Homology of Balanced Complexes via the Fourier Transform

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Abstract

Let $G_0, \ldots, G_k$ be finite abelian groups and let $G_0 * \cdots * G_k$ be the join of the 0-dimensional complexes G_i. We give a characterization of the the integral k-coboundaries of subcomplexes of $G_0 * \cdots * G_k$ in terms of the Fourier transform on the group $G_0 \times \cdots \times G_k$. This provides a short proof of an extension of a recent result of Musiker and Reiner on a topological interpretation of the cyclotomic polynomial.

1 Introduction

Let $G_0, \ldots, G_k$ be finite abelian groups with the discrete topology and let $N = \prod_{i=0}^{k}(|G_i| - 1)$. The simplicial join $Y = G_0 * \cdots * G_k$ is homotopy equivalent to a wedge of N k-dimensional spheres (see e.g. Theorem 1.3 in [1]). Subcomplexes of Y are called *balanced complexes* (see [5]). Denote the $(k-1)$-dimensional skeleton of Y by $Y^{(k-1)}$. Let A be a subset of $G_0 \times \cdots \times G_k$. Regarding each $a \in A$ as an oriented k-simplex of Y, we consider the balanced complex

$$X(A) = X_{G_0,\ldots,G_k}(A) = Y^{(k-1)} \cup A.$$

In this note we characterize the integral k-coboundaries of $X(A)$ in terms of the Fourier transform on the group $G_0 \times \cdots \times G_k$. As an application we

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give a short proof of an extension of a recent result of Musiker and Reiner [4] on a topological interpretation of the cyclotomic polynomial.

We recall some terminology. Let $R[G]$ denote the group algebra of a finite abelian group G with coefficients in a ring R. By writing $f=\sum_{x\in G} f(x)x \in R[G]$ we identify elements of $R[G]$ with R-valued functions on G. For a subset $A\subset G$ let $R[A]=\{f\in R[G] : \mathrm{supp}(f)\subset A\}$. A character of G is a homomorphism of G into the multiplicative group $\mathbb{C}-\{0\}$. Let $\widehat{G}$ be the character group of G and let $\mathbf{1}$ be the trivial character of G. The orthogonality relation asserts that for $\chi\in\widehat{G}$

$$\sum_{g\in G}\chi(g)=|G|\cdot\delta(\chi,\mathbf{1}) \tag{1}$$

where $\delta(\chi,\mathbf{1})=1$ if $\chi=\mathbf{1}$ and is zero otherwise. The Fourier transform is the linear bijection $\mathcal{F}:\mathbb{C}[G]\rightarrow\mathbb{C}[\widehat{G}]$ given on $f\in\mathbb{C}[G]$ and $\chi\in\widehat{G}$ by

$$\mathcal{F}(f)(\chi)=\widehat{f}(\chi)=\sum_{x\in G} f(x)\chi(x)\ .$$

Let $G=G_0\times\cdots\times G_k$ then $\widehat{G}=\widehat{G}_0\times\cdots\times\widehat{G}_k$. For $0\leq i\leq k$ let

$$L_i=G_0\times\cdots\times G_{i-1}\times G_{i+1}\times\cdots\times G_k.$$

We identify the group of integral k-cochains $C^k(X(A);\mathbb{Z})$ with $\mathbb{Z}[A]$ and the group of integral $(k-1)$-cochains $C^{k-1}(X(A);\mathbb{Z})=C^{k-1}(X(G);\mathbb{Z})$ with the $(k+1)$-tuples $\psi=(\psi_0,\ldots,\psi_k)$ where $\psi_i\in\mathbb{Z}[L_i]$. The coboundary map

$$d_{k-1}:C^{k-1}(X(G);\mathbb{Z})\rightarrow C^k(X(G);\mathbb{Z})$$

is given by

$$d_{k-1}\psi(g_0,\ldots,g_k)=\sum_{i=0}^{k}(-1)^i\psi_i(g_0,\ldots,g_{i-1},g_{i+1},\ldots,g_k).$$

For $0\leq i\leq k$ let $\mathbf{1}_i$ denote the trivial character of G_i and let

$$\widehat{G}^{+}=(\widehat{G}_0-\{\mathbf{1}_0\})\times\cdots\times(\widehat{G}_k-\{\mathbf{1}_k\}).$$

For $A\subset G$ and $f\in\mathbb{Z}[G]$ let $f_{|A}\in\mathbb{Z}[A]$ be the restriction of f to A. The group

$$\mathrm{B}^k(X(A);\mathbb{Z})=\{d_{k-1}\psi_{|A}:\psi\in C^{k-1}(X(G);\mathbb{Z})\}$$

of integral k-coboundaries of $X(A)$ is characterized by the following

Proposition 1.1. *For any $A \subset G$*

$$\mathrm{B}^k(X(A);\mathbb{Z}) = \{f_{|A} : f \in \mathbb{Z}[G] \ \ \textit{such that} \ \ \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+\}.$$

As an application of Proposition 1.1 we study the homology of a family of balanced complexes introduced by Musiker and Reiner [4]. Let $p_0, \ldots, p_k$ be distinct primes and for $0 \le i \le k$ let $G_i = \mathbb{Z}/p_i\mathbb{Z} = \mathbb{Z}_{p_i}$. Writing $n = \prod_{i=0}^k p_i$ let

$$\theta : \mathbb{Z}_n \to G = G_0 \times \cdots \times G_k$$

be the standard isomorphism given by

$$\theta(x) = (x(\bmod\ p_0), \ldots, x(\bmod\ p_k)).$$

For any ℓ let $\mathbb{Z}_\ell^\times = \{m \in \mathbb{Z}_\ell : \gcd(m,\ell) = 1\}$. Let $\varphi(n) = |\mathbb{Z}_n^\times| = \prod_{i=0}^k (p_i - 1)$ be the Euler function of n and let $A_0 = \{\varphi(n)+1, \varphi(n)+2, \ldots, n-2, n-1\}$. For $A \subset \{0, \ldots, \varphi(n)\}$ consider the complex

$$K_A = X(\theta(A \cup A_0)) \subset \mathbb{Z}_{p_0} * \cdots * \mathbb{Z}_{p_k}\ .$$

Let $\omega = \exp(\frac{2\pi i}{n})$ be a fixed primitive n-th root of unity. The n-th cyclotomic polynomial (see e.g. [2]) is given by

$$\Phi_n(z) = \prod_{j \in \mathbb{Z}_n^\times} (z - \omega^j) = \sum_{j=0}^{\varphi(n)} c_j z^j \in \mathbb{Z}[z].$$

Musiker and Reiner [4] discovered the following remarkable connection between the coefficients of $\Phi_n(z)$ and the homology of the complexes $K_{\{j\}}$.

Theorem 1.2 (Musiker and Reiner)**.** *For any $j \in \{0, \ldots, \varphi(n)\}$*

$$\tilde{\mathrm{H}}_i(K_{\{j\}};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & i = k-1 \\ \mathbb{Z} & i = k \text{ and } c_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The next result extends Theorem 1.2 to general K_A's. Let

$$c_A = (c_j : j \in A) \in \mathbb{Z}^A$$

and

$$d_A = \begin{cases} \gcd(c_A) & c_A \neq 0 \\ 0 & c_A = 0. \end{cases}$$

Theorem 1.3. *For any* $A \subset \{0, \dots, \varphi(n)\}$

$$\tilde{\mathrm{H}}^i(K_A;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = k-1 \text{ and } d_A = 0 \\ \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z} & i = k \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{\mathrm{H}}_i(K_A;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/d_A\mathbb{Z} & i = k-1 \\ \mathbb{Z}^{|A|} & i = k \text{ and } d_A = 0 \\ \mathbb{Z}^{|A|-1} & i = k \text{ and } d_A \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.1 is proved in Section 2. It is then used in Section 3 to obtain an explicit form of the k-coboundaries of K_A (Proposition 3.1) that directly implies Theorem 1.3.

2 k-Coboundaries and Fourier Transform

Proof of Proposition 1.1. It suffices to consider the case $A = G$. Let $\psi = (\psi_0, \dots, \psi_k) \in C^{k-1}(X(G);\mathbb{Z})$. Using (1) it follows for any $\chi = (\chi_0, \dots, \chi_k) \in \widehat{G}$

$$\widehat{d_{k-1}\psi}(\chi) = \sum_{g=(g_0,\dots,g_k)\in G} d_{k-1}\psi(g)\chi(g) =$$

$$\sum_{(g_0,\dots,g_k)} \sum_{i=0}^{k} (-1)^i \psi_i(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_k) \prod_{j=0}^{k} \chi_j(g_j) =$$

$$\sum_{i=0}^{k} (-1)^i \sum_{(g_0,\dots,g_{i-1},g_{i+1},\dots,g_k)} \psi_i(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_k) \prod_{j\neq i} \chi_j(g_j) \sum_{g_i} \chi_i(g_i) =$$

$$\sum_{i=0}^{k} (-1)^i \widehat{\psi}_i(\chi_0, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_k)|G_i|\delta(\chi_i, \mathbf{1}_i).$$

Therefore $\mathrm{supp}(\widehat{d_{k-1}\psi}) \subset \widehat{G} - \widehat{G}^+$ and so

$$U_1 \stackrel{\text{def}}{=} \mathrm{B}^k(X(G);\mathbb{Z}) \subset \{f \in \mathbb{Z}[G] : \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+\} \stackrel{\text{def}}{=} U_2.$$

Since $X(G)$ is homotopy equivalent to a wedge of $\prod_{i=0}^{k}(|G_i|-1) = |\widehat{G}^+|$ k-dimensional spheres, it follows that $\mathrm{H}^k(X(G);\mathbb{Z}) = \mathbb{Z}[G]/U_1$ is free of rank $|\widehat{G}^+|$ and hence $rank\ U_1 = |\widehat{G}| - |\widehat{G}^+|$. On the other hand, the injectivity of the Fourier transform implies that

$$rank\ U_2 \leq \dim_{\mathbb{C}}\{f \in \mathbb{C}[G] : \mathrm{supp}(\widehat{f}) \subset \widehat{G} - \widehat{G}^+\} = |\widehat{G}| - |\widehat{G}^+|$$

and therefore $rank\ U_2/U_1 = 0$. Since $U_2/U_1 \subset \mathrm{H}^k(X(G);\mathbb{Z})$ is free it follows that $U_1 = U_2$. □

3 The Homology of K_A

Recall that, in the context of Theorems 1.2 and 1.3, one chooses $G = \mathbb{Z}_{p_0} \times \cdots \times \mathbb{Z}_{p_k}$ and $n = \prod_{j=0}^{k} p_j$. For $h \in \mathbb{Z}[G]$ let $\theta^* h \in \mathbb{Z}[\mathbb{Z}_n]$ be the pullback of h given by $\theta^* h(x) = h(\theta(x))$. For any ℓ we identify the character group $\widehat{\mathbb{Z}_\ell}$ with $\mathbb{Z}_\ell$ via the isomorphism $\eta_\ell : \mathbb{Z}_\ell \to \widehat{\mathbb{Z}_\ell}$ given by $\eta_\ell(y)(x) = \exp(2\pi i xy/\ell)$. The Fourier transform on $\mathbb{Z}_\ell$ is then regarded as the automorphism of $\mathbb{C}[\mathbb{Z}_\ell]$ given by

$$\widehat{f}(y) = \sum_{x \in \mathbb{Z}_\ell} f(x) \exp(\frac{2\pi i xy}{\ell}).$$

Proposition 1.1 implies the following characterization of the integral k-coboundaries of K_A. For $A \subset \{0, \ldots, \varphi(n)\}$ let θ_A denote the restriction of θ to $A \cup A_0$ and let θ_A^* be the induced isomorphism from $\mathbb{Z}[\theta(A \cup A_0)]$ to $\mathbb{Z}[A \cup A_0]$. Let

$$\mathcal{B}(A) = \{f_{|A \cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \ \ such\ that\ \ \widehat{f}(1) = 0\}.$$

Proposition 3.1.

$$\theta_A^*(\mathrm{B}^k(K_A;\mathbb{Z})) = \mathcal{B}(A).$$

Proof. We first examine the relation between the Fourier transforms on $\mathbb{Z}_n$ and on G. Let

$$\lambda = \sum_{j=0}^{k} \prod_{t \neq j} p_t \in \mathbb{Z}_n^\times.$$

For any $h \in \mathbb{Z}[G]$ and $m \in \mathbb{Z}_n$

$$\widehat{\theta^* h}(\lambda m) = \sum_{x \in \mathbb{Z}_n} \theta^* h(x) \exp(\frac{2\pi i x \lambda m}{n}) =$$

$$\sum_{x\in\mathbb{Z}_n} h(\theta(x))\exp(\sum_{j=0}^{k}\frac{2\pi i x m}{p_j}) = \widehat{h}(\theta(m)). \tag{2}$$

Noting that

$$\theta^{-1}(\widehat{G}^{+}) = \theta^{-1}(\mathbb{Z}_{p_0}^{\times}\times\cdots\times\mathbb{Z}_{p_k}^{\times}) = \mathbb{Z}_n^{\times} = \lambda\mathbb{Z}_n^{\times}\ ,$$

it follows from Proposition 1.1 and Eq. (2) that

$$\mathrm{B}^k(K_A;\mathbb{Z}) = \{h_{|\theta(A\cup A_0)} : h\in\mathbb{Z}[G]\ \ \textit{such that}\ \ \mathrm{supp}(\widehat{h})\subset\widehat{G}-\widehat{G}^{+}\} =$$

$$(\theta_A^*)^{-1}\{f_{|A\cup A_0} : f\in\mathbb{Z}[\mathbb{Z}_n]\ \ \textit{such that}\ \ \mathrm{supp}(\widehat{f})\subset\mathbb{Z}_n-\mathbb{Z}_n^{\times}\}. \tag{3}$$

Let $\mathcal{P}_n = \{\omega^m : m\in\mathbb{Z}_n^{\times}\}$ be the set of primitive n-th roots of 1. The Galois group $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ acts transitively on $\mathcal{P}_n$. Hence, by Eq. (3):

$$\theta_A^*(\mathrm{B}^k(K_A;\mathbb{Z})) = \{f_{|A\cup A_0} : f\in\mathbb{Z}[\mathbb{Z}_n]\ \ \textit{such that}\ \ \mathrm{supp}(\widehat{f})\subset\mathbb{Z}_n-\mathbb{Z}_n^{\times}\} =$$

$$\{f_{|A\cup A_0} : f\in\mathbb{Z}[\mathbb{Z}_n]\ \textit{such that}\ \widehat{f}(m) = \sum_{x\in\mathbb{Z}_n} f(x)\omega^{mx} = 0\ \ \textit{for all}\ \ m\in\mathbb{Z}_n^{\times}\} =$$

$$\{f_{|A\cup A_0} : f\in\mathbb{Z}[\mathbb{Z}_n]\ \ \textit{such that}\ \ \widehat{f}(1) = 0\} = \mathcal{B}(A).$$

□

Corollary 3.2. θ_A^* *induces an isomorphism between* $\mathrm{H}^k(K_A;\mathbb{Z})$ *and*

$$\mathcal{H}(A) \overset{\mathrm{def}}{=} \mathbb{Z}[A\cup A_0]/\mathcal{B}(A).$$

□

For $j\in A\cup A_0$ let $g_j\in\mathbb{Z}[A\cup A_0]$ be given by $g_j(i) = 1$ if $i = j$ and $g_j(i) = 0$ otherwise. Let $[g_j]$ be the image of g_j in $\mathcal{H}(A)$. The computation of $\mathcal{H}(A)$ depends on the following

Claim 3.3.

(i) $\mathcal{H}(A)$ *is generated by* $\{[g_j] : j\in A\}$.

(ii) The minimal relation between $\{[g_j]\}_{j\in A}$ *is* $\sum_{j\in A} c_j[g_j] = 0$.

Proof of (i). Let $t \in A_0$. There exist $u_0, \ldots, u_{\varphi(n)-1} \in \mathbb{Z}$ such that

$$\sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0.$$

Let $f \in \mathbb{Z}[\mathbb{Z}_n]$ be given by

$$f(\ell) = \begin{cases} u_\ell & 0 \le \ell \le \varphi(n) - 1 \\ 1 & \ell = t \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\widehat{f}(1) = \sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0 \ ,$$

it follows that

$$\sum_{j \in A} u_j g_j + g_t = f_{|A \cup A_0} \in \mathcal{B}(A).$$

Hence $[g_t] = -\sum_{j \in A} u_j [g_j]$.

Proof of (ii). Let $f \in \mathbb{Z}[\mathbb{Z}_n]$ be given by $f(\ell) = c_\ell$ if $0 \le \ell \le \varphi(n)$ and zero otherwise. Since $\widehat{f}(1) = \Phi_n(\omega) = 0$, it follows that

$$\sum_{j \in A} c_j g_j = f_{|A \cup A_0} \in \mathcal{B}(A).$$

Hence $\sum_{j \in A} c_j [g_j] = 0$. Conversely, suppose that $\sum_{j \in A} \alpha_j [g_j] = 0$ for integers $\{\alpha_j\}_{j \in A}$. Then there exists an $h \in \mathbb{Z}[\mathbb{Z}_n]$ such that $\widehat{h}(1) = 0$ and $h_{|A \cup A_0} = \sum_{j \in A} \alpha_j g_j$. In particular $h(\ell) = 0$ for $\ell \ge \varphi(n) + 1$. Let $p(z) = \sum_{\ell=0}^{\varphi(n)} h(\ell) z^\ell$ then $p(\omega) = \widehat{h}(1) = 0$. Hence $p(z) = r\Phi_n(z)$ for some $r \in \mathbb{Z}$. Therefore $\alpha_j = h(j) = rc_j$ for all $j \in A$.

□

Proof of Theorem 1.3. Corollary 3.2 and Claim 3.3 imply that

$$\mathrm{H}^k(K_A; \mathbb{Z}) \cong \mathcal{H}(A) = \mathbb{Z}[A]/\mathbb{Z}c_A \cong \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z} \ . \tag{4}$$

The remaining parts of Theorem 1.3 are formal consequences of (4) and the universal coefficient theorem (see e.g. [3]):

$$0 \leftarrow \mathrm{Hom}(\mathrm{H}_p(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow \mathrm{H}^p(K_A;\mathbb{Z}) \leftarrow \mathrm{Ext}(\mathrm{H}_{p-1}(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow 0\ . \quad (5)$$

First consider the case $c_A = 0$. By (4) and (5)

$$0 \leftarrow \mathrm{Hom}(\mathrm{H}_k(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow \mathbb{Z}^{|A|} \leftarrow \mathrm{Ext}(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow 0\ .$$

Therefore $\mathrm{H}_k(K_A;\mathbb{Z}) \cong \mathbb{Z}^{|A|}$ and $\mathrm{H}_{k-1}(K_A;\mathbb{Z})$ is torsion free. The Euler-Poincaré relation

$$rank\ \mathrm{H}_k(K_A;\mathbb{Z}) = rank\ \tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) + |A| - 1 \quad (6)$$

then implies that $\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) \cong \mathbb{Z}$ and

$$\tilde{\mathrm{H}}^{k-1}(K_A;\mathbb{Z}) \cong \mathrm{Hom}(\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}.$$

Next assume that $c_A \neq 0$. By (4) and (5)

$$0 \leftarrow \mathrm{Hom}(\mathrm{H}_k(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z} \leftarrow \mathrm{Ext}(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}) \leftarrow 0\ .$$

Therefore $\mathrm{H}_k(K_A;\mathbb{Z}) \cong \mathbb{Z}^{|A|-1}$ and $\mathrm{Ext}(\mathrm{H}_{k-1}(K_A;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}/d_A\mathbb{Z}$. It follows by (6) that $rank\ \tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) = 0$. Hence $\tilde{\mathrm{H}}_{k-1}(K_A;\mathbb{Z}) = \mathbb{Z}/d_A\mathbb{Z}$ and $\tilde{\mathrm{H}}^{k-1}(K_A;\mathbb{Z}) = 0$.

□

Remark: In the proof of (ii) it was observed that the function $f \in \mathbb{Z}[\mathbb{Z}_n]$ given by $f(\ell) = c_\ell$ if $0 \leq \ell \leq \varphi(n)$ and zero otherwise, is the image under θ^* of a k-coboundary of $X(G)$. This fact also appears (with a different proof) in Proposition 24 of [4] and is attributed there to D. Fuchs.

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