Additive Bases of Vector Spaces over Prime Fields

N. Alon*

Department of Mathematics, Sackler Faculty of Exact Sciences,
Tel Aviv University, Tel Aviv, Israel and
Bellcore, Morristown, New Jersey 07960

N. Linial

Department of Computer Science, Hebrew University of Jerusalem,
Jerusalem, Israel and
IBM Almaden Research Center,
650 Harry Road, San Jose, California 95120

AND

R. Meshulam'

RUTCOR, Rutgers University, New Brunswick,
New Jersey 08803 and Department of Mathematics,
M.I.T., Cambridge, Massachusetts 02139

Communicated by the Managing Editors

Received November 1, 1988

It is shown that for any \( t > c_p \log n \) linear bases \( B_1, \ldots, B_t \) of \( \mathbb{Z}_p^n \) their union (with repetitions) \( \bigcup_{i=1}^t B_i \) forms an additive basis of \( \mathbb{Z}_p^n \); i.e., for any \( x \in \mathbb{Z}_p^n \) there exist \( A_1 \subseteq B_1, \ldots, A_t \subseteq B_t \) such that \( x = \sum_{i=1}^t \sum_{j \in A_i} x_j \).


I. INTRODUCTION

Let \( \mathbb{Z}_p^n \) be the \( n \)-dimensional linear space over the prime field \( \mathbb{Z}_p \). An additive basis of \( \mathbb{Z}_p^n \) is a multisett \( \{x_1, \ldots, x_m\} \subseteq \mathbb{Z}_p^n \), such that any \( x \in \mathbb{Z}_p^n \) is representable as a \( 0 \)-1 combination of the \( x_i \)'s. Let \( f(p, n) \) denote the minimal integer \( t \), such that for any \( t \) (linear) bases \( B_1, \ldots, B_t \) of \( \mathbb{Z}_p^n \), the union (with repetitions) \( \bigcup_{i=1}^t B_i \) forms an additive basis of \( \mathbb{Z}_p^n \).

The problem of determining or estimating \( f(p, n) \), besides being interesting in its own right, is naturally motivated by the study of universal

* Research supported in part by Allon Fellowship and by a Bat Sheva de Rothschild Grant.

† Research supported in part by Air Force Office of Scientific Research Grant AFOSR-0271.

203

0097-3165/91 $3.00

Copyright © 1991 by Academic Press, Inc.
All rights of reproduction in any form reserved.
flows in graphs (see [JLPT]). The authors of [JLPT] conjectured that \( f(p, n) \) is bounded above by a function of \( p \) alone.

Clearly \( f(p, n) \geq p - 1 \), as the union of \( p - 2 \) identical copies of the same basis does not form an additive basis. For \( p \geq 3 \) and \( n \geq 2 \), this trivial lower bound may be improved to \( f(p, n) \geq p \). It clearly suffices to show this for \( n = 2 \). Let \( \{a_1, a_2\} \) be any basis of \( \mathbb{Z}_p^n \), and consider \( p - 2 \) copies of \( \{a_1, a_2\} \) and one copy of \( \{a_1 + a_2, a_1 - a_2\} \). As \( -a_2 \) is not in the additive span of these \( p - 1 \) bases we obtain \( f(p, 2) \geq p \).

In this paper we give two proofs of the following.

**Theorem 1.1.** \( f(p, n) \leq c(p) \log n \).

In Section 2 we use exponential sums to show that \( f(p, n) \leq 1 + (p^2/2) \log 2pn \). The algebraic method in Section 3 gives the somewhat better bound \( f(p, n) \leq (p - 1) \log n + p - 2 \). The final Section 4 contains some concluding remarks and open problems.

### 2. ADDITIVE SPANNING AND EXPONENTIAL SUMS

Let \( B_1, \ldots, B_t \) be any \( t > (p^2/2) \log 2pn \) bases of \( \mathbb{Z}_p^n \). Denote by \( \{x_1, \ldots, x_m\}, m = tn \), their union with repetitions, and for any \( x \in \mathbb{Z}_p^n \), let \( N(x) = \{(e_1, \ldots, e_m) : \sum_{j=1}^m e_jx_j = x, e_j \in \{0, 1\}\} \).

We shall show that \( N(x) > 0 \) for all \( x \in \mathbb{Z}_p^n \). For \( x, y \in \mathbb{Z}_p^n \), \( x \cdot y \) is their standard inner product, and for \( a \in \mathbb{Z}_p \), let \( e(a) = e^{2\pi i a/p} \).

Following Baker and Schmidt [BS, p. 471] we represent \( N(x) \) as an exponential sum,

\[
N(x) = \sum_{x \in \{0, 1\}^m} \frac{1}{p^m} \sum_{y \in \mathbb{Z}_p^n} e \left( y \cdot \left( \sum_{j=1}^m e_jx_j - x \right) \right) \\
= \frac{1}{p^m} \sum_{y \in \mathbb{Z}_p^n} e(y \cdot x) \sum_{x \in \{0, 1\}^m} e \left( y \cdot \sum_{j=1}^m e_jx_j \right) \\
= \frac{1}{p^m} \sum_{y \in \mathbb{Z}_p^n} e(y \cdot x) \sum_{x_0=0}^1 \cdots \sum_{x_m=0}^1 e(x_0y \cdot x_m) \\
= \frac{2^m}{p^m} \sum_{y \in \mathbb{Z}_p^n} e(y \cdot x) \prod_{j=1}^m \left| 1 + e(y \cdot x_j) \right|/2.
\]

Therefore

\[
|N(x) - \frac{2^m}{p^m} \sum_{y \in \mathbb{Z}_p^n} \prod_{j=1}^m \left| 1 + e(y \cdot x_j) \right|/2 | 
\tag{2.1}
\]
(The same estimate is also used in [BS].) Next we estimate the right hand side of (2.1). For any fixed basis $B$ of $Z_p^n$, and $y \in Z_p^n$ let $P_B(y) = \prod_{b \in B} (1 + e(y \cdot b))/2$.

Since $P_B(y)$ depends only on the list of inner products $(y \cdot b : b \in B)$, it follows that the multiset $\{P_B(y) : y \in Z_p^n\}$ is independent of the choice of the basis $B$. Choosing $B = \{b_1, \ldots, b_n\}$ to be the standard basis of $Z_p^n$, and noting that for $y = (y_1, \ldots, y_n)$

$$\left| \frac{1 + e(b_j \cdot y)}{2} \right| = \frac{1 + e(y_j)}{2} = \left| \cos \frac{\pi y_j}{p} \right|,$$

we obtain

$$\sum_{y \in Z_p^n} \left| \prod_{j=1}^n \frac{1 + e(y \cdot b_j)}{2} \right| = \sum_{y \in Z_p^n} \prod_{j=1}^n P_B(y) \leq \sum_{y \in Z_p^n} P_B(y)' = \sum_{y \in Z_p^n} \prod_{j=1}^n \left| \cos \frac{\pi y_j}{p} \right| \leq \left( \sum_{k=0}^{p-1} \left| \cos \frac{\pi k}{p} \right|^n \right)^{1/p} \leq \left( 1 + (p-1) \cos \frac{2\pi}{p} \right)^n \leq \left( 1 + \frac{1}{2n} \right)^n \leq e^{1/2}. \quad (2.2)$$

Combining (2.1) and (2.2) we obtain

$$\left| N(x) - \frac{2^n}{p^n} \right| \leq \frac{2^n}{p^n} (e^{1/2} - 1) < \frac{2^n}{p^n}.$$

Hence $N(x) > 0$ for all $x \in Z_p^n$.  

3. PERMANENTS AND VECTOR SUMS

In this section we present a second proof of Theorem 1.1, with a somewhat better estimate for $c(p)$. Specifically, we prove the following proposition.

**Proposition 3.1.** Let $A_1 = \{g^{11}, \ldots, g^{1n}\}$, $A_2 = \{g^{21}, \ldots, g^{2n}\}$, $\ldots$, $A_l = \{g^{l1}, \ldots, g^{ln}\}$ be $l$ bases of the vector space $Z_p^n$. If

$$\left( 1 - \frac{1}{p-1} \right)^{l-2 + \frac{2}{n}} < 1 \quad (3.1)$$
then for any vector \( b \in \mathbb{Z}_p^n \), there are \( c_{ij} \in \{0, 1\} \) (\( 1 \leq i \leq l, 1 \leq j \leq n \)), such that \( \sum_{i,j} c_{ij} a_{ij} = b \). In particular, the conclusion holds provided \( l \geq (p-1) \log n + p - 2 \).

The proof presented here differs considerably from the one given in Section 2 and is based on some simple properties of permanents over finite fields. The basic method resembles the one used in [AT], but several additional ideas are incorporated.

It is convenient to split the proof into several lemmas. We start with the following simple lemma (which appears in a similar context in [AFK]).

**Lemma 3.2.** Let \( P = P(x_1, ..., x_m) \) be a multilinear polynomial with \( m \) variables \( x_1, ..., x_m \) over a commutative ring with identity \( R \); i.e., \( P = \sum_{U \subseteq \{1, ..., m\}} a_U \prod_{i \in U} x_i \), where \( a_U \in R \). If \( P(x_1, ..., x_m) = 0 \) for each \( (x_1, ..., x_m) \in \{0, 1\}^m \) then \( P \equiv 0 \), i.e., \( a_U = 0 \) for all \( U \subseteq \{1, ..., m\} \).

**Proof.** We apply induction on \( m \). The result is trivial for \( m = 1 \). Assuming it holds for \( m-1 \) we prove it for \( m \). Clearly \( P(x_1, ..., x_m) = P_1(x_1, ..., x_{m-1})x_m + P_2(x_1, ..., x_{m-1}) \), where \( P_1 \) and \( P_2 \) are multilinear polynomials in \( x_1, ..., x_{m-1} \). Moreover, it is easy to see that \( P_1 \) and \( P_2 \) satisfy the hypotheses of the lemma for \( m-1 \). By the induction hypothesis \( P_1 \equiv P_2 \equiv 0 \), completing the proof. \( \square \)

The next lemma shows a connection between a permanent of a matrix and the possible sums of subsets of its set of columns. This connection is crucial for our proof.

**Lemma 3.3.** Let \( A = (a_{ij}) \) be an \( m \times m \) matrix over the finite prime field \( \mathbb{Z}_p \). Suppose that \( \text{Per}(A) \neq 0 \) (over \( \mathbb{Z}_p \)). Then for any vector \( \epsilon = (\epsilon_1, ..., \epsilon_m) \in \mathbb{Z}_p^m \) there are \( \epsilon_1, ..., \epsilon_m \in \{0, 1\} \) such that \( \sum_{i=1}^m \epsilon_i a_{ij} \neq \epsilon_j \), for all \( 1 \leq i \leq m \). In other words, for any vector \( \epsilon \) there is a subset of the columns of \( A \) whose sum differs from \( \epsilon \) in each coordinate.

**Proof.** Suppose the lemma is false and let \( A = (a_{ij}) \) and \( \epsilon \) be a counter-example. Consider the polynomial \( P = P(x_1, ..., x_m) = \prod_{i=1}^m (\sum_{j=1}^n a_{ij}x_j - \epsilon_j) \). By assumption, \( P(x_1, ..., x_m) = 0 \) for each \( (x_1, ..., x_m) \in \{0, 1\}^m \). Let \( \overline{P} = \overline{P}(x_1, ..., x_m) \) be the multilinear polynomial obtained from \( P \) by writing \( P \) as a sum of monomials and replacing each monomial \( a_U \prod_{i \in U} x_i \) where \( U \subseteq \{1, ..., m\} \) and \( \delta_i > 0 \), by \( a_U \prod_{i \in U} x_i \). Clearly \( \overline{P}(x_1, ..., x_m) = \overline{P}(x_1, ..., x_m) = 0 \) for each \( (x_1, ..., x_m) \in \{0, 1\}^m \). By Lemma 3.2 we conclude that \( \overline{P} \equiv 0 \). However, this is impossible, since the coefficient of \( \prod_{i=1}^m x_i \) in \( \overline{P} \) (which equals the coefficient of that product in \( P \)) is \( \text{Per}(A) \neq 0 \). This completes the proof. \( \square \)

For a (column) vector \( \bar{v} = (v_1, ..., v_m) \in \mathbb{Z}_p^n \) let us denote by \( \bar{v}^* = \bar{v}^*(p) \) the (column) vector in \( \mathbb{Z}_p^{(p-1)n} \) defined by \( \bar{v}^*_i = \bar{v}_i \) for all \( 1 \leq i \leq p-1 \).
1 ≤ j ≤ n. Thus g^* is simply the tensor product of g with a vector of (p - 1) 1's. Clearly g^* = g^*(p) depends on p as well as on p, but since p remains fixed during this section we usually omit it and simply write g^*.

A simple corollary of Lemma 3.3 is the following.

**Corollary 3.4.** Let g^1, ..., g^((p-1)n) be (p - 1)n vectors in Z^*_p. Let A be the (p - 1)n by (p - 1)n matrix whose columns are the vectors g^1*, ..., g^((p-1)n)*. If Per A ≠ 0 then any vector b ∈ Z^*_p is a sum of a certain subset of the vectors g^1, ..., g^((p-1)n).

**Proof.** Let c = (c^1, ..., c^((p-1)n)) ∈ Z^((p-1)n)_p be a vector satisfying \{c_{i(p-1)+j}: 1 ≤ i ≤ p - 1\} = Z^*_p \{b\} for each j, 1 ≤ j ≤ n. By Lemma 3.3 there are \(e_i, ..., e_{(p-1)n}\) ∈ \{0, 1\} such that for any 1 ≤ i ≤ p - 1 and any 1 ≤ j ≤ n

\[\sum_{i=1}^{(p-1)n} e_i g^{i*}_{(i-1)n+j} \neq c_{(i-1)n+j}.\]

However, since the left hand side in the last equality is simply \(\sum_{i=1}^{(p-1)n} e_i g^{i}_j\), this shows that \(\sum_{i=1}^{(p-1)n} e_i g^{i}_j \neq Z^*_p \{b\}\) for each 1 ≤ j ≤ n. Consequently, \(\sum_{i=1}^{(p-1)n} e_i g^{i} = b\), completing the proof.

The last corollary implies that in order to prove Proposition 3.1 it suffices to show that from any sequence of l n vectors consisting l bases of Z^*_p one can choose (p - 1)n distinct members \(g^1, ..., g^{(p-1)n}\) of the sequence such that the permanent of the matrix whose columns are \(g^1*, ..., g^{(p-1)n*}\) is nonzero (over Z^*). In what follows we show that this is always possible provided (3.1) holds.

**Lemma 3.5.** Let D = \{d^1, ..., d^n\} be a basis of Z^*_p, and let A_D be a (p - 1)n by (p - 1)n matrix whose columns are the vectors d^1*, ..., d^n*, each appearing p - 1 times. Then Per A_D ≠ 0.

**Proof.** Let E = \{e^1, ..., e^n\} be the standard basis of Z^*_p, and let A_E be the (p - 1)n by (p - 1)n matrix whose columns are e^1*, ..., e^n*, each appearing (p - 1) times. One can easily check that Per A_E is simply the number of perfect matchings in the union of n pairwise disjoint complete bipartite graphs K_{p-1,p-1}, which is ((p-1)!)^n ≠ 0 (in Z^*_p). Since D is a basis, each column of A_E is a linear combination of the columns of A_D. By the multilinearity of the permanent function it follows that Per A_E is a linear combination (over Z^*_p) of permanents of matrices whose columns are columns of A_D. Since Per A_E ≠ 0, we conclude that there is a (p - 1)n by (p - 1)n matrix M, each column of which is d^i* for some 1 ≤ i ≤ n, satisfying Per M ≠ 0. However, if the same column appears in M p times or more,
than Per $M$ is divisible by $p!$, and is thus 0. It follows that no column appears in $M$ more than $(p-1)$ times, and hence $M$ equals $A_0$ up to a permutation of the columns. Thus $\text{Per } A_0 = \text{Per } M \neq 0$, completing the proof.

**Lemma 3.6.** Let $A_i = \{g^{i1}, g^{i2}, \ldots, g^{in}\}, \ldots, A_l = \{g^{l1}, g^{l2}, \ldots, g^{ln}\}$ be $l$ bases of $\mathbb{Z}_p^n$ and let $S = (s_1, \ldots, s_n)$ be the sequence of length $l \cdot n$ of vectors in $\mathbb{Z}_p^{(p-1)n}$ given by $s_i j = g^{ij}$ for all $1 \leq i \leq l, 1 \leq j \leq n$. Suppose that for some integer $h$

$$\left(1 - \frac{1}{p-1}\right)^{l-h} \cdot (p-1) \cdot n < h + 1. \tag{3.2}$$

Then there are $l \cdot n$ distinct indices $1 \leq i_1 < i_2 < \cdots < i_{(p-1)n} \leq ln$ such that the matrix whose columns are $\{s_{i_j}; 1 \leq j \leq (p-1)n\}$ has a nonzero permanent.

**Proof.** Given a $(p-1)n$ by $(p-1)n$ matrix $B$ whose columns are members of $S$, we call a column of $B$ a repeated column if the same member of $S$ appears in at least one additional column of $B$. Let $c(B)$ denote the total number of repeated columns of $B$. Our objective is to construct a matrix with no repeated columns whose permanent is nonzero. To this end, we construct a sequence of matrices $B_1, B_2, \ldots$, with nonzero permanents as follows. Let $B_1$ be the $(p-1)n$ by $(p-1)n$ matrix whose columns are $s_1, \ldots, s_n$, each appearing $(p-1)$ times. By Lemma 3.5 Per $B_1 \neq 0$, and clearly, all the $(p-1)n$ columns of $B_1$ are repeated columns. Since $A_2$ is a basis, each column of $B_1$ is a linear combination of $s_{n+1}, \ldots, s_{2n}$. Let us replace all but one of the $p-1$ occurrences of each $s_i$ in $B_1$ by the linear combination of $s_{n+1}, \ldots, s_{2n}$ expressing it. By the multilinearity of the permanent function, this enables us to write Per $B_1 \neq 0$ as a linear combination of permanents of matrices whose columns are all from the set $\{s_1, \ldots, s_{2n}\}$. Obviously, at least one of these matrices has a nonzero permanent.

Let $B_2$ be such a matrix. Then, there are at least $n$ nonrepeated columns of $B_2$, since each of the $n$ vectors $s_1, \ldots, s_n$ appears precisely once in it. Hence, $c(B_2) \leq (1-1/(p-1))(p-1)n$. It is also clear that no $s_i$ appears more than $p-1$ times as a column of $B_2$, as Per$(B_2) \neq 0$. Assume, by induction, that we have already constructed, for each $i \leq k$, a $(p-1)n$ by $(p-1)n$ matrix $B_{k+1}$, each column of which belongs to the set $s_1, \ldots, s_{(k+1)n}$, satisfying

$$\text{Per}(B_{k+1}) \neq 0 \quad \text{and} \quad c(B_{k+1}) \leq \left(1 - \frac{1}{p-1}\right)^{k} (p-1)n. \tag{3.3}$$

Let us show that if $k + 2 \leq l$ we can construct a matrix $B_{k+2}$ with the same properties. If $c(B_{k+1}) = 0$ simply take $B_{k+2} = B_{k+1}$. Otherwise, replace
each occurrence of each repeated column of $B_{k+1}$ but one, by a linear combination of \( s_{(k+1)n+1}, \ldots, s_{(k+2)n} \), and apply, as before, multilinearity to obtain a matrix $B_{k+2}$ with a nonzero permanent. Since no repeated column can appear in $B_{k+1}$ more than $p-1$ times, we conclude that
\[
c(B_{k+2}) \leq \left(1 - \frac{1}{p-1}\right) c(B_{k+1}) \leq \left(1 - \frac{1}{p}\right)^{k+1} (p-1)n.
\]

In particular, taking $i = i - h$, it follows from (3.2) and (3.3) that there is a matrix $B_{i-h+1}$, each column of which belongs to the set $s_{1}, \ldots s_{(i-h+1)n}$ such that $\text{per} (B_{i-h+1}) \neq 0$ and $c(B_{i-h+1}) \leq (1 - 1/(p-1))^{i-h} (p-1)n < h+1$.

Thus $B_{i-h+1}$ has at most $h$ repeated columns. Denote these columns by $b^i, b^{i-1}, \ldots, b^{i-h+1}$. For each $i$, $0 \leq i \leq h-2$, let us express $b^{i-i}$ as a linear combination of $s_{(i-j)n+1}, \ldots, s_{(i-j)n}$. Applying multilinearity once more we obtain a matrix with nonzero permanent and no repeated columns. This completes the proof.

We are now ready to prove Proposition 3.1. Given the $l$ bases $A_1, \ldots, A_l$, where $l$ satisfies (3.1), we apply Lemma 3.6 with $h = p-2$ to conclude that there is a set $I$ of $(p-1)n$ distinct double indices $ij$ such that the matrix whose columns are $\{g^{ij} : ij \in I\}$ has a nonzero permanent. By Corollary 3.4, this implies that for any vector $\tilde{b} \in Z_p^n$ there are $e_i \in \{0, 1\}$, $(ij \in I)$, such that $\sum_{ij \in I} e_{ij} g^{ij} = \tilde{b}$. This completes the proof of Proposition 3.1. Observe that we actually proved a somewhat stronger result; if $l$ satisfies (3.1) then it is possible to choose a fixed set of $(p-1)n$ of our vectors such that any $\tilde{b} \in Z_p^n$ is a sum of a subset of this fixed set.

4. CONCLUDING REMARKS AND OPEN PROBLEMS

The main open problem is, of course, whether the union of any $c(p)$ linear bases of $Z_p^n$ is an additive basis, where $c(p)$ depends on $p$ alone. The following two results, which follow from our previous proofs of Theorem 1.1, suggest that this, indeed, may be the case.

**PROPOSITION 4.1.** For any $l$ bases $B_1, \ldots, B_l$ of $Z_p^n$, when $l \geq p \log (pn)$ there are subsets $A_i \subseteq B_i$ $(1 \leq i \leq l)$, such that $\sum_{i=1}^{l} |A_i| \leq (p-1)n$ and $\bigcup_{i=1}^{l} A_i$ (with repetitions) is an additive basis of $Z_p^n$.

**PROPOSITION 4.2.** Let $S = (s_1, s_2, \ldots, s_j)$ be a sequence of vectors in $Z_p^n$ and suppose that each subsequence of $l - (p-1)n$ members of $S$ linearly spans $Z_p^n$. Then $S$ is an additive basis of $Z_p^n$.  

The following conjecture about permanents would imply, if true, that $f(p, n) \leq p$.

**Conjecture 4.3.** For any $p$ nonsingular $n$ by $n$ matrices $A_1, A_2, ..., A_p$ over $\mathbb{Z}_p$, there is an $n$ by $p \cdot n$ matrix $C$ such that

$$\begin{bmatrix}
A_1 & A_2 & \cdots & A_p \\
A_1 & A_2 & \cdots & A_p \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_p \\
C
\end{bmatrix} \neq 0.$$

**References**


Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium