SOLUTION TO THE MOED BET EXAM OF 27/3/2009

Version 2. 30/4/2009

This version may need some revision in some places but I do not have time to read it again just now.

(No one is infallible. There is always some possibility of a mistake here.

We may perhaps write a new version of this solution later, even if there are no mistakes, if we see after reading your solutions that some things need to be discussed in more detail.

Please note that this solution is written in some detail, with extra discussion of some points. It also may sometimes describe more than one way of solving some of the problems.So parts of it might be considerably longer and sometimes more elaborate than the solutions which we would expect students to write during a test. It is designed to also help students in future semesters prepare for their tests.

If it bothers you that it is written in English, please stop and ask yourself, how are you going to manage in your future professional career if you cannot read technical explanations in English? Please make the effort NOW to get used to reading technical English. Later it will be harder for you.

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1. ALEF. The line ℓ_2 lies on the plane $x-z=1$ so it must be perpendicular (ניצב) to the vector $[1,0,-1]$. It also lies on the plane $x-y+2z=2$ so it must be perpendicular to the vector $[1,-1,2]$. It follows that ℓ_2 must be parallel to the vector

$$[1,0,-1]\times[1,-1,2]=[-1,-3,-1]\,.$$

So it is also parallel to the vector $[1,3,1]$.

Now let us find a point on ℓ_2, i.e., a point (x,y,z) which satisfies both of the equations $x-z=1$ and $x-y+2z=2$. There are of course infinitely many such points. Let us try, for example, to find one where $z=0$. Then the two equations become $x=1$ and $x-y=2$. So $y=x-2=-1$ and we have the point $(x,y,z)=(1,-1,0)$.

Now we know that the point $P=(x,y,z)$ lies on the line ℓ_2 if and only if its position vector $\vec{OP}$ is of the form

$$\vec{OP}=[1,-1,0]+t\,[1,3,1]$$

for some scalar t. Since $[1,-1,0]+t\,[1,3,1]=[1+t,-1+3t,t]$ we see that the three equations

$$x=1+t,\ y=-1+3t,\ z=t\,\text{for}\ -\infty<t<\infty$$

are parametric equations for ℓ_2. Of course this is not the only correct answer. There are also infnitely many other different sets of equations which are parametric equation for this same line.

BET. From the parametric equations for ℓ_1 we see that every point P on ℓ_1 must have the position vector

$$\vec{OP}=[2,1,1]+t\,[1,2,1]$$

for some scalar t. So the vector $[1,2,1]$ is parallel to ℓ_1. Since the plane Π is parallel to both the lines ℓ_1 and ℓ_2, its normal vector $[A,B,C]$ must be perpendicular to any vector which is parallel to ℓ_1 or to ℓ_2. So $[A,B,C]$ is perpendicular to $[1,2,1]$ and also to $[1,3,1]$. This means that $[A,B,C]$ is parallel to

$$[1,2,1]\times[1,3,1]=[-1,0,1]$$

so we can simply choose $[A,B,C]=[-1,0,1]$ or, alternatively $[A,B,C]=[1,0,-1]$. Since we are given that the point $(1,0,-2)$ lies on the plane Π, we see that any point (x,y,z) lies on the plane if and only if the vector $[x,y,z]-[1,0,-2]$ is either the zero vector or is perpendicular to $[1,0,-1]$. This condition is the same as the equation $[x-1,y,z+2]\cdot[1,0,-1]=0$, which we can rewrite as $x-1-z-2=0$. So the equation

$$x-z=3$$

is an equation for the plane Π. Of course there are many correct answers to part BET. They will all be equations of the form $cx-cz=3c$ for some constant $c\neq 0$.

GIMEL. The two lines ℓ_1 and ℓ_2 intersect if and only if there exist two is numbers s and t for which

$$(2+s,1+2s,1+s)=(1+t,-1+3t,t)\,. \tag{1}$$

So we have to try to solve the three equations $2+s=1+t$, $1+2s=-1+3t$, and $1+s=t$. If we substitute the formula for t from third equation into the first equation, we get that $2+s=1+(1+s)$ and this will hold for every choice of s. Now, if we substitute $t=1+s$ into the second equation, we get that $1+2s=-1+3+3s$ which has the solution $s=-1$. So $t=1-1=0$, and when we substitute in (1) we obtain that the point $(1,-1,0)$ is the intersection of the two lines.

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2. It is intuitively clear (and we will explain this more carefully later) that to make the area of the rectangle as big as possible we should put all its vertices on the boundary of the ellipse and that one of these vertices (x, y) will satisfy $x > 0$ and $y > 0$.

In that case the other vertices (קודקודים) will be $(-x, y)$, $(x, -y)$ and $(-x, -y)$ and the area of the rectangle will be $4xy$. So we need to find the maximum value of the function $f(x, y) = xy$ as the point (x, y) ranges over the curve

$$C = \left\{ (x, y) : x \geq 0,\, y \geq 0,\, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} .$$

(Throughout this solution C will ALWAYS denote this particular curve.)

Obviously whatever point (x, y) gives a maximum value on this curve to the function xy will also give a maximum value to the function $4xy$, so we can forget about the factor of 4.

Since f is a continuous function and C is a closed bounded set, we know that f really does attain a maximum value on C.

We can find this maximum, and the point where it is attained, using the method of Lagrange multipliers with the help of the function $g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$. We should first check that the gradient $\vec{\nabla} g(x, y) = \left[\frac{2x}{a^2}, \frac{2y}{b^2}\right]$ vanishes only at the point $(0, 0)$ which does not lie on the curve C. Then we should also check the values of f at the endpoints of C. These are the points $(a, 0)$ and $(0, b)$ and f vanishes at both of them. So it remains to look for points (x, y) on C which satisfy the equation $\vec{\nabla} f + \lambda \vec{\nabla} g = \vec{0}$ for some number λ. This means we have to solve the system of three equations

$$\begin{aligned} y + \frac{2\lambda x}{a^2} &= 0 \\ x + \frac{2\lambda y}{b^2} &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 &= 0 \,. \end{aligned}$$

Except for the two endpoints $(a, 0)$ and $(0, b)$ of C which we have already considered, we have x and y both non zero. So, if we eliminate λ from the first two equations, we obtain that $\frac{a^2 y}{x} = \frac{b^2 x}{y}$ which in turn gives us that $\frac{y^2}{x^2} = \frac{b^2}{a^2}$, so that $y^2 = \frac{b^2 x^2}{a^2}$. If we now substitute this in the third equation we obtain that $\frac{2x^2}{a^2} = 1$ and so $x = \pm a/\sqrt{2}$ and $y = \pm b/\sqrt{2}$. We know that x and y are both non negative on C. So we get the $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. Considering the values of f at all relevant "suspected points" (נקודות חשודות) that we have found in this calculation we conclude that the maximum value of f on C is $\frac{ab}{2}$ which is attained at the point $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and the minimum value is 0 which is attained at $(a, 0)$ and $(0, b)$. We conclude that the rectangle with maximum area which has its sides parallel to the axes and is enclosed in the given ellipse has vertices at the points $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$, $\left(\frac{a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$, $\left(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and $\left(\frac{-a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$. Its length and width are $\sqrt{2}a$ and $\sqrt{2}b$.

Alternative ways to find the maximum of f on C would be to consider the maximum of the function $x \mapsto f\left(x, b\sqrt{1 - x^2/a^2}\right) = bx\sqrt{1 - x^2/a^2}$ on the interval $[0, a]$ or the maximum of the function $\theta \mapsto f(a \cos\theta, b \sin\theta) = ab \sin\theta \cos\theta$ on the interval $[0, \pi/2]$. In fact this second option would be quite quick and easy, easier than our calculation above.

If you did not realize that the maximum of f is attained on the boundary of the ellipse, then you can consider f on the set

$$E = \left\{ (x, y) : x \geq 0,\, y \geq 0,\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} . \tag{2}$$

It is clear that f does not have any critical points in the interior of E and also that f vanishes on all points of the boundary of E which are not part of the curve C. So the maximum of f on E is the same as the maximum of f on C.

Finally, here is a more precise explanation of why the rectangle of maximum area has to have vertices of the form $(\pm x, \pm y)$ *for some* $(x, y) \in C$. *(We allowed you to consider this fact as intuitively obvious, and did not expect you to give this explanation.) Let us consider any rectangle contained in the set* E *with sides parallel to the axes. Its sides must be segments (קטעים) of the lines* $x = x_1$, $x = x_2$ *and* $y = y_1$ *and* $y = y_2$. *This means that the vertices of the rectangle are the poins* (x_1, y_1), (x_1, y_2), (x_2, y_1) *and* (x_2, y_2). *We use the obvious fact that, if* (x, y) *is any point* *in the ellipse E, then the points $(-x,y)$ and $(x,-y)$ and $(-x,-y)$. are also all in E. If $\alpha = \max\{|x_1|, |x_2|\}$ we see that (α, y_1) and (α, y_2) are both in E. Then, if $\beta = \max\{|y_1|, |y_2|\}$ we deduce that (α,β) is also in E. It follows that the rectangle whose vertices are (α,β), $(-\alpha,\beta)$, $(\alpha,-\beta)$, and $(-\alpha,-\beta)$, is in E, and this rectangle contains the rectangle that we started with. Finally, if the positive numbers α and β satisfy $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}<1$, then replace α by a bigger number α_1 so that $\frac{\alpha_1^2}{a^2}+\frac{\beta^2}{b^2}=1$. Then the rectangle whose vertices are $(x,y)=(\alpha_1,\beta)$ and its three "reflections" is contained in E has all its vertices on the boundary of E and has area larger than the rectangle that we started with.*

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3. As we shall see, it is rather difficult to calculate this surface integral directly. It is easier to calculate it with the help of the Gauss Divergence Theorem. The given surface S is not the boundary of some set on which we could apply the divergence theorem. But we can "close" it by adding some other surfaces on which the calculation of the integral is quite easy. More precisely, we want to consider the "first octant" of the unit sphere, namely the set

$$V = \left\{(x,y,z) : x^2+y^2+z^2 \le 1,\ x\ge 0,\ y\ge 0,\ z\ge 0\right\}.$$

The set ∂V, i.e., the boundary of this set, is the union of the four surfaces S, S_1, S_2 and S_3, where S is the given surface in the formulation of question 3, and the surfaces S_1, S_2 and S_3 are the parts of ∂V which intersect with the planes $x=0$, $y=0$ and $z=0$.

The surface S_1 is simply the quarter disk $\left\{(0,y,z) : y^2+z^2\le 1,\ y\ge 0,\ z\ge 0\right\}$. So we can use the parametric representation $\vec{r}(y,z)=[0,y,z]$ where (y,z) ranges over the planar set $D=\left\{(y,z) : y^2+z^2\le 1,\ y\ge 0,\ z\ge 0\right\}$. We have $\vec{r}\,'_y=\hat{\mathbf{j}}$ and $\vec{r}\,'_z=\hat{\mathbf{k}}$. So $\vec{r}\,'_y\times\vec{r}\,'_z=\hat{\mathbf{i}}$. When we substitute in the formula for direct calculation of surface integrals we obtain that

$$\iint_{S_1}\vec{F}\cdot d\vec{S}=\iint_D\left[0+4y^2+4z^2,y^2,z^2\right]\cdot\hat{\mathbf{i}}dydz=\iint_D 4y^2+4z^2dydz\,.$$

We can use polar coordinates of the form $y=r\cos\theta$ and $z=r\sin\theta$ to obtain that this integral equals the repeated integral

$$\int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{1}4r^3dr\right)d\theta=\int_{\theta=0}^{\pi/2}\left(\left.\frac{4r^4}{4}\right|_{r=0}^{1}\right)d\theta=\frac{\pi}{2}\,.$$

In fact this answer has to be multiplied by -1 because our normal vector $\hat{\mathbf{i}}$ points into the set V. When we apply the Gauss divergence theorem the normal vectors on ∂V have to point outwards from V. Hence we finally conclude that $\iint_{S_1}\vec{F}\cdot d\vec{S}=-\frac{\pi}{2}$.

The normal on the surface S_2 is parallel to $\hat{\mathbf{j}}$ and on S_2 we have $\vec{F}(x,y,z)=\vec{F}(x,0,z)=\left[x^2+4z^2,0,z^2\right]$. So, no matter what parametric representation $[X(u,v),Y(u,v),Z(u,v)]$ we choose for S_2 we will have $Y(u,v)=0$ identically, and $\vec{r}\,'_u\times\vec{r}\,'_v$ parallel to $\hat{\mathbf{j}}$, and so $\vec{F}\left(X(u,v),0,Z(u,v)\right)$ and $\vec{r}\,'_u\times\vec{r}\,'_v$ will be orthogonal (**ניצבים**) to each other. This means that $\iint_{S_2}\vec{F}\cdot d\vec{S}=0$. Similarly the normal on S_3 is parallel to $\hat{\mathbf{k}}$ and on S_3 we have $\vec{F}(x,y,z)=\vec{F}(x,y,0,)=\left[x^2+4y^2,y^2,0\right]$ which is orthogonal to $\hat{\mathbf{k}}$. So for exactly analogous reasons to those just given for S_2, we have $\iint_{S_3}\vec{F}\cdot d\vec{S}=0$.

Now we have to calculate the integral $\iiint_V\vec{\nabla}\cdot\vec{F}dxdydz$. We have $\vec{\nabla}\cdot\vec{F}=2x+2y+2z$. We will use standard spherical polar coordinates r,θ,ϕ, related to x,y,z by $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$. The relevant Jacobian is $r^2\sin\theta$ and the set V corresponds to the "box" $\left\{(r,\theta,\phi) : 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}\right\}$. So

$$\begin{aligned}
&\iiint_V\vec{\nabla}\cdot\vec{F}dxdydz=2\iiint_V x+y+z\,dxdydz\\
=\;&2\int_{\theta=0}^{\pi/2}\left(\int_{\phi=0}^{\pi/2}\left(\int_{r=0}^{1}\left(r\sin\theta\cos\phi+r\sin\theta\sin\phi+r\cos\theta\right)r^2\sin\theta dr\right)d\phi\right)d\theta\\
=\;&2\int_{\theta=0}^{\pi/2}\left(\int_{\phi=0}^{\pi/2}\left(\sin^2\theta\cos\phi+\sin^2\theta\sin\phi+\cos\theta\sin\theta\right)\left(\int_{r=0}^{1}r^3dr\right)d\phi\right)d\theta\,.
\end{aligned}$$

Since $\int_0^1 r^3 dr = \frac{1}{4}$ and $\int_0^{\pi/2} \sin\phi d\phi = \int_0^{\pi/2} \cos\phi d\phi = 1$, this last integral equals

$$\begin{aligned} & \frac{1}{2}\int_{\theta=0}^{\pi/2} \sin^2\theta + \sin^2\theta + \frac{\pi}{2}\cos\theta\sin\theta\, d\theta \\ = & \frac{1}{2}\int_{\theta=0}^{\pi/2} 2\sin^2\theta + \frac{\pi}{4}\sin 2\theta\, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} 1 - \cos 2\theta + \frac{\pi}{4}\sin 2\theta\, d\theta \\ = & \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2} - \frac{\pi}{8}\cos 2\theta\Big|_0^{\pi/2}\right) = \frac{1}{2}\left(\frac{\pi}{2} - 0 + 0 + \frac{\pi}{8} + \frac{\pi}{8}\right) = \frac{3\pi}{8}\,. \end{aligned}$$

We can now combine all our calculations with the divergence theorem to obtain the value of the integral $\iint_S \vec{F}\cdot d\vec{S}$. Before we use the divergence theorem we should check that the conditions for using it are fulfilled. First, since the components of $\vec{F}$ are all polynomials, they must have continuous partial derivative of first order at all points in an open set containing the set V (in fact in ALL of $\mathbb{R}^3$). Then the set V is x-simple, y-simple, and also z-simple. (It would be enough if V were a finite union of sets with this property.) So indeed we can apply the divergence theorem. It gives us that

$$\begin{aligned} \iiint_V \vec{\nabla}\cdot\vec{F} dxdydz &= \iint_{\partial V} \vec{F}\cdot d\vec{S} \\ &= \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} \end{aligned}$$

where the normal vectors in the surface integrals all point outwards from V. Substituting the integrals we have calculated, we obtain that

$$\frac{3\pi}{8} = \iint_S \vec{F}\cdot d\vec{S} - \frac{\pi}{2} + 0 + 0$$

and so $\iint_S \vec{F}\cdot d\vec{S} = \frac{3\pi}{8} + \frac{\pi}{2} = \boxed{\dfrac{7\pi}{8}}$.

Let us now see what happens if we try to calculate this same integral directly. We can obtain a parametric representation for the surface S by using spherical coordinates and setting $r = 1$. Thus we have

$$\vec{r}(\theta,\phi) = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]\,,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\,.$$

A standard calculation (whose result you can simply quote) gives us that

$$\begin{aligned} \vec{r}'_\theta \times \vec{r}'_\phi &= \sin\theta\vec{r}(\theta,\phi) \\ &= \left[\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \cos\theta\sin\theta\right]\,. \end{aligned}$$

Before we substitute into the standard formula for calculating surface integrals we can slightly simplify things by using the fact that for all points (x, y, z) on the surface S the first component of $\vec{F}$ equals $x^2+4y^2+4z^2 = 1+3y^2+3z^2$. We obtain that $\iint_S \vec{F}\cdot d\vec{S}$ equals

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left[1 + 3\sin^2\theta\cos^2\phi + 3\cos^2\theta, \sin^2\theta\cos^2\phi, \cos^2\theta\right]\cdot\left[\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \cos\theta\sin\theta\right] d\theta d\phi\,.$$

When we calculate the inner product in the integrand we will get a complicated formula containing various powers of sines and cosines. In fact our integrand here is

$$\sin^2\theta\cos\phi + 3\sin^4\theta\cos^3\phi + 3\sin^2\theta\cos^2\theta\cos\phi + \sin^4\theta\sin\phi\cos^2\phi + \cos^3\theta\sin\theta\,.$$

It is certainly possible to calculate the double integral of this function, using changes of variable such as $u = \sin\theta$ and half angle formulae such as $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ and $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ .But it is a rather long story and we will not do it in this version of the solution. It is not surprising that no student during this exam was able to successfully do the calculation in this way.

Remark: *Since y and z appear rather symmetrically in this vector field, or at least in the rewritten field* $\left[1 + 3y^2 + 3z^2, y^2, z^2\right]$ *on S, and since on S itself y and z have rather symmetric roles, it might perhaps be easier to do this direct calculation using the alternative parametric representation*

$$\vec{r}(\rho,\varphi) = \left[\sqrt{1-\rho^2}, \rho\cos\varphi, \rho\sin\varphi\right]\,,\ 0 \le \rho \le \pi/2\,.$$

I have not tried this yet.

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4. Let us first try to calculate this integral directly. In $\mathbb{R}^3$ the equation $x^2+z^2=1$ defines an infinite cylindrical surface whose axis is the y axis and whose radius is 1, and the equation $y=1$ defines a plane parallel to the xz plane. This plane is perpendicular to the axis of the cylindrical surface, so the curve L, which is its intersection with the cylindrical surface, must be a circle. Every point on the cylindrical surface can be written in the form $(\cos t, y, \sin t)$ for some suitable values of the real parameters t and y. So every point on L can be written in the form $(\cos t, 1, \sin t)$, and, as t ranges from 0 to 2π, the point $(\cos t, 1, \sin t)$ moves over the whole circle L. This gives us a parametric representation for L. When we use this representation in the standard formula for calculating line integrals we see that our integral here equals

$$\int_0^{2\pi}\left[\sin^3 t, 3, -\cos^3 t\right]\cdot[-\sin t, 0, \cos t]\,dt =$$

$$= \int_0^{2\pi} -\sin^4 t - \cos^4 t dt = -\int_0^{2\pi} \sin^4 t + \cos^4 t dt\,. \tag{3}$$

Later we shall see that it is not too too difficult to calculate this integral directly. But, for now, let us see how to solve this same question using Stokes' theorem.

First we have to choose a surface S whose edge (קצה) is the circle (מעגל) L. There are infinitely many possible choices. But the most natural choice is to take S to be the disk (עיגול)

$$S=\left\{(x,1,z): x^2+z^2\le 1\right\} \tag{4}$$

which lies in the plane $y=1$. Just by looking at the formula (4) we can write down a parametric representation for S. The natural variables to take in the roles of u and v are x and z. We can write $\vec{r}(x,z)=[x,1,z]$ with $(x,z)\in D$, where D is the disk $\left\{(x,z): x^2+z^2\le 1\right\}$. *(We can of course also write all these formulae with u in place of x and v in place of z. It will still be the same calculation.)*

The *curl* or the *rotor* of our field $\vec{F}=\left[z^3, 3y^2, -x^3\right]$ is given by $\vec{\nabla}\times\vec{F}=\left[0, 3x^2+3z^2, 0\right]$.

We have $\vec{r}'_x=[1,0,0]=\hat{\mathbf{i}}$ and $\vec{r}'_z=[0,0,1]$. So $\vec{r}'_x\times\vec{r}'_z=\hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}}=[0,-1,0]$. This is the vector which will be used in our calculation as the normal vector to S, and we need to know whether it has the correct sense (מגמה). We will check this separately later.

Meanwhile we can use the standard formula for calculating surface integrals to obtain that

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}$$

$$= \iint_D \left[0, 3(x^2+z^2), 0\right]\cdot[0,-1,0]\,dxdz = -\iint_D 3(x^2+z^2)dxdz$$

and, if we use the change of variables $x=r\cos\theta$, $z=r\sin\theta$, then we have to replace D by the set $\widetilde{D}=\{(r,\theta): 0\le r\le 1,\ 0\le\theta\le 2\pi\}$ and the previous integral is equal to

$$-\iint_{\widetilde{D}} 3r^2\cdot r dr d\theta = -\int_0^{2\pi}\left(\int_0^1 3r^3 dr\right)d\theta = -\int_0^{2\pi}\left(\frac{3}{4}\right)d\theta = -\frac{3\pi}{2}\,.$$

As we will explain below, we have to multiply this answer by -1 to get $\frac{3\pi}{2}$.

We have used Stokes' theorem here. So we should also check that the conditions which allow us to use this theorem are fulfilled:

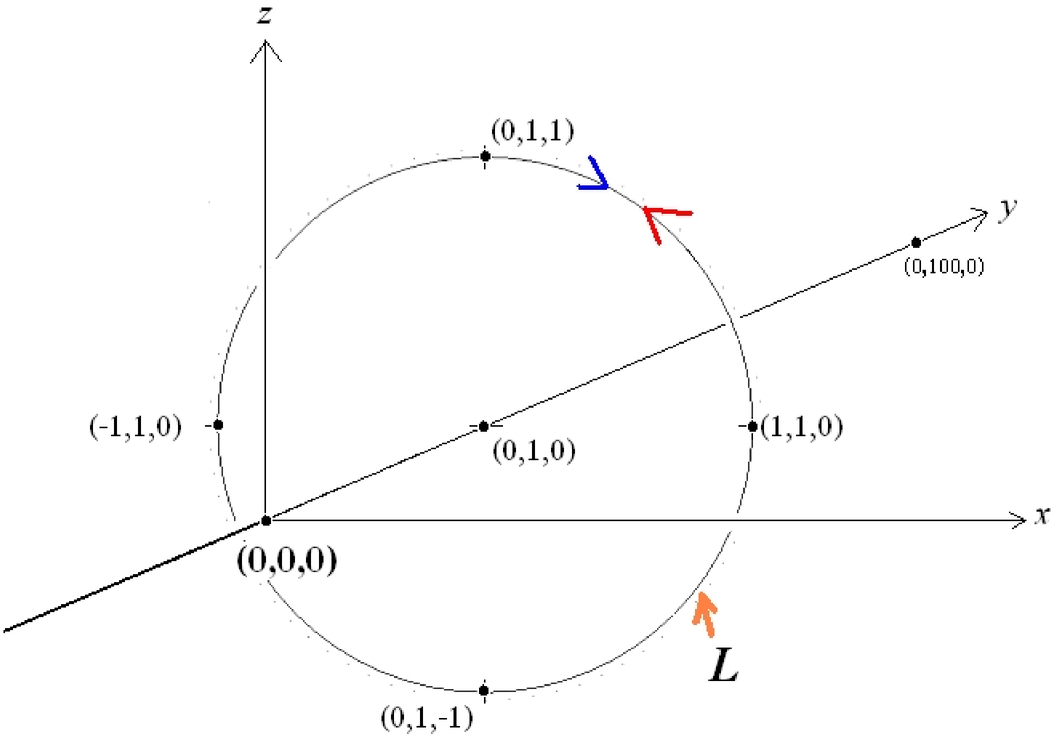
- First we see that the components of the field $\vec{F}=\left[z^3, 3y^2, -x^3\right]$ are all polynomials, so they all have continuous derivatives of first order at every point of an open set which contains the surface S. In fact in our case here we can take that open set to be the whole of $\mathbb{R}^3$.
- Then we see that the surface S which we used is orientable (i.e., it has two sides) and it has a parametric representation by functions which have continuous first derivatives. Alternatively, we can use the following justification regarding S and L: The disk D is a set in the plane for which Green's theorem holds, and we have a parametric representation (the one we used in our calculation) with continuous first derivatives which maps D onto S and maps the boundary of D onto L.

Now let us go back and see how to directly calculate the integral (3). It is possible to do this with the help of the formulae $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. Using several applications of these formulae we obtain that

$$\begin{aligned}\sin^4 t + \cos^4 t &= \left(\frac{1-\cos 2\theta}{2}\right)^2 + \left(\frac{1+\cos 2\theta}{2}\right)^2 \\ &= \frac{2+2\cos^2 2t}{4} = \frac{1+\cos^2 2t}{2} \\ &= \frac{1}{2} + \frac{1}{4}\left(1+\cos 4t\right).\end{aligned}$$

It follows that our integral equals $-\int_0^{2\pi} \frac{3}{4} + \frac{1}{4}\cos 4t\, dt = -\frac{3\pi}{2}$. However, as we shall explain below, because the direction of integration for our representation of L is opposite to the direction specified in the question, the answer to part Alef is $\frac{3\pi}{2}$.

Finally let us discuss the direction of integration on L and the sense (מגמה) of the normal vector on S. As we have done below, please draw a picture of the circle L and mark the points $(\cos t, 1, \sin t)$ on L, when t takes the values $t = 0, \pi/2, \pi, 3\pi/2$ and 2π. The motion along L through these points, in the order that we have listed them (the direction shown by the red arrow in our drawing), will appear to be anticlockwise if we look at L from a point $(0, c, 0)$ when the constant c satisfies $c < 1$. (In the drawing below we are looking at L from such a point.) But if $c > 1$, for example if $c = 100$, then, from the point $(0, c, 0)$, this same motion along L will appear to be clockwise. This means that the direction of integration given by the particular parametric representation that we chose here (shown by the red arrow) is *opposite* to the direction (shown by the blue arrow) that we were asked to use for integrating in the statement of the question. That is why we multiplied the value of the integral that we calculated by -1.



Now please look again at the same picture of L and remember that S is the disk which has L as its edge. Imagine a little man running along L so that his left hand points towards S, i.e., towards the point $(0, 1, 0)$. If he is running in the direction that we were asked to use on L (i.e., in the direction of the blue arrow) then his head has to be pointing in the direction of the POSITIVE y axis (i.e., INTO the page in the picture above). So, when we calculate the surface integral $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$, the normal vector to S also has to pointing in the direction of the positive y axis, i.e., its $\hat{\mathbf{j}}$ component must be positive. Since in our calculation above, our normal vector was $-\hat{\mathbf{j}} = [0, -1, 0]$

it has the opposite sense to the sense prescribed by Stokes' theorem, so we have to change the sign of the integral that we calculated.

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5. From the definition of differentiability, together with the theorem about differentiability implying existence of partial derivatives, we know that f is differentiable at $(0,0)$ if and only if

- both its first order partial derivatives exist at $(0,0)$, and
- the function $\epsilon(h,k)$ which is DEFINED in some neighbourhood of $(0,0)$ by the formula

$$f(h,k) = f(0,0) + \frac{\partial f}{\partial x}(0,0) \cdot h + \frac{\partial f}{\partial y}(0,0) \cdot k + \sqrt{h^2+k^2}\epsilon(h,k)$$

satisfies $\lim_{(h,k)\to(0,0)} \epsilon(h,k) = 0$.

As with various other "popular" functions that appear in standard exercise and examples, the function f here has the special property that $f(x,0) = 0$ for ALL x including $x = 0$ and also $f(0,y) = 0$ for all y including $y = 0$. So we obtain that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0$$

and a similar explanation shows that $\frac{\partial f}{\partial y}(0,0) = 0$.

It follows that the function $\epsilon(h,k)$ that we need to consider here is given by the formula $\epsilon(h,k) = \frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{h^2k}{(h^2+k^2)^{3/2}}$. This is a bounded function, but it is not difficult to see that it does not have a limit at $(0,0)$. More relevantly for our purposes. it is easy to see that, even if it had a limit, that limit could not be 0. We can do this by considering the ray (קרן) of points $\{(h,h) : h > 0\}$. This ray contains points arbitrarily close to $(0,0)$ and at every point of this ray we have $\epsilon(h,k) = \epsilon(h,h) = \frac{h^3}{(2h^2)^{3/2}} = 2^{-3/2}$ identically. This last fact is enough to show that the limit $\lim_{(h,k)\to(0,0)} \epsilon(h,k)$, if it exists, cannot be 0. So this shows that the function f cannot be differentiable at $(0,0)$.

To see that the limit $\lim_{(h,k)\to(0,0)} \epsilon(h,k)$ *does not even exist, we can look also at what happens on other rays, for example, on the ray* $\{(h,h) : h < 0\}$ *where* $\epsilon(h,k)$ *identically equals* $-2^{-3/2}$*. So, although, as* n *tends to infinity, the sequence of points* $(1/n, (-1)^n/n)$ *tends to* $(0,0)$*, the sequence of numbers* $\epsilon\left(1/n, (-1)^n/n\right) = (-1)^n\, 2^{-3/2}$ *does not converge to a limit. By a suitable version of Heine's theorem, this shows that* $\lim_{(h,k)\to(0,0)} \epsilon(h,k)$ *does not exist.*

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6. ALEF. Let us calculate the "curl" or "rotor" of the field $\vec{u}$. It is given by

$$\vec{\nabla} \times \vec{u} = [(2+a)x - 3ax, 3y - (2+a)y, 5 + 3az - 5a - 3z]$$

The first component of this vector, $(2+a)x - 3ax$, equals $(a-1)x$ and it will vanish at all points (x,y,z) if and only if $a = 1$. In that case the second component equals $3y - 3y = 0$ and the third component equals $5 + 3z - 5 - 3z = 0$. Thus we have shown that if (and only if) the constant a equals 1, then the field $\vec{u}$ satisfies $\vec{\nabla} \times \vec{u} = \vec{0}$ at every point $(x,y,z) \in \mathbb{R}^3$.

In the remaining parts of this question we shall always take a to equal 1, in accordance with the formulation of the question.

BET. The following fact is known:

THEOREM A. *If* V *is a simply connected subset of* $\mathbb{R}^3$ *or of* $\mathbb{R}^2$ *and if some vector field* $\vec{u}$*, whose components all have continuous first order partial derivatives, satisfies* $\vec{\nabla} \times \vec{u} = \vec{0}$ *at every point of* V*, then* $\vec{u}$ *is a conservative field on* V*. There are also examples where* V *is not simply connected, and* $\vec{u}$ *is not conservative on* V *even though* $\vec{\nabla} \times \vec{u} = \vec{0}$ *at every point of* V*.*

In our particular case the set V is not simply connected. We can see that because, for example, the closed curve $\{(2\cos\theta, 2\sin\theta, 0) : 0 \le \theta \le 2\pi\}$ lies completely in V but there is no surface completely contained in V whose edge is S. So we cannot apply Theorem A directly. But we can apply it indirectly. The set $\mathbb{R}^3$ is of course simply connected, and $\vec{\nabla} \times \vec{u} = \vec{0}$ at every point of $\mathbb{R}^3$. So Theorem A tells us that $\vec{u}$ is a conservative field on $\mathbb{R}^3$. This implies that $\vec{u}$ is also a conservative field on every open subset of $\mathbb{R}^3$, and in particular on the subset V.

Let me give a more detailed explanation of this last claim. Suppose that U *and* W *are open subsets of* $\mathbb{R}^n$*. (Here* n *can be* 2 *or* 3*.) A field* $\vec{F}$ *defined on* W *is conservative on* W *if its components are all continuous functions on* W *and if there exists some function* $\phi : W \to \mathbb{R}$ *which satisfies* $\vec{F} = \vec{\nabla}\phi$ *at every point of* W*. (This is one of the equivalent definitions of a conservative field.) Now suppose that* $\vec{F}$ *is indeed conservative on* W *and that* $V \subset W$*.*

Then of course the same function ϕ which satisfies $\vec{F} = \vec{\nabla}\phi$ at every point of W automatically satisfies the same equation at every point of V. I.e., the function ϕ, or, more pedantically, the function which is the restriction of ϕ to the smaller set V is the "potential function" which shows that $\vec{F}$ is also conservative on V.

GIMEL. Since, as already explained in the solution of part BET, $\vec{u}$ is conservative on $\mathbb{R}^3$, there has to be a function $f : \mathbb{R}^3 \to \mathbb{R}$ which satisfies $\vec{\nabla} f = \vec{u}$ at all points of $\mathbb{R}^3$. But we have to find this function. There are several ways to do this.

- One way is to look for a solution of the three equations corresponding to $\vec{\nabla} f = \vec{u}$, namely

$$\frac{\partial f}{\partial x} = x^2 + 5y + 3yz,\ \frac{\partial f}{\partial y} = 5x + 3xz - 2,\ \frac{\partial f}{\partial z} = 3xy - 4z\,. \tag{5}$$

Here we are doing "partial integration" instead of partial differentiation. The fact that $\frac{\partial f}{\partial z} = 3xy - 4z$ implies that $f(x, y, z) = 3xyz - 2z^2 + c$ where c is a "constant". Well it is not exactly a constant, but rather an unknown function of x and y. So it is more correct to write that

$$f(x, y, z) = 3xyz - 2z^2 + g(x, y) \tag{6}$$

for some unknown function $g(x, y)$, which we still have to find. If we differentiate this last equation partially with respect to x and substitute for $\frac{\partial f}{\partial x}$ from the first equation in (5) we obtain that $x^2 + 5y + 3yz = 3yz + \frac{\partial g}{\partial x}(x, y)$. Similarly, if differentiate (6) partially with respect to y and substitute for $\frac{\partial f}{\partial y}$ from the second equation in (5), then we obtain that $5x+3xz-2 = 3xz+\frac{\partial g}{\partial y}(x, y)$. So now we have two equations for determining g, namely $\frac{\partial g}{\partial x} = x^2+5y$ and $\frac{\partial g}{\partial y} = 5x - 2$. The first of these equations gives us that

$$g(x, y) = \frac{1}{3}x^3 + 5xy + h(y) \tag{7}$$

for some unknown function h depending only on y. We differentiate this last equation partially with respect to y to obtain that $\frac{\partial g}{\partial y} = 5x + h'(y)$. In view of the previous equation for $\frac{\partial g}{\partial y}$ this gives us that $h'(y) = -2$ and so $h(y) = -2y + c$ for some constant c. Substituting in (7) and then in (6) we obtain that

$$f(x, y, z) = 3xyz - 2z^2 + \frac{1}{3}x^3 + 5xy - 2y + c\,.$$

We can choose any value that we wish for the constant c.

There is another way to find the function f, namely to use line integrals. In general if $\vec{F}$ is a conservative field in some open connected set $W \subset \mathbb{R}^3$ then we can find a function $\phi : W \to \mathbb{R}$ which satisfies $\vec{F} = \vec{\nabla}\phi$ at every point of W by the following procedure: Choose a particular point P in W. Then, for every point (x, y, z) we define $\phi(x, y, z)$ to be the line integral $\int_{C_{x,y,z}} \vec{F} \cdot d\vec{r}$ where $C_{x,y,z}$ is a curve, any curve, lying completely in W and connecting P to $C_{x,y,z}$.

In our case here, since W is all of $\mathbb{R}^3$ we can choose P and each $C_{x,y,z}$ in quite simple ways. For example we can take P to be $(0, 0, 0)$ and we can let $C_{x,y,z}$ be a straight line segment from $(0, 0, 0)$ to (x, y, z). Let us use these choices to calculate the value of ϕ, or rather the function f at some general point in $\mathbb{R}^3$ which we will denote by (a, b, c) instead of (x, y, z). (Why do you think we change our notation here?). The curve $C_{a,b,c}$ has parametric representation

$$\vec{r}(t) = [at, bt, ct]\,,\ 0 \le t \le 1$$

and so the line integral $\int_{C_{a,b,c}} \vec{u} \cdot d\vec{r}$ is equal to

$$\begin{aligned} & \int_0^1 \vec{u}(at, bt, ct) \cdot [a, b, c]\,dt \\ = & \int_0^1 \left[a^2t^2 + 5bt + 3bct^2, 5at + 3act^2 - 2, 3abt^2 - 4ct\right] \cdot [a, b, c]\,dt\,. \end{aligned}$$

The function that we have to integrate here is

$$\begin{aligned} & a^3t^2 + 5abt + 3abct^2 + 5abt + 3abct^2 - 2b + 3abct^2 - 4c^2t^2 \\ = \ & a^3t^2 + 9abct^2 + 10abt - 4c^2t - 2b\,. \end{aligned}$$

When we integrate this function from $t = 0$ to $t = 1$ we obtain

$$\frac{a^3}{3} + 3abc + 5ab - 2c^2 - 2b\,. \tag{8}$$

We have shown that $f(a,b,c)$ equals the expression in (8), and of course this is the same as saying that $f(x,y,z)$ equals $\frac{x^3}{3} + 3xyz + 5xy - 2z^2 - 2y$. This is of course the same answer as we obtained by the previous method.

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7. We are given that the function f has continuous partial derivatives at every point of $\mathbb{R}^3$, that $\vec{\nabla} f\Big|_{(2,1,8)} = \widehat{\mathbf{i}} + 3\widehat{\mathbf{j}} + \widehat{\mathbf{j}}$ and $\frac{\partial^9}{\partial x^4 \partial z^5} f(2,1,8) = 33$.

The first of these conditions might tempt you to assume that $f(x,y,z) = x + 3y + z + C$ for some constant C and all $(x,y,z) \in \mathbb{R}^3$. But this is entirely wrong, and the condition $\frac{\partial^9}{\partial x^4 \partial z^5} f(2,1,8) = 33$ shows conclusively that it has to be wrong. If you wrongly assumed that $f(x,y,z) = x + 3y + z + C$ then you will get the correct answer for $\vec{\nabla} g(1,2)$ but your method is completely wrong, so, unfortunately, your score for this question will be zero.

In this calculation you have to be careful not to be confused by the fact that the variables x and y and z appear in unexpected places. There is danger of ambiguity when you look for example at the notation $\frac{\partial f}{\partial x}(x^2y, x, x^3y^3)$. (See the remark at the end of this solution for more discussion of this difficulty.)

One way to avoid or reduce confusion is to give other new names, instead of x and y, to the variables in the definition of g. So it is quite correct to say that $g(u,v) = f(u^2v, u, u^3v^3)$ for all $(u,v) \in \mathbb{R}^2$. The continuity of the partial derivatives of f ensures that f is differentiable and so it is correct the apply the chain rule. (You were required to provide this justification for using the chain rule.) The chain rule gives us that

$$\begin{aligned}
\frac{\partial g}{\partial u}(u,v) &= g_1'(u,v) \\
&= \frac{\partial f}{\partial x}(u^2v, u, u^3v^3)\frac{\partial\left(u^2v\right)}{\partial u} + \frac{\partial f}{\partial y}(u^2v, u, u^3v^3)\frac{\partial u}{\partial u} + \frac{\partial f}{\partial z}(u^2v, u, u^3v^3)\frac{\partial (u^3v^3)}{\partial u} \\
&= \frac{\partial f}{\partial x}(u^2v, u, u^3v^3)2uv + \frac{\partial f}{\partial y}(u^2v, u, u^3v^3) + \frac{\partial f}{\partial z}(u^2v, u, u^3v^3)3u^2v^3.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial g}{\partial v}(u,v) &= g_2'(u,v) \\
&= \frac{\partial f}{\partial x}(u^2v, u, u^3v^3)\frac{\partial\left(u^2v\right)}{\partial v} + \frac{\partial f}{\partial y}(u^2v, u, u^3v^3)\frac{\partial u}{\partial v} + \frac{\partial f}{\partial z}(u^2v, u, u^3v^3)\frac{\partial (u^3v^3)}{\partial v} \\
&= \frac{\partial f}{\partial x}(u^2v, u, u^3v^3)u^2 + 0 + \frac{\partial f}{\partial z}(u^2v, u, u^3v^3)3u^3v^2.
\end{aligned}$$

In particular, at the point $(u,v) = (1,2)$ we have $(u^2v, u, u^3v^3) = (2,1,8)$ and so, substituting in the above formulæ gives us that

$$g_1'(1,2) = \frac{\partial f}{\partial x}(2,1,8) \times 4 + \frac{\partial f}{\partial y}(2,1,8) + \frac{\partial f}{\partial z}(2,1,8) \times 24 = 4 + 3 + 24$$

and

$$g_2'(1,2) = \frac{\partial f}{\partial y}(2,1,8) + \frac{\partial f}{\partial z}(2,1,8) \times 12 = 1 + 12.$$

It follows that

$$\begin{aligned}
\vec{\nabla} g(1,2) &= g_1'(1,2)\widehat{\mathbf{i}} + g_2'(1,2)\widehat{\mathbf{j}} \\
&= 31\widehat{\mathbf{i}} + 13\widehat{\mathbf{j}}.
\end{aligned}$$

Remark. *What do we mean by the notation* $\frac{\partial f}{\partial x}(x^2y, x, x^3y^3)$*? I think it should mean this: Take the original function* $f(x,y,z)$ *and differentiate it partially with respect to* x*. This will give you a formula which depends on* x*,* y *and* z*. Take that formula make a new formula in the following way: In every place that you have* x*, replace it by* x^2y*, in every place that you have* y *(before you did the first substitution) replace it by* x *and in every place that you have* z *replace it by* x^3y^3*. This is* $\frac{\partial f}{\partial x}(x^2y, x, x^3y^3)$*. But some people could understand* $\frac{\partial f}{\partial x}(x^2y, x, x^3y^3)$ *to have this different meaning. Take the formula for* $f(x,y,z)$ *and use it to get a new function of* x*,* y *and* z *by replacing each* x *by* x^2y*, every* y *(from before you did the first substitution) by* x*, and every* z *by* x^3y^3*. Then, AFTER you have created this new function, differentiate it partially with respect to* x*. If we use the notation* $f_1'(x^2y, x, x^3y^3)$ *instead* *of* $\frac{\partial f}{\partial x}(x^2y, x, x^3y^3)$ *then it should be rather clearer that we intend the first of these two interpretations, and not the second. But it is probably even better to change the names of the variables as we did in the solution above.*

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8. There are probably quite a few misunderstandings about the topic of this question. So this solution will have a rather long introduction.

The standard change of variables formula for double integrals is

$$\iint_A f(x,y)dxdy = \iint_B f\left(X(u,v), Y(u,v)\right) \left| \frac{\partial X}{\partial u}(u,v)\frac{\partial Y}{\partial v}(u,v) - \frac{\partial X}{\partial v}(u,v)\frac{\partial Y}{\partial u}(u,v) \right| dudv \tag{9}$$

where the sets A and B are maybe connected by the formula

$$B = \left\{(u,v) \in \mathbb{R}^2 : (X(u,v), Y(u,v)) \in A\right\} \tag{10}$$

or maybe by some other formula. (We will discuss the connection between A and B more precisely in a moment.) So this means that the function $Z(u,v)$ in the statement of Question 8 has to be the ABSOLUTE VALUE of the determinant of the matrix

$$\left(\begin{array}{cc} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{array} \right) . \tag{11}$$

There are various theorems which give sufficent conditions for the formula (9) to hold. We will state one of these theorems a bit later. These theorems require the map which takes points (u,v) in the subset B of the uv plane to points $(X(u,v), Y(u,v))$ in the subset A of the xy plane, to be a one to one map. Or, in more sophisticated versions, it is still OK if the one to one condition is violated on some subset of A which has area 0.

Here is another possible formula for making the connection between A and B.

$$A = \{(X(u,v), Y(u,v)) : (u,v) \in B\} \ . \tag{12}$$

In some cases the two formulae (10) and (12) mean exactly the same thing. But in other cases they do not.

If the map $(u,v) \mapsto (X(u,v), Y(u,v))$ is a one to one map of all of $\mathbb{R}^2$ onto all of $\mathbb{R}^2$ then, indeed, any two sets A and B satisfy (10) if and only if they satisfy (12). For example this will happen if $X(u,v) = \alpha u + \beta v + c$ and $Y(u,v) = au + bv + c$ and the matrix $\left(\begin{array}{cc} \alpha & \beta \\ a & b \end{array} \right)$ has non zero determinant.

One of the simplest cases and best known cases where (10) and (12) do not mean exactly the same thing, occurs for polar coordinates. Here we can write the formulae for these coordinates as $X(u,v) = u\cos v$ and $Y(u,v) = u \sin v$. Suppose, for example, that we start by choosing B to be the rectangle $\{(u,v) : 1 \le u \le 2,\ 0 \le v < 2\pi\}$ and we let the set A be defined by (12). Then it follows that A is the annulus (טבעת) $A = \left\{(x,y) : 1 \le x^2 + y^2 \le 4\right\}$ and the map $(u,v) \mapsto (u\cos v, u \sin v)$ is a one to one map from B onto A. On the other hand, if we START by choosing A to be the annulus $A = \left\{(x,y) : 1 \le x^2 + y^2 \le 4\right\}$ and want B to be the set defined by (10) then we see that B contains the point (u,v) for $1 \le u \le 2$ and EVERY choice of v, not just in the interval $[0, 2\pi)$. Even more, B also contains all points (u,v) for all u satisfying $-2 \le u \le -1$. So B consists of two infinitely long vertical strips, each of length 1. In this case the map $(u,v) \mapsto (u\cos v, u\sin v)$ is definitely NOT a one to one map from B onto A. Note that in this example the Jacobian $\frac{\partial X}{\partial u}(u,v)\frac{\partial Y}{\partial v}(u,v) - \frac{\partial X}{\partial v}(u,v)\frac{\partial Y}{\partial u}(u,v)$ is u and so is non zero at every point of B. So this example shows that, in general, the condition $\frac{\partial X}{\partial u}(u,v)\frac{\partial Y}{\partial v}(u,v) - \frac{\partial X}{\partial v}(u,v)\frac{\partial Y}{\partial u}(u,v) \neq 0$ does NOT guarantee (as people sometimes mistakenly think[1]) that the map $(u,v) \mapsto (X(u,v), Y(u,v))$ is one to one on B.

In fact when we use polar coordinates to calculate a double integral $\iint_A f(x,y)dxdy$ for a given set A and a given function f and want to write the integral in the form $\iint_B f(r\cos\theta, r\sin\theta)rdrd\theta$, we should not use the formula (10) to specify the set B. Instead we should use the formula

$$B = \{(u,v) : u \ge 0,\ 0 \le v \le 2\pi,\ (X(u,v), Y(u,v)) \in A\}$$

or, more explicitly,

$$B = \{(r,\theta) : 0 \le r,\ 0 \le \theta \le 2\pi,\ (r\cos\theta, r\sin\theta) \in A\}$$

and sometimes it is more convenient to replace the condition $0 \le \theta \le 2\pi$ by $-\pi \le \theta \le \pi$.

These extra conditions on r and θ ensure that the map $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$ is one to one from B to A except for any subset of B contained in the line $r = 0$ or the line $\theta = 2\pi$. Such a subset has zero area and does not have any effect when we calculate double integrals.

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[1] A non vanishing Jacobian only guarantees that the map is one to one on some small, (perhaps very very small) neighbourhood of the point where we are given that the Jacobian is non zero.

Now we are ready to state one version of the theorem for change of variables in double integrals:

THEOREM B. *Let A be a bounded subset of $\mathbb{R}^2$. Suppose that the boundary of A consists of finitely many curves which are each either of the form $\{(x,\phi(x)) : a \le x \le b\}$ or of the form $\{(\phi(y),y) : a \le y \le b\}$ for suitable continuous functions ϕ on suitable intervals $[a,b]$.*

Suppose that B is another bounded subset of $\mathbb{R}^2$ whose boundary satisfies the same condition as the boundary of A.

Let $X(u,v)$ and $Y(u,v)$ be functions which are defined and have continuous partial derivatives of first order at every point of some open subset containing the closure of B. Suppose that $A = \{(X(u,v), Y(u,v)) : (u,v) \in B\}$ and that the map $(u,v) \mapsto (X(u,v), Y(u,v))$ is one to one from B onto A.

Suppose that f is a function which is defined and continuous at every point of some open set containing the closure of A. Then the formula (9) holds.

Remark: In this theorem it is possible to replace the conditions on the sets A and B and the function f by other weaker conditions. Also, as already mentioned, we can allow the map $(u,v) \mapsto (X(u,v), Y(u,v))$ to fail to be one to one on certain suitably small subsets of B. However, after such modifications, the statement of the theorem is rather more complicated, or its conditions are more difficult to actually check in practice. One option is to replace the condition about the form of the boundary of B by the requirement that the function (often called χ_B) which equals 1 on B and 0 on $\mathbb{R}^2 \backslash B$ is Riemann integrable. Then the other conditions imposed on the functions $X(u,v)$ and $Y(u,v)$ are enough to automatically guarantee that the set A has a similar property and there is no need to impose any condition on the boundary of A.

Now to Question 8 itself:

ALEF. The theorem that you state here can be Theorem B or some reasonable variant of it. We expect you to know that the map $(u,v) \mapsto (X(u,v), Y(u,v))$ has to be one to one from B onto A and that its component functions both have to have continuous first order derivatives. We also expect you to know that the function $Z(u,v)$ is the absolute value of the Jacobian. We will not take off points if you did not notice that the formula (10), which appeared in the statement of the question to connect A and B, has to be replaced by something else, (unless, for example, $(u,v) \mapsto (X(u,v), Y(u,v))$ is known to be defined and one to one from $\mathbb{R}^2$ onto $\mathbb{R}^2$). We will also not take off points if you do not mention a condition about the boundary of A and B or the integrability of χ_B, but we do expect you to know that the set B has to be bounded. The set A should also be bounded, but this will follow automatically from the boundedness of B and the continuity of the functions $X(u,v)$ and $Y(u,v)$ on the closure of B.

BET. In this particular case the matrix (11) becomes $\begin{pmatrix} 1 & 0 \\ 0 & 2v \end{pmatrix}$ and so its determinant is $2v$, which means that the function $Z(u,v) = 2|v|$.

From the fact that here $A = \{(x,y) : 1 \le 2,\ 1 \le y \le x^2\}$ and the stated connection between A and B, and the formulae for $X(u,v)$ and $Y(u,v)$ we see that

$$B = \{(u,v) \in \mathbb{R}^2 : (u,v^2) \in A\} = \{(u,v) \in \mathbb{R}^2 : 1 \le u \le 2,\ 1 \le v^2 \le u^2\} .$$

In the uv plane, the condition $1 \le u \le 2$ means that the point (u,v) lies in the strip between the two vertical lines $u = 1$ and $u = 2$. Then, since the preceding condition means that u is positive, we see that the remaining condition $1 \le v^2 \le u^2$ can be rewritten as $1 \le |v| \le u$, which means that v has to satisfy either $1 \le v \le u$ or $-u \le v \le -1$. The first of these conditions means that the point (u,v) lies above the line $v = 1$ and underneath the line $v = u$. The second of these conditions means that the point (u,v) lies above the line $v = -u$ and underneath the line $v = -u$. (Of course in these kinds of considerations we are more used to writing x and y in place of u and v, but we should be able to adjust to these small changes of notation.) We conclude that B is the union of two triangles. (Please draw a picture.) One of them, which we can call B_+, has vertices $(1,1)$, $(2,2)$ and $(2,1)$, and the other, which we can call B_-, has vertices $(1,-1)$, $(2,-2)$ and $(2,-1)$. (Since there is no curve lying completely in B which joins any given point of B_+ to any given point of B_- we see that B indeed is not a connected set.)

GIMEL. The condition which is not fulfilled is that the map $(u,v) \mapsto (u,v^2)$ is not one to one from B to A.

To see this, note, for example the point $\left(7/5, \sqrt{6/5}\right)$ is in A but it is the image of two different points in B namely $\left(7/5, (6/5)^{1/4}\right)$ and $\left(7/5, -(6/5)^{1/4}\right)$. Indeed EVERY point (a,b) in A is the image of two different points $\left(a, \sqrt{b}\right)$ and $\left(a, -\sqrt{b}\right)$ in B.

DALET. In our particular context, the formula

$$\iint_A f(x,y)dxdy = \frac{1}{2}\iint_B f\left(X(u,v),Y(u,v)\right)Z\left(u,v\right)dudv$$

can be rewritten as

$$\int_{x=1}^{2}\left(\int_{y=1}^{x^2} f(x,y)dy\right)dx = \frac{1}{2}\iint_{B_-\cup B_+} f\left(u,v^2\right)2\left|v\right|dudv\,. \tag{13}$$

Since the two sets B_+ and B_- are disjoint, the integral $\iint_{B_-\cup B_+} f\left(u,v^2\right)2\left|v\right|dudv$ is equal to

$$\begin{aligned} &\iint_{B_-} f\left(u,v^2\right)2\left|v\right|dudv + \iint_{B_+} f\left(u,v^2\right)2\left|v\right|dudv \\ = &\int_{u=1}^{2}\left(\int_{v=-u}^{-1} f\left(u,v^2\right)2\left|v\right|dv\right)du + \int_{u=1}^{2}\left(\int_{v=1}^{u} f\left(u,v^2\right)2\left|v\right|dv\right)du\,. \end{aligned} \tag{14}$$

Now we the formula from Hedva 1M for changing variables in an integral of a function of *one* variable. If we set $t=v^2$, then, for each positive v we have $\frac{dt}{dv}=2v=2\left|v\right|$ and the integral $\int_{v=1}^{u} f\left(u,v^2\right)2\left|v\right|dv$ equals $\int_{t=1}^{u^2} f(u,t)dt$.

We can use the same change of variable $t=v^2$ for v in the negative range $-u\le v\le -1$. For each negative v we have $\frac{dt}{dv}=2v=-2\left|v\right|$ and the integral $\int_{v=-u}^{-1} f\left(u,v^2\right)2\left|v\right|dv$ equals $-\int_{t=u^2}^{1} f(u,t)dt = \int_{t=1}^{u^2} f(u,t)dt$.

These calculations show that the two repeated integrals whose sum we have written in (14) are equal to each other, and their sum equals $2\int_{u=1}^{2}\left(\int_{t=1}^{u^2} f(u,t)dt\right)du$ which is of course the same thing as $2\int_{x=1}^{2}\left(\int_{y=1}^{x^2} f(x,y)dy\right)dx$. So we simply have to divide by 2 and this gives us a proof of the formula (13).

We remark that our proof here of (13) did not use the condition $f(x,y)=f(x,-y)$ anywhere. In fact, in both sides of (13) the values of $f(x,y)$ for negative values of y simply do not appear or have any role. We conclude that (13) is true for all continuous function $f:\mathbb{R}^2\to\mathbb{R}$. In fact it is also true for all functions which are only defined on the closure of A, provided they are also continuous on the closure of A.

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