $E \subset F^n$.*

Proof. Let $\mathscr{A} = \{A_1, \ldots, A_m\}$ be a basis of W. As in Theorem 2, we may assume that $p(A_1), \ldots, p(A_m)$ are all distinct. W does not contain a matrix of rank $>r$, therefore by Theorem 1, $\rho(\mathscr{A}) \leq r$. Choose r lines which cover $p(A_1), \ldots, p(A_m)$. Since each line covers at most n of the $p(A_i)$'s, it follows that the lines are pairwise disjoint, and that each of them consists entirely of $p(A_i)$'s.

Hence, either all r lines are rows, or all r lines are columns.

We shall assume the first case—that is: $p(A_1), \ldots, p(A_m)$ form r rows. (The case of columns is treated similarily).

Next we note that if $Q_1, Q_2 \in M_n(F)$ are non-singular, then the maximal rank in Q_1WQ_2 is equal to the maximal rank in W, and $W = E_1 \otimes E_2$ for some $E_1, E_2 \subset F^n$ iff $Q_1WQ_2 = (Q_1E_1) \otimes (E_2Q_2)$.

In particular, by performing the same row permutation on all matrices in W, we may assume that $p(A_1), \ldots, p(A_m)$ consist of the first r rows.

Clearly, by gaussian elimination on $A_1, \ldots, A_m$ (regarded as vectors in F^{n^2}), we may obtain a new basis $\{B_{ij}: 1 \leq i \leq r, 1 \leq j \leq n\}$ of W, such that $B_{ij}(i, j) = 1$ and $B_{ij}(k, l) = 0$ for all $1 \leq k \leq r$, $1 \leq l \leq n$ such that $(k, l) \neq (i, j)$.

Claim 2. B_{ij} is zero, except for the jth column.

Proof. We have to show that $B_{ij}(k, l) = 0$ for $l \neq j$ and $r+1 \leq k \leq n$ (for $1 \leq k \leq r$ this is known). Since our claim is invariant under row and column permutations, it suffices to prove it for specific i, j, k, l (which satisfy $l \neq j$ and $r+1 \leq k \leq n$), say $i = j = r$, $k = l = r+1$. That is, we show that $B_{rr}(r+1, r+1) = 0$. let $C_{ij} = B_{ij}([r+1] \mid [r+1]) \in M_{r+1}(F)$, and define $E_{ij} \in M_r(F)$ for $1 \leq i, j \leq r$ by $E_{ij}(k, l) = \delta_{ik}\delta_{jl}$.

Let $P \subset [r] \times [r]$. As $C_p(i, r+1) = 0$ for all $p \in P$, $1 \leq i \leq r$, we have:

$$\det\Big(\sum_{p \in P} C_p\Big) = \det\Big(\sum_{p \in P} E_p\Big)\Big(\sum_{p \in P} C_p(r+1, r+1)\Big). \tag{3}$$

Since W does not contain a matrix of rank $>r$, it follows that $\det(\sum_{p \in P} C_p) = 0$, and so if $P \subset [r] \times [r]$ satisfies:

$$\det\Big(\sum_{p \in P} E_p\Big) \neq 0 \tag{4}$$

Then $\sum_{p \in P} C_p(r+1, r+1) = 0$.

It is clear that the sets $P = \{(1, 1), (2, 2), \ldots, (r-2, r-2), (r-1, r), (r, r-1)\}$ ($P = \{(1, 1)\}$ for $r = 1$), and $P_1 = P \cup \{(r, r)\}$, both satisfy (4), and so:

$$\sum_{p\in P} C_p(r+1, r+1) = \sum_{p\in P_1} C_p(r+1, r+1) = 0.$$

This implies $B_{rr}(r+1, r+1) = C_{rr}(r+1, r+1) = 0$. ■

We complete the proof of Theorem 3, by showing that for every $1 \leq i \leq r$ there exists $x_i \in F^n$, such that for every $1 \leq j \leq n$ $B_{ij} = x_i \otimes e_j$ (e_j is the jth unit vector in F^n).

In view of Claim 2, we only have to show that for $1 \leq j_1, j_2 \leq n$, the j_1th column of B_{ij_1} is equal to the j_2th column of B_{ij_2}. Again by permuting rows and columns it suffices to prove (for example) that $B_{11}(r+1, 1) = B_{12}(r+1, 2)$. Using the notations of Claim 2, let

$$C = C_{11} + C_{12} + (C_{23} + C_{34} + \cdots + C_{rr+1})$$

By Claim 2: $C(r+1, 1) = B_{11}(r+1, 1)$, $C(r+1, 2) = B_{12}(r+1, 2)$. C, being an $r+1 \times r+1$ minor of a matrix in W is singular, because W has no member of rank $>r$. On the other hand it is clear that:

$$\det(C) = (-1)^r(C(r+1, 1) - C(r+1, 2))$$

Therefore $C(r+1, 1) = C(r+1, 2)$ and so: $B_{11}(r+1, 1) = B_{12}(r+1, 2)$. ■

Remark. Atkinson and Lloyd [**1**] have extended Flanders' classification, by proving that if $W \subset M_n(F)$ does not contain a matrix of rank $> r$, $\dim W \geq rn - r + 1$ and $|F| \geq r+1$, then W is r-decomposable (that is: $W \subset E_1 \otimes F^n + F^n \otimes E_2$ for some subspaces $E_1, E_2 \subset F^n$ such that $\dim E_1 + \dim E_2 = r$).

Contrary to Theorems 2 and 3, this result does depend on the field, as the following example, which has been suggested by the referee, indicates: Let W be the 5-dimensional space of all

$$\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & a+b \end{pmatrix} \tag{5}$$

over $GF(2)$. Clearly W does not contain a non-singular matrix, yet W is not 2-decomposable. For otherwise W'—the space of all matrices of the form (5) over say, $GF(4)$—would also be 2-decomposable, which is impossible since W' contains non-singular matrices.

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