HEDVA 2M
SOLUTION TO THE EXAMINATION OF 20/3/2007 (MOED BET).

Version 1.62, 18/02/09.

The only change from version 1.61 of 27/4/2007 is that the graphics files which generate the pictures have been slightly improved.

As usual I remark that no one is infallible, and that there is always a small possibility of a mistake here. Indeed a smll misprint in 1 part BET has was corrected in an earlier version (f(0,0)=1). If there is still another mistake in this solution, then the first student to send me an e-mail reporting it may get a (modest) tsiyun magen, if I am still teaching the course.

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Certain parts of this solution are explained in rather more detail than we would expect from most students. We also sometimes give several different ways of answering questions, whereas we only expected one way from you.

I repeat my usual words to those of you who have some difficulty reading technical material in English: I could apologize for not writing a Hebrew solution. But it is in your very best interests to learn to read this sort of material in English NOW. Next year you will be older and it will be harder for you to learn. It is impossible, or almost impossible to have a reasonable career in engineering or science without being able to read English well.

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1. In this question we study two functions $f(x,y) = e^{\sqrt{x^2+y^2}}$ and $g(x,y) = \arctan\frac{1}{x^2+y^2}$. *(In some versions of the exam we reverse the notation and write* $g(x,y) = e^{\sqrt{x^2+y^2}}$ *and* $f(x,y) = \arctan\frac{1}{x^2+y^2}$ *.)*

ALEF. First consider g. We know from Hedva 1m that $\lim_{t\to+\infty}\arctan t = \frac{\pi}{2}$. Obviously $\frac{1}{x^2+y^2}$ tends to $+\infty$ as (x,y) tends to $(0,0)$. So we see that $\lim_{(x,y)\to(0,0)} g(x,y) = \frac{\pi}{2}$. So, according the rules of the game in this question we have to choose $g(0,0) = \frac{\pi}{2}$, and, with this choice, g is continuous at $(0,0)$.

Now consider f. Using standard results about continuous functions of continuous functions and products and sums of continuous functions we see that $\lim_{(x,y)\to(0,0)} e^{\sqrt{x^2+y^2}} = e^{\sqrt{0^2+0^2}} = e^0 = 1$. So again, the rules of this question require us to choose $f(0,0) = 1$ and to conclude that f is continuous at $(0,0)$.

BET. Here we investigate the partial derivatives of f and g. Since $f(y,x) = f(x,y)$ and $g(y,x) = g(x,y)$ it does not matter if we consider the partial derivatives with respect to x or with respect to y.

We now know that $g(0,0) = \frac{\pi}{2}$. It follows that, by definition, $\frac{\partial g}{\partial x}(0,0) = \lim_{x\to0}\frac{g(x,0)-g(0,0)}{x} = \lim_{x\to0}\frac{1}{x}(\arctan\frac{1}{x^2} - \pi/2)$ if this limit exists. We shall attempt to use L'Hôpital's rule. First we see that $\arctan\frac{1}{x^2} - \pi/2$ and x both tend to 0 when x tends to 0. Then we see that $\frac{d}{dx}\left(\arctan\frac{1}{x^2} - \frac{\pi}{2}\right) = \frac{1}{1+\frac{1}{x^4}} \cdot \frac{-2}{x^3} = \frac{-2}{x^3+1/x} = \frac{-2x}{x^4+1}$ for all $x \neq 0$ and of course $\frac{d}{dx}x = 1$. So all the conditions for applying L'Hôpital's rule are fulfilled and we have

$$\lim_{x\to0}\frac{\arctan\frac{1}{x^2} - \pi/2}{x} = \lim_{x\to0}\frac{\frac{-2x}{x^4+1}}{1} = \lim_{x\to0}\frac{-2x}{x^4+1} = 0. \tag{1}$$

We have shown that $\frac{\partial g}{\partial x}(0,0) = 0$. In exactly the same way we can obtain that $\frac{\partial g}{\partial y}(0,0) = 0$, which is essentially the question that was asked in some versions where f and g were interchanged.

For the function f things turn out differently. In part ALEF we found that that $f(0,0) = 1$. We also have $f(x,0) = e^{|x|}$. So $\frac{\partial f}{\partial x}(0,0) = \lim_{x\to0}\frac{e^{|x|}-1}{x}$ if the limit exists. Since the function $x \to |x|$ is not differentiable at $x = 0$ we also suspect that the function $e^{|x|}$ might fail to be differentiable at $x = 0$. Here is one way to see that exactly. First we observe that, by Taylor's theorem, or by L'Hôpital's rule, we have $\lim_{t\to0}\frac{e^t-1}{t} = 1$. This means that $\lim_{x\to0}\frac{e^{|x|}-1}{|x|} = 1$. When x is positive $\frac{e^{|x|}-1}{x} = \frac{e^{|x|}-1}{|x|}$ and so the one sided limit from the right, $\lim_{x\searrow0}\frac{e^{|x|}-1}{x}$ exists and equals 1. When x is negative $\frac{e^{|x|}-1}{x} = -\frac{e^{|x|}-1}{|x|}$ and so the one sided limit from the left, $\lim_{x\searrow0}\frac{e^{|x|}-1}{x}$ exists and equals -1. Since these two one sided limits are not equal it follows that the (usual two sided) limit $\lim_{x\to0}\frac{e^{|x|}-1}{x}$ does not exist. So $\frac{\partial f}{\partial x}(0,0)$ does not exist. Similarly $\frac{\partial f}{\partial y}(0,0)$ does not exist. Either of these last two facts is sufficient to also tell us that f cannot be differentiable at $(0,0)$. (If f is differentiable at $(0,0)$ then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both have to exist at $(0,0)$.)

GIMEL. We have just seen that f is not differentiable at $(0,0)$. To check if g is differentiable at $(0,0)$ we have to check whether $\lim_{(h,k)\to(0,0)}\varepsilon(h,k) = 0$, where $g(h,k) = g(0,0) + h\frac{\partial g}{\partial x}(0,0) + k\frac{\partial g}{\partial y}(0,0) + \sqrt{h^2+k^2}\varepsilon(h,k)$. In our case this means that

$$\varepsilon(h,k) = \frac{g(h,k) - g(0,0)}{\sqrt{h^2+k^2}} = \frac{\arctan\frac{1}{h^2+k^2} - \pi/2}{\sqrt{h^2+k^2}}.$$

It is not at first so clear to calculate the limit $\lim_{(h,k)\to(0,0)} \varepsilon(h,k)$. If we could somehow replace $\sqrt{h^2+k^2}$ by x then we could use (1) to obtain that $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$. It turns out that we CAN replace $\sqrt{h^2+k^2}$ by x.

Here are two different ways of doing this:

First method: Use Heine's theorem twice. In the first application (in the style of Hedva 1m) we know that:

(A) If $\{x_n\}_{n\in\mathbb{N}}$ is ANY sequence of real numbers such that $x_n \neq 0$ and $\lim_{n\to\infty} x_n = 0$ then $\lim_{n\to\infty} \dfrac{\arctan \frac{1}{x_n^2} - \pi/2}{x_n} = 0$.

Now consider a sequence $\{(h_n,k_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $\lim_{n\to\infty}(h_n,k_n) = (0,0)$ and $(h_n,k_n) \neq (0,0)$ for all $n \in \mathbb{N}$. Set $x_n = \sqrt{h_n^2+k_n^2}$. Then $\varepsilon(h_n,k_n) = \frac{\arctan \frac{1}{x_n^2} - \pi/2}{x_n}$. We clearly have $x_n \neq 0$ for each n and also $\lim_{n\to\infty} x_n = 0$.

So, by (A) we have $\lim_{n\to\infty} \varepsilon(h_n,k_n) = \lim_{n\to\infty} \dfrac{\arctan \frac{1}{x_n^2} - \pi/2}{x_n} = 0$. Since this is true for EVERY sequence $\{(h_n,k_n)\}_{n\in\mathbb{N}}$ with the $\lim_{n\to\infty}(h_n,k_n) = (0,0)$ and $(h_n,k_n) \neq (0,0)$ for all $n \in \mathbb{N}$, it follows from Heine's theorem in the Hedva 2m version for double limits that $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$.

Second method: Use the continuity of composed continuous functions. First let $u : \mathbb{R} \to \mathbb{R}$ be the function defined by $u(t) = \frac{\arctan \frac{1}{t^2} - \pi/2}{t}$ for all $t \neq 0$ and $u(0) = 0$. By (1) we know that u is continuous at $t = 0$ and of course it is also continuous at every other point $t \in \mathbb{R}$.

We also know that the function $v(x,y) = \sqrt{x^2+y^2}$ is continuous at every point $(x,y) \in \mathbb{R}^2$. So the composed function $w(x,y) = u(v(x,y))$ must be continuous at every point $(x,y) \in \mathbb{R}^2$. In particular $\lim_{(h,k)\to(0,0)} w(h,k) = w(0,0) = u(v(0,0)) = u(0) = 0$. But $w(h,k)$ is the function $\varepsilon(h,k)$ and so we have shown that $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$.

Thus both of these methods show that g is differentiable at $(0,0)$.

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2. ALEF. *Some of you solved this question in another way, using methods that you learned in a physics course. If this was done correctly, you got full credit for such a solution.*

Since one of our aims in this course is to teach you about double integrals and test your knowledge of them, we will attempt in the future to avoid giving questions like this one which can be solved in other ways.

In our case, since ρ is constant, the general formula $\iint_\Omega \rho(x,y)dxdy$ for the mass of Ω becomes simply MA where A is the area of Ω. We observe that the area of Ω is given by $\pi 2^2 - \pi 1^2 = 3\pi$. So $\iint_\Omega \rho(x,y)dxdy = 3M\pi$.

(We could of course also calculate the double integral $\iint_\Omega M dxdy$ more explicitly, perhaps via polar coordinates, similarly to the calculations that we will now see for part BET.)

BET. According to a standard formula, the centre of mass of the region Ω is the point $(\overline{x},\overline{y})$ where $\overline{x} = \frac{\iint_\Omega x\rho(x,y)dxdy}{\iint_\Omega \rho(x,y)dxdy}$ and $\overline{y} = \frac{\iint_\Omega y\rho(x,y)dxdy}{\iint_\Omega \rho(x,y)dxdy}$. We have $\rho(x,y) \equiv M$ and $\Omega = \{(x,y) : 2x \leq x^2+y^2 \leq 4\}$. The condition $2x \leq x^2+y^2$ is equivalent to $0 \leq x^2 - 2x + y^2$, i.e. to $1 \leq x^2 - 2x + 1 + y^2$. So Ω consists of all the points which our outside the circle $(x-1)^2 + y^2 = 1$ and inside the circle $x^2+y^2 = 4$. Since both of these circles are symmetric with respect to the x axis, and since ρ is constant, it is reasonable to guess that $\overline{y} = 0$. But, anyway, we were not asked to calculate $\overline{y}$.

It is convenient to divide Ω into two parts Ω_- and Ω_+, where Ω_- is the part of Ω lying on the left side of the y axis and Ω_+ is the part on the right side. These two sets are disjoint except perhaps for an overlap on the y axis. So, for any integrable function f on Ω we have

$$\iint_\Omega f(x,y)dxdy = \iint_{\Omega_-} f(x,y)dxdy + \iint_{\Omega_+} f(x,y)dxdy. \tag{2}$$

Calculating our integrals on Ω_- is relatively simple. Using polar coordinates, we have

$$\iint_{\Omega_-} xMdxdy = M\iint_E r\cos\theta \cdot rdrd\theta$$

where $E = \{(r,\theta) : 0 \le r \le 2, \pi/2 \le \theta \le 3\pi/2\}$. So

$$\begin{aligned}
\iint_{\Omega_-} xM dxdy &= M \int_{\pi/2}^{3\pi/2} \left(\int_{r=0}^{2} r^2 \cos\theta dr \right) d\theta \\
&= M \int_{\pi/2}^{3\pi/2} \cos\theta \left(\frac{r^3}{3}\Big|_{r=0}^{2} \right) d\theta \\
&= \frac{8M}{3} \int_{\pi/2}^{3\pi/2} \cos\theta d\theta = \frac{8M}{3}\left(\sin\theta|_{\pi/2}^{3\pi/2}\right) = \frac{8M}{3}(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}) = -\frac{16M}{3}.
\end{aligned}$$

By drawing a picture of Ω_+ and using simple trigonometry and the fact that the angle in a semicircle is $\frac{\pi}{2}$ we can see that Ω_+ is the set of all points $(x,y) = (r\cos\theta, r\sin\theta)$ with $-\pi/2 \le \theta \le \pi/2$ and $2\cos\theta \le r \le 2$. We can also get this last inequality by substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the inequality $2x \le x^2 + y^2 \le 4$ and dividing by 2 or by r. So, via a change of variable to polar coordinates, we obtain that

$$\iint_{\Omega_+} xM dxdy = M \iint_E r\cos\theta \cdot r dr d\theta$$

where this time $E = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2, 2\cos\theta \le r \le 2\}$. So the previous double integral equals the repeated integral

$$\begin{aligned}
M \int_{-\pi/2}^{\pi/2} \left(\int_{r=2\cos\theta}^{2} r^2 \cos\theta dr \right) d\theta &= M \int_{-\pi/2}^{\pi/2} \left(\frac{r^3\cos\theta}{3}\Big|_{r=2\cos\theta}^{2} \right) d\theta \\
&= \frac{M}{3} \int_{-\pi/2}^{\pi/2} \left(8\cos\theta - 8\cos^4\theta\right) d\theta \\
&= \frac{8M}{3} \int_{-\pi/2}^{\pi/2} \left(\cos\theta - \left(\frac{1+\cos 2\theta}{2}\right)^2 \right) d\theta \\
&= \frac{2M}{3} \int_{-\pi/2}^{\pi/2} 4\cos\theta - 1 - 2\cos 2\theta - \cos^2 2\theta \; d\theta \\
&= \frac{2M}{3} \int_{-\pi/2}^{\pi/2} 4\cos\theta - 1 - 2\cos 2\theta - \frac{1+\cos 4\theta}{2} \; d\theta \\
&= \frac{2M}{3} \left(4\sin\theta - \theta - \sin 2\theta - \frac{\theta + \frac{1}{4}\sin 4\theta}{2} \Big|_{-\pi/2}^{\pi/2} \right) \\
&= \frac{2M}{3} \left(4 - \pi/2 - 0 - \frac{\pi}{4} - \left(-4 + \pi/2 + 0 + \frac{\pi}{4} \right) \right) \\
&= \frac{2M}{3}(8 - 3\pi/2) = \left(\frac{16}{3} - \pi \right) M.
\end{aligned}$$

Combining the previous calculations and using (2), we see that

$$\overline{x} = \frac{-16M/3 + (16/3 - \pi)M}{3M\pi} = -\frac{1}{3}.$$

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3[1]. In this question you are asked to find the maximum and minimum of the function $f(x,y,z) = ax + by + cz$ on the surface $S = \left\{ (x,y,z) : \frac{x^2+y^2}{4} + z^2 = 1 \right\}$.

The constants a, b and c take different values in different versions.

We let $g : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $g(x,y,z) = \frac{x^2+y^2}{4} + z^2 - 1$.

[1] An exercise which is quite similar to this question appears in some material that you were asked to read during the semester. See the remark at the end of this document.

We want to use the method of Lagrange multipliers. To do this we first have to check that f and g both have continuous first order partial derivatives at every point of $\mathbb{R}^3$. This is clear because both these functions are simply polynomials in x, y and z.

Then we also have to check if there are any points (x,y,z) on S where $\vec{\nabla}g(x,y,z) = \vec{0}$. The Lagrange method has to consider the possibility that the extremum of f on the set S is attained at such points. Here we can exclude this possibility. This is because $\vec{\nabla}g(x,y,z) = \frac{x}{2}\widehat{\mathbf{i}} + \frac{y}{2}\widehat{\mathbf{j}} + 2z\widehat{\mathbf{k}}$ and so the only point where this gradient vanishes is $(0,0,0)$, which is not on S.

One other possibility that has to be considered is that the extremum of f on S is attained at some point on the "edge" of S. But here S is a complete ellipsoid and so it does not have an "edge".

Having eliminated all these possibilities, we now know that the extremum of f on S has to be obtained at some point (x,y,z) which satisfies the equation $\vec{\nabla}f(x,y,z) + \lambda\vec{\nabla}g(x,y,z) = \vec{0}$ for some number λ. This vector equation corresponds to the three scalar equations $a + x\lambda/2 = 0$, $b + y\lambda/2 = 0$, and $c + 2z\lambda = 0$ and we also of course have the fourth equation $\frac{x^2+y^2}{4} + z^2 - 1 = 0$.

In all versions of this question, at least one of the three constants a, b and c is non zero. This means that $\lambda \neq 0$. So the first three equations give us that

$$x = -2a/\lambda, y = -2b/\lambda, \text{ and } z = -c/2\lambda. \tag{3}$$

Now let us substitute for x, y and z in the fourth equation. This gives us that $\frac{4a^2+4b^2}{4\lambda^2} + \frac{c^2}{4\lambda^2} = 1$. It follows that $\lambda^2 = \frac{4a^2+4b^2}{4} + \frac{c^2}{4} = a^2 + b^2 + \frac{c^2}{4}$ and so

$$\lambda = \pm\sqrt{a^2 + b^2 + \frac{c^2}{4}}. \tag{4}$$

In each version of the exam, a, b and c have been chosen so that this square root is easy to evaluate and gives reasonably simple rational or integer value for λ, or rather two such values, the same number with opposite signs.

In view of (4) and (3) we have only two points where f can attain an extremum on S. These are $\left(\frac{2a}{\sqrt{a^2+b^2+\frac{c^2}{4}}}, \frac{2b}{\sqrt{a^2+b^2+\frac{c^2}{4}}}, \frac{c}{2\sqrt{a^2+b^2+\frac{c^2}{4}}}\right)$ and $\left(\frac{-2a}{\sqrt{a^2+b^2+\frac{c^2}{4}}}, \frac{-2b}{\sqrt{a^2+b^2+\frac{c^2}{4}}}, \frac{-c}{2\sqrt{a^2+b^2+\frac{c^2}{4}}}\right)$ The values of f at these two points are, respectively $\frac{2a^2+2b^2+c^2/2}{\sqrt{a^2+b^2+\frac{c^2}{4}}} = 2\sqrt{a^2+b^2+\frac{c^2}{4}}$ and $-2\sqrt{a^2+b^2+\frac{c^2}{4}}$.

So the maximum and minimum of f must be $2\sqrt{a^2+b^2+\frac{c^2}{4}} = 2|\lambda|$ and $-2\sqrt{a^2+b^2+\frac{c^2}{4}} = -2|\lambda|$ respectively.

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4. In this question a, b and γ are constants which take different values in different versions. You are asked to calculate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(\frac{\cos^9\left(e^{-y^2}\right)}{(1+y^8)(5+y^{12})} + bx\right)\widehat{\mathbf{j}}$. The curve C is the intersection of the cylindrical surface $\left\{(x,y,z) : x^2 + z^2 = a^2\right\}$ with the plane $y = z + \gamma$.

Direct calculation of this integral looks rather impossible because of the presence of the very nasty function $\frac{\cos^9\left(e^{-y^2}\right)}{(1+y^8)(5+y^{12})}$ in the vector field $\vec{F}$.

(Maybe in some other problem, some other time, you might need to calculate the line integral of some other nicer vector field along this same curve C. See below for some information relevant for doing that.)

But since this very nasty function depends only on y and is in the $\widehat{\mathbf{j}}$component of the field, it will not have any effect on the rotor of $\vec{F}$. Indeed $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times bx\widehat{\mathbf{j}} = b\widehat{\mathbf{k}}$. This strongly suggests trying to use Stokes' theorem to calculate J.

The curve C is the "edge" (katzeh) of the ellipse S which is the part of the plane $y = z + \gamma$ enclosed in the cylinder $x^2 + z^2 \leq a^2$.

By Stokes' theorem the line integral J equals the surface integral $\iint_S b\widehat{\mathbf{k}}\cdot d\vec{S}$.

We will have to decide what is the sense ("megamah") of the normal vector on S for this integration. But we will delay doing this until almost the end of our calculation.

BET. Here we apparently have to give a parametric representation for the surface S. The easiest way of getting such a representation is to consider S as the graph $y = f(x, z)$ of the function $f(x, z) = z + \gamma$, where (x, z) ranges over the disk

$$D = \left\{(x, z) : x^2 + z^2 \le a^2\right\}. \tag{5}$$

This means we take $x = u$ and $z = v$ and our parametric representation is $\vec{r}(x, z) = x\widehat{\mathbf{i}} + (z + \gamma)\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}$ or $\vec{r}(u, v) = u\widehat{\mathbf{i}} + (v + \gamma)\widehat{\mathbf{j}} + v\widehat{\mathbf{k}}$.

Another natural possibility is to use polar coordinates in the xz plane, so $x = u\cos v$ and $z = u\sin v$. Then our parametric representation is $\vec{r}(u, v) = u\cos v\widehat{\mathbf{i}} + (u\sin v + \gamma)\widehat{\mathbf{j}} + u\sin v\widehat{\mathbf{k}}$, where this time the relevant set D of all points (u, v) is

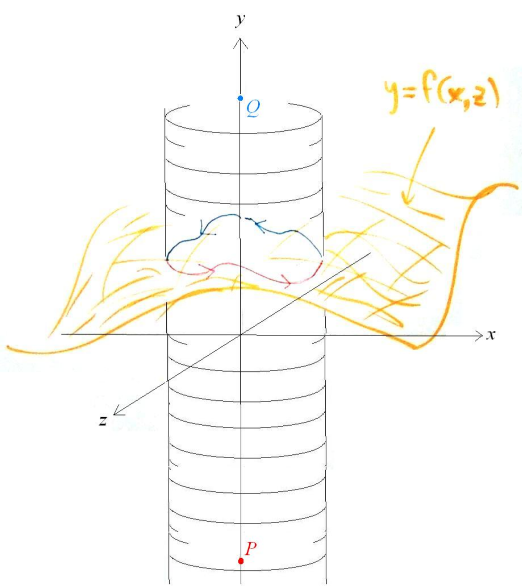
$$D = \{(u, v) : 0 \le u \le a, 0 \le v \le 2\pi\} \ . \tag{6}$$

(If we were free to choose, we might feel more natural and comfortable if we could write other letters here, e.g. perhaps ρ in place of u and ϕ in place of v. But we have not been given this choice this time.)

GIMEL and DALET. In part GIMEL we apparently have to choose option (ii)

If we use the first of the above two parametric representations, then $\vec{r}_x = \widehat{\mathbf{i}}$ and $\vec{r}_z = \widehat{\mathbf{j}} + \widehat{\mathbf{k}}$ and so $\vec{r}_x \times \vec{r}_k = \widehat{\mathbf{k}} - \widehat{\mathbf{j}}$.

It follows that $\iint_S b\widehat{\mathbf{k}} \cdot d\vec{S} = \iint_D b\widehat{\mathbf{k}} \cdot (\widehat{\mathbf{k}} - \widehat{\mathbf{j}})dxdz = \iint_D bdxdz = b\iint_D dxdz$. So the function $g(u, v)$ or $g(x, z)$ which you were asked to write down in part GIMEL, is simply the constant function b. Or, as explained below, it might be $-b$.) In this case D is the disk, defined by (5), whose area is πa^2 and so we immediately deduce in part DALET that $J = \pi a^2 b$. Or perhaps $J = -\pi a^2 b$, and we should have written above that $\iint_S b\widehat{\mathbf{k}} \cdot d\vec{S} = -\iint_D b\widehat{\mathbf{k}} \cdot (\widehat{\mathbf{k}} - \widehat{\mathbf{j}})dxdz$. The choice between these two answers depends on what is the correct sense (megama) for the normal vector to S. Now is the time to think about that, with the help of the following picture.



The x and y axes are in the plane of our picture. So the z axis must be pointing towards us. The cylindrical surface $x^2 + z^2 = a^2$ is shown in black. Another surface, which is the graph $y = f(x, z)$ of some continuous function f of two variables, is shown in orange. In our particular case f would be the function $f(x, z) = z + \gamma$ and the surface would be a plane. But in this discussion we will reach the same conclusions for ***every*** choice of the function f.

(Of course it is more usual to talk about graphs $z = f(x, y)$ of functions of x and y, i.e., sets of points of the form $(x, y, f(x, y))$. Here we have interchanged the "traditional" roles of x, y and z and are considering a set of points of the form $(x, f(x, z), z)$, and of course we are allowed to do that.)

We have drawn a curve, call it C, which is the intersection of the cylindrical surface and the orange surface.

The part of the curve which is nearer to us (in the region where $z \geq 0$) is shown in red, and the part of the curve which is further away from us (in the region where $z < 0$) is shown in light blue.

We are going to move along the curve C in the direction shown by the arrows. To an observer located at the point P on the negative y axis, far below the curve, it will appear that we are moving along C in a clockwise direction. To an observer located at the point Q on the positive y axis, far above the curve, it will appear that we are moving along C in an anti-clockwise direction. (From both P and Q the curve will look almost like a circle.)

If S is the part of the orange surface enclosed by the cylinder $x^2 + z^2 = a^2$, and we are moving along C with our heads pointing upwards, then S is on our left side. This means that, if we want to apply Stokes' theorem to the surface S and the curve C using the direction that we have chosen for moving along C, then the normal to S will have to be, like our heads, pointing upwards. In general it will not be exactly vertical, but its $\widehat{\mathbf{j}}$ component will have to be positive.

Returning to our special case, where $f(x, z) = z + \gamma$, we conclude that the answer $J = \pi a^2 b$ which corresponds to the normal vector $\vec{r}_x \times \vec{r}_k = \widehat{\mathbf{k}} - \widehat{\mathbf{j}}$, i.e., with negative $\widehat{\mathbf{j}}$ component, is correct when the direction of C from the point $P = (0, -100, 0)$ appears to be, davka, anti-clockwise. If the direction from the point $Q = (0, 100, 0)$ appears to be anti-clockwise, then the correct answer will be $J = -\pi a^2 b$.

Suppose that we decide to use the second parametric representation mentioned above. Then D is given by (6). We have $\vec{r}_u = \cos v\widehat{\mathbf{i}} + \sin v\widehat{\mathbf{j}} + \sin v\widehat{\mathbf{k}}$ and $\vec{r}_v = -u \sin v\widehat{\mathbf{i}} + u \cos v\widehat{\mathbf{j}} + u \cos v\widehat{\mathbf{k}}$. So $\vec{r}_u \times \vec{r}_v = -u\widehat{\mathbf{j}} + u\widehat{\mathbf{k}}$. We have to write a suitable function $g(u, v)$ in option (ii) of part GIMEL. This function must be $g(u, v) = \vec{\nabla} \times \vec{F}(u \cos v, u \sin v + \gamma, u \sin v) \cdot \vec{r}_u \times \vec{r}_v = b\widehat{\mathbf{k}} \cdot (-u\widehat{\mathbf{j}} + u\widehat{\mathbf{k}})$. So $g(u, v) = bu$. Or maybe $g(u, v) = -bu$, depending on the direction of integration along C. Then in part DALET we have to calculate the double integral $\int_{v=0}^{2\pi} \left(\int_{u=0}^{a} budu\right) dv = 2\pi b \int_0^a udu = 2\pi b \cdot \frac{a^2}{2}$. So of course we get the same value $\pi a^2 b$ as when we use the other parametric representation.

Perhaps some students declared in part ALEF that they wanted to, davka, calculate J directly, despite the impossibly nasty function $\frac{\cos^9\left(e^{-y^2}\right)}{(1+y^8)(5+y^{12})}$ which would appear in their integral.
What should we do in their case?

This would be considered a wrong answer since it is definitely not reasonable (savir) to use this way, unless these students showed that they could in fact somehow do this calculation. Can you?

But if they made this choice then of course, yes, they would have to use a parametric representation, but of the curve C instead of a surface whose "edge" is C.

They would be given credit in part BET if they wrote a correct parametric representation for C. Let us find such a representation. Anyway, as we already said above, you might need it for some other question some other time.

Of course every curve has infinitely many possible parametric representations. But perhaps the most natural one for C can be obtained by first observing that the projection of C on to the xz plane is the circle Γ whose parametric representation can be given by $\vec{r}(t) = a \cos t\widehat{\mathbf{i}} + 0\widehat{\mathbf{j}} + a \sin t\widehat{\mathbf{k}}$ with $0 \leq t \leq 2\pi$. Now we shall imagine that there are two brothers running (or climbing or flying) along the curves Γ and C. At each "time" t the first brother is on Γ at the point $(a \cos t, 0, a \sin t)$. The second brother moves along C so that he is always directly "above" or "below" his brother. Here by "above" or "below" we mean what would be above or below in the usual way if we rotated all the axes so that the y axis was pointing upwards instead of to one side. In other words, at each moment, both brothers have the same x coordinate and the same z coordinate, but usually their y coordinates are different.

Since the second brother has to move on the surface of the plane $y = z + \gamma$, this forces him to be at the point $(a \cos t, \gamma + a \sin t, a \sin t)$ whenever the first brother is at $(a \cos t, 0, a \sin t)$. So this gives us a natural parametric representation for C, namely

(7) $$\vec{r}(t) = a \cos t\widehat{\mathbf{i}} + (\gamma + a \sin t)\widehat{\mathbf{j}} + a \sin t\widehat{\mathbf{k}} \text{ again with } 0 \leq t \leq 2\pi.$$

In part GIMEL, the students who chose this option would now have to choose option (i). If they had chosen the parametric representation (7), then $[\alpha, \beta]$ would be $[0, 2\pi]$. (Or it could be some other interval of length 2π.) The function $u(t)$ would be a real mess, quite impossible to fit in the little square provided. So this might be a good clue that a better option should be tried. Anyway, if (7) had been chosen, $u(t)$ here would have to be

$$\left(\frac{\cos^9\left(e^{-(\gamma+a\sin t)^2}\right)}{(1+(\gamma+a\sin t)^8)\,(5+(\gamma+a\sin t)^{12})} + b\cos t \right) a\cos t$$

Actually you might be surprised to learn that the integral $\int_0^{2\pi} u(t)dt$ for this function u is not as impossible to calculate as you might first think. Let's do it. We can write u in the form $u(t) = g(\sin t)\cos t + ab\cos^2 t$ where g is a horrible but continuous function of one variable. Let me show you that $\int_0^{2\pi} g(\sin t)\cos t dt = 0$. Then the rest will be easy. *How did we think of doing this? Well, it helped us very much that we knew what the answer should be, from our calculation with Stokes' theorem above.* Here we go:

Define a new function $G : \mathbb{R} \to \mathbb{R}$ by the formula $G(t) = \int_0^t g(s)ds$. Since g is such a complicated function there is absolutely no hope of finding a formula for G. But all we need to know about G is that it is differentiable, and $G'(t) = g(t)$ for every $t \in \mathbb{R}$. It follows that the function $t \mapsto G(\sin t)$ is also differentiable, and $\frac{d}{dt}G(\sin t) = G'(\sin t)\cos t = g(\sin t)\cos t$. Now we apply the fundamental theorem of integral calculus (i.e., the Newton–Leibniz formula) to see that

$$\int_0^{2\pi} g(\sin t)\cos t dt = \int_0^{2\pi} \frac{d}{dt}G(\sin t)dt = G(\sin 2\pi) - G(\sin 0) = G(0) - G(0) = 0.$$

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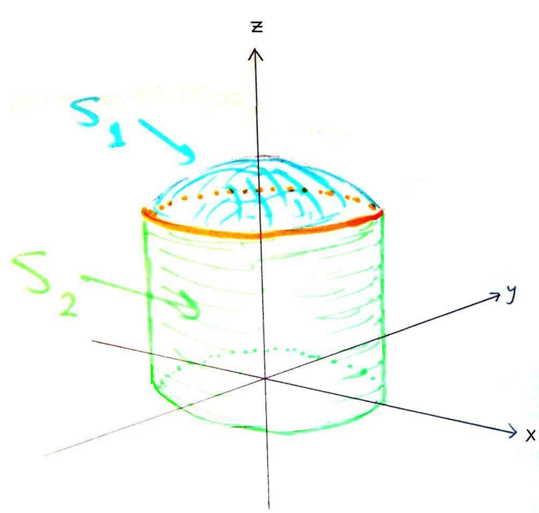
5. In this question a, b and c and γ are constants which take different values in different versions. We are asked to calculate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ where

$$\vec{F}(x,y,z) = \left(axz + \frac{z\sin^7\left(\log(1+z^4)\right)}{\log(\gamma+y^8)} \right)\widehat{\mathbf{i}} + byz\widehat{\mathbf{j}} + c\widehat{\mathbf{k}}$$

and S is the union of two surfaces S_1 and S_2. We show these surfaces in the following picture.



S_2 is shown in green and is the part between the two planes $z = 0$ and $z = 1$ of the infinite cylindrical surface of radius 1 whose axis is the z-axis.

S_1 is shown in blue and is the part of the spherical surface $x^2 + y^2 + z^2 = 2$ which lies above the plane $z = 1$.

These two surfaces fit together along a "seam" (kav tefer) which is the circle, shown in orange, of radius 1 in the plane $z = 1$ with centre at $(0, 0, 1)$.

Although the definition of $\vec{F}$ involves some very complicated functions, the divergence of $\vec{F}$ is quite simple. In fact it is given by

$$\vec{\nabla} \cdot \vec{F} = az + bz + 0 = (a+b)z.$$

This suggests trying to use the Gauss divergence theorem to calculate . The problem is that S is not a closed surface. So we want to "close" S. More precisely, we find add another surface S_3 so that $S \cup S_3$, the union of these two surfaces will be a closed surface. The most natural way to do this is to choose S_3 to be the disk of radius 1 in the xy plane centred at $(0,0,0)$. Then the set V which has $S \cup S_3$ as its boundary, is given by $V = \left\{(x,y,z) : x^2+y^2 \le 1, 0 \le z \le \sqrt{2-x^2-y^2}\right\}$. Since the set V is x-simple, y-simple and z-simple, and since the components of $\vec{F}$ all have continuous partial derivatives of first order in an open set containing V, all the conditions required for applying the Gauss divergence theorem are fulfilled[2].

The Gauss theorem gives us that

$$\iiint_V (a+b)z dxdydz = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S}, \tag{8}$$

where the normal vectors on S and on S_3 have to point outward from V. The direction (sense) of the normal specified on S in the statement of the question is "away from the z axis" and this indeed is the same as pointing outwards from V. On S_3 the normal will have to point downwards. We have

$$\begin{aligned}
\iiint_V (a+b)z dxdydz &= (a+b)\iint_{\{x^2+y^2\le 1\}} \left(\int_{z=0}^{\sqrt{2-x^2-y^2}} z dz\right) dxdy \\
&= \frac{a+b}{2}\iint_{\{x^2+y^2\le 1\}} \left(z^2\Big|_0^{\sqrt{2-x^2-y^2}}\right) dxdy \\
&= \frac{a+b}{2}\iint_{\{x^2+y^2\le 1\}} \left(2-x^2-y^2\right) dxdy \\
&= \frac{a+b}{2}\int_{r=0}^{1} \left(\int_{\theta=0}^{2\pi} \left(2-r^2\right) r d\theta\right) dr \\
&= \frac{a+b}{2}\int_{r=0}^{1} \left(\int_{\theta=0}^{2\pi} 2r - r^3 d\theta\right) dr \\
&= 2\pi \cdot \frac{a+b}{2} \cdot \left(r^2 - \frac{r^4}{4}\Big|_0^1\right) = \frac{3\pi(a+b)}{4}.
\end{aligned}$$

A convenient parametric representation for S_3 is given by $\vec{r}(x,y) = x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + 0\widehat{\mathbf{k}}$ where (x,y) ranges over the disk $D = \left\{(x,y) : x^2+y^2\right\}$. We have $\vec{r}_x \times \vec{r}_y = \widehat{\mathbf{k}}$. This normal vector is pointing upwards instead of downwards. We will correct for this by multiplying our surface integral by -1. We thus have

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = -\iint_D \vec{F}(x,y,0)\cdot \widehat{\mathbf{k}} dxdy = -\iint_D c dxdy.$$

Since the area of D is π, this integral equals $-c\pi$. We can now see from (8) and the preceding calculations that

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (a+b)z dxdydz - \iint_{S_3} \vec{F}\cdot d\vec{S} = \left(\frac{3(a+b)}{4} + c\right)\pi.$$

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6. ALEF. As t varies from -1 to 1, the point $(u(t), v(t), w(t))$ moves along the curve C. At each point $(x,y,z) = (u(t), v(t), w(t))$ the value of g is of course $g(u(t), v(t), w(t))$. So the function $\phi(t) = g(u(t), v(t), w(t))$ assumes exactly the values on $[-1,1]$ that g assumes on the curve C.

This function ϕ can be very helpful in part GIMEL. Of course there are also other correct, but less natural ways to choose other functions ϕ on other intervals with property that we required here.

BET. The answer here is quite obvious. Since, by definition, $g(x,y,z) = M$ at every point of S, if $C \subset S$ this means that $g(x,y,z) = M$ at every point of C. So the maximum and minimum values of g on C are both M.

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[2] Of course the divergence theorem also holds under weaker assumptions. In general V does not have to have the property of being x, y and z –simple. It is enough if V is a finite union of such sets. More precisely, it is enough if V is a finite union of sets, $V = V_1 \cup V_2 \cup .....V_N$ obtained by "cutting V into pieces" by planes. where the interiors of the sets V_j are disjoint from each other and each V_j is x, y and z –simple.

GIMEL[3]. The tangent to the curve $C = \{(u(t), v(t), w(t)) : -1 \le t \le 1\}$ at some point $(u(t_*), v(t_*), w(t_*))$, where $t_* \in [-1, 1]$, is parallel to the vector $u'(t_*)\widehat{\mathbf{i}} + v'(t_*)\widehat{\mathbf{j}} + w'(t_*)\widehat{\mathbf{k}}$. Our conditions on the functions u, v and w guarantee that this vector is always non zero, so it really does give us the direction of the tangent at $(u(t_*), v(t_*), w(t_*))$. In particular, setting $t_* = 0$, we have that the tangent to C at the point (x_0, y_0, z_0) is parallel to the non zero vector $u'(0)\widehat{\mathbf{i}} + v'(0)\widehat{\mathbf{j}} + w'(0)\widehat{\mathbf{k}}$.

To show that $\vec{\nabla} g(x_0, y_0, z_0)$ is perpendicular to the tangent, it is equivalent to show that the relevant scalar product satisfies

$$\vec{\nabla} g(x_0, y_0, z_0) \cdot \left(u'(0)\widehat{\mathbf{i}} + v'(0)\widehat{\mathbf{j}} + w'(0)\widehat{\mathbf{k}} \right) = 0.$$

In view of the definition of $\vec{\nabla} g$ and the fact that $(x_0, y_0, z_0) = (u(0), v(0), w(0))$, this is the same as

$$\frac{\partial g}{\partial x}(u(0), v(0), w(0))u'(0) + \frac{\partial g}{\partial y}(u(0), v(0), w(0))v'(0) + \frac{\partial g}{\partial z}(u(0), v(0), w(0))w'(0) = 0. \tag{9}$$

The main idea in our proof is to use the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = g((u(t), v(t), w(t)))$.
(Hopefully you already met this function in part ALEF.)

Since g is differentiable at the point $(x_0, y_0, z_0) = (u(0), v(0), w(0))$, we can apply the chain rule to obtain that the derivative $\phi'(t)$ exists for $t = 0$ and is given by the formula

$$\phi'(0) = \frac{\partial g}{\partial x}(u(0), v(0), w(0))u'(0) + \frac{\partial g}{\partial y}(u(0), v(0), w(0))v'(0) + \frac{\partial g}{\partial z}(u(0), v(0), w(0))w'(0).$$

On the other hand, since each point $(u(t), v(t), w(t))$ lies on C and C is contained in S we have that $\phi(t) = g((u(t), v(t), w(t))) = M$ for each $t \in [-1, 1]$.
(Hopefully part BET emphasized this property of ϕ for you.)

So $\phi'(t)$ exists and equals 0 for each t. In particular this means that

$$0 = \frac{\partial g}{\partial x}(u(0), v(0), w(0))u'(0) + \frac{\partial g}{\partial y}(u(0), v(0), w(0))v'(0) + \frac{\partial g}{\partial z}(u(0), v(0), w(0))w'(0).$$

This gives us (9) and completes our proof.

Here is an alternative "almost proof" which uses, or tries to use the implicit function theorem:

First, since the vector $\vec{\nabla} g(0,0,0)$ is non zero, at least one of its components must be non zero. So we can suppose, for example, that $\frac{\partial g}{\partial z}0, 0, 0) \neq 0$. In the other two cases the proof is analogous, with permutations of the roles of x and y.

Here is where we have a **problem.** *We only know that g is differentiable at $(0,0,0)$. To apply the implicit function theorem we would need to assume more, namely that g has continuous first order partial derivatives in some neighbourhood of $(0,0,0)$.*

Well, let us be "greedy" and add this extra assumption, even though we were ***not*** *told that we have it.*

If we do this, then the implicit function theorem will give us that, at least in a small neighbourhood of $(0,0,0)$, the set S coincides with the graph of some function $z = \psi(x, y)$ which satisfies $\psi(0,0) = 0$ and has derivatives which satisfy $\psi'_x(0,0) = -g'_x(0,0,0)/g'_z(0,0,0)$ and $\psi'_y(0,0) = -g'_y(0,0,0)/g'_z(0,0,0)$. So, at $(0,0,0)$ the surface S has a tangent plane Π whose normal is $-\psi'_x(0,0)\widehat{\mathbf{i}} - \psi'_y(0,0)\widehat{\mathbf{j}} + \widehat{\mathbf{k}}$. Since C lies on this surface, its tangent must lie in the plane Π and is therefore perpendicular to the vector

$$\vec{V} = -\psi'_x(0,0)\widehat{\mathbf{i}} - \psi'_y(0,0)\widehat{\mathbf{j}} + \widehat{\mathbf{k}} = g'_x(0,0,0)/g'_z(0,0,0)\widehat{\mathbf{i}} + g'_y(0,0,0)/g'_z(0,0,0)\widehat{\mathbf{j}} + \widehat{\mathbf{k}}.$$

*Since $\vec{V} = \frac{1}{g'_z(0,0,0)} \vec{\nabla} g(0,0,0)$, we see that $\vec{V}$ is parallel to $\vec{\nabla} g(0,0,0)$, and so our "**almost** proof" is complete.*

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7. This question is exactly the same as Question 6 of the Moed Alef exam of 18/3/2007. So please read the solution of that question in the document:

`http://www.math.technion.ac.il/~mcwikel/h2m/soln207.pdf`

That document also includes quite a lot of extra comments about that question.

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[3]Some proofs which are very similar to the one that you were asked to provide here appear in some material that you were asked to read during the semester. See the remark at the end of this document.

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Further remarks about Questions 6 and 3.

Results very very similar to Question 3 and to part GIMEL of Question 6 appeared in the material that you were asked to read about extrema of functions and Lagrange multipliers.

For example you can go to the web page

`http://www.math.technion.ac.il/~mcwikel/h2m/books-o05.html`

and choose the link on the third line of that page which is to the postscript file

http://www.math.technion.ac.il/courses/104011/targ/104011ta2.ps

On page 25 line 10 of that postscript file you will see a proof of a result for functions of two variables which is very similar to 6 GIMEL.

Then on page 27, 19 lines from the bottom of the page, you will see a proof of a result for functions of three variables which is even more similar to 6 GIMEL.

Via other links from

`http://www.math.technion.ac.il/~mcwikel/h2m/books-o05.html`

you can get to the pdf file

`http://www.math.technion.ac.il/courses/Math2H/Hoveret1.pdf`

which is the original handwritten version of the above postscript file. The same two proofs mentioned above can also be found in this file. When using "Acrobat reader" press Shift+Control N and choose page 64. This will bring you to the top of the page with a number 61 in a triangle at the top. The first proof begins at the top of that page. To see the second proof, press Shift+Control N and choose page 69. (This is the page with the number 66 in a triangle at the top of the page.) The relevant proof begins four lines from the bottom of the page.

(In between these two proofs you will also see an exercise about finding the extremum of a function on a half ellipsoid. Question 3 of this exam is a considerably simpler variant of that exercise.)

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