HEDVA 2M
SOLUTION TO THE EXAMINATION OF 18/2/2007 (MOED ALEF).

Version 4, 17/9/2007.

There is only one small correction since version 3 (of 5/3/07): A minus sign has been added on line -4 of page 10. Thank you Rami.

As usual I remark that no one is infallible, and that there is always a small possibility of a mistake here, and indeed there were mistakes in earlier versions. Thanks to those of you who noticed them! If there is *still* a mistake in this solution, then the first student to send me an e-mail reporting it will still get a (modest) tsiyun magen.

If you find the letters here too small to comfortably read, then I can provide you with another version of this document with larger fonts. Just send an email to mcwikel@math.technion.ac.il

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Certain parts of this solution are explained in rather more detail than we would expect from most students. We also sometimes give several different ways of answering questions, whereas we only expected one way from you.

I repeat my usual words to those of you who have some difficulty reading technical material in English: I could apologize for not writing a Hebrew solution. But it is in your very best interests to learn to read this sort of material in English NOW. Next year you will be older and it will be harder for you to learn. It is impossible, or almost impossible to have a reasonable career in engineering or science without being able to read English well.

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1. ALEF. As happens in many other exercises, here we cannot use the formula for the derivative of a quotient (="manah"). (Why?) So we have to calculate the partial derivatives of f at $(0,0)$ from the definition. So

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly

$$f'_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

BET. To see that f is continuous at $(0,0)$ we write it as a product of two functions $f(x,y) = \frac{|x|}{\sqrt{x^2+y^2}} \cdot y$. The first function $\frac{|x|}{\sqrt{x^2+y^2}}$ is of course bounded. In fact $0 \le \frac{|x|}{\sqrt{x^2+y^2}} \le 1$. The second function y tends to 0 and (x,y) tends to $(0,0)$. So the product of these two functions tends to 0 which equals $f(0,0)$.

GIMEL. The directional derivative of f at $(0,0)$ in the direction of any given unit vector $\widehat{\nu} = \nu_1\widehat{\mathbf{i}} + \nu_2\widehat{\mathbf{j}}$ is defined by the formula $D_{\widehat{\nu}}f(0,0) = \lim_{h\to 0}\frac{1}{h}\left(f(\nu_1 h, \nu_2 h) - f(0,0)\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{|\nu_1 h|\nu_2 h}{\sqrt{\nu_1^2h^2+\nu_2^2h^2}} - 0\right) = \lim_{h\to 0}\frac{|\nu_1|\nu_2}{\sqrt{\nu_1^2+\nu_2^2}} = |\nu_1|\,\nu_2$. In particular, if $\nu_1 = \cos\alpha$ and $\nu_2 = \sin\alpha$, then $D_{\widehat{\nu}}f(0,0) = |\cos\alpha|\sin\alpha$.

DALET. The result of part GIMEL shows that function f does not satisfy the formula $D_{\widehat{\nu}}f(0,0) = \vec{\nabla}f(0,0)\cdot\widehat{\nu}$, which is satisfied by all functions which are differentiable at $(0,0)$. So f cannot be differentiable at $(0,0)$.

Alternatively, (but it is a pity to do unnecessary work) this can also be seen via the definition of differentiability, by considering the function

$$\varepsilon(h,k) = \frac{1}{\sqrt{h^2+k^2}}\left(f(h,k) - f(0,0) - f'_x(0,0)h - f'_y(0,0)k\right). \tag{1}$$

Clearly $\varepsilon(h,k) = \frac{|h|k}{h^2+k^2}$ and this function does not tend to 0 at $(0,0)$. Perhaps the quickest way to see this is to look at the values of $\varepsilon(h,k)$ on the line $h = k$. *In fact by looking at the values of* $\varepsilon(h,k)$ *on this line on both sides of* $(0,0)$ *we also see that the limit* $\lim_{(h,k)\to(0,0)}\varepsilon(h,k)$ *does not exist.*

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2. In this question b and c are positive constants which take different values in different versions of the exam.

S is the surface

$$S = \left\{(x,y,z) : x^2+y^2 = c^2, y \ge 0, -b \le z \le b\right\}$$

and the given parametric representation is

$$\vec{R}(\theta,z) = c\cos\theta\widehat{\mathbf{i}} + c\sin\theta\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}.$$

So the set D must consist of all points (θ,z) such that $\theta\in[0,\pi]$ and $z\in[-b,b]$.

We have $\vec{R}_\theta = -c\sin\theta\widehat{\mathbf{i}} + c\cos\theta\widehat{\mathbf{j}}$ and $\vec{R}_z = \widehat{\mathbf{k}}$. So $\vec{R}_\theta \times \vec{R}_z = -c\sin\theta\widehat{\mathbf{i}} \times \widehat{\mathbf{k}} + c\cos\theta\widehat{\mathbf{j}} \times \widehat{\mathbf{k}} = c\cos\theta\widehat{\mathbf{i}} + c\sin\theta\widehat{\mathbf{j}}$. At the point $(0, c, 0)$ we have $z = 0$ and $\theta = \frac{\pi}{2}$ so the y component of $\vec{R}_\theta \times \vec{R}_z$ is c, which is positive.

(It is not hard to draw a picture of S. It is half of a cylinder of radius c and height $2b$. Its radius is the z axis, and it is the half of the cylinder which is on the "positive" side of the xz plane, i.e., the side where $y \geq 0$.)

For our particular choice of $X(\theta, z)$, $Y(\theta, z)$ and $Z(\theta, z)$ we have $\vec{F}(X(\theta,z), Y(\theta,z), Z(\theta,z)) = cz\cos\theta\widehat{\mathbf{i}} + cz\sin\theta\widehat{\mathbf{j}} + z^2\widehat{\mathbf{k}}$.

So $\iint_S \vec{F} \cdot \widehat{n} dS$ (which is of course the same as $\iint_S \vec{F} \cdot d\vec{S}$) is apparently given by the formula

$$\begin{aligned}
& \iint_D \vec{F}(X(\theta,z), Y(\theta,z), Z(\theta,z)) \cdot \vec{R}_\theta \times \vec{R}_z d\theta dz \\
= & \iint_D \left(cz\cos\theta\widehat{\mathbf{i}} + cz\sin\theta\widehat{\mathbf{j}} + z^2\widehat{\mathbf{k}}\right) \cdot \left(c\cos\theta\widehat{\mathbf{i}} + c\sin\theta\widehat{\mathbf{j}}\right) d\theta dz \\
= & \iint_D c^2 z\cos^2\theta + c^2 z\sin^2\theta d\theta dz = \int_{z=-b}^{b} \left(\int_{\theta=0}^{\pi} c^2 z d\theta\right) dz \\
= & \pi c^2 \left(\frac{z^2}{2}\right)\Big|_{-b}^{b} = 0.
\end{aligned}$$

In those versions of the exam where it was required that the y component of the normal vector to S at the point $(0, c, 0)$ should be **positive**, the steps of the calculation are exactly as written above. In the other versions where the same y component had to be **negative**, we see that $\vec{R}_\theta \times \vec{R}_z$ is pointing in the opposite direction (i.e., opposite sense or "megama") to the one that was requested. This means that we have to multiply the function appearing in the integrand by -1, even though, in this case, this will not change the actual final value, 0, of the integral.

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3. We are given that $\rho(x,y,z) = L\sqrt{x^2+y^2+z^2}$ where L is a positive constant (L is replaced by other letters in different versions of the exam.)

The ball V is defined by the inequality $x^2 + y^2 + z^2 \leq z$ which is equivalent to $x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$, i.e., $x^2 + y^2 + (z - \frac{1}{2})^2 = \left(\frac{1}{2}\right)^2$. So the ball rests on the plane $z = 0$. We will use spherical coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, $z = r\cos\theta$. To get to all the points in V we have to let θ range from 0 to $\pi/2$ (since the ball rests on the plane $z = 0$) and ϕ ranges from 0 to 2π.

(To be really convinced that this is true, consider the intersection of V with the plane Π_ϕ which contains the z axis and makes an angle ϕ with the x axis. Do this for each value of ϕ.)

For each value of θ and ϕ in these ranges, r ranges from 0 to some number depending on θ (and maybe ϕ?). We can find this number by using simple geometry (the angle in a semicircle is $\pi/2$) and trigonometry. In fact it is $\cos\theta$.

(Again, you will see this more clearly by considering the intersection of V with each plane Π_ϕ.)

If you didn't notice that, then instead you can substitute in the inequality $x^2+y^2+z^2 \leq z$ to get $r^2 \leq r\cos\theta$, which (since r is positive) is the same as $r \leq \cos\theta$.

So now we can calculate the mass of the ball, which is given by $\iiint_V L\sqrt{x^2+y^2+z^2} dxdydz$. When we change to spherical coordinates and then change from a triple integral to a repeated integral this becomes

$$\begin{aligned}
& L\int_{\phi=0}^{2\pi} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{\cos\theta} r \cdot r^2 \sin\theta dr\right) d\theta\right) d\phi \\
= & L\int_{\phi=0}^{2\pi} \left(\int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^4}{4}\Big|_0^{\cos\theta}\right) d\theta\right) d\phi = \frac{2\pi L}{4}\int_0^{\pi/2} \cos^4\theta\sin\theta d\theta.
\end{aligned}$$

When we make the change of variable $t = -\cos\theta$ this integral becomes

$$\frac{\pi L}{2}\int_{-1}^{0} t^4 dt = \frac{\pi}{2} \cdot \frac{1}{5} = \frac{\pi L}{10}.$$

So the mass of V is of course $\pi L/10$.

Let $(\overline{x}, \overline{y}, \overline{z})$ denote the centre of mass of V. We were only asked to calculate $\overline{z}$. *(It is not too difficult to show that $\overline{x}$ and $\overline{y}$ are both 0.)*

To find $\overline{z}$ we first calculate $\iiint_V zL\sqrt{x^2+y^2+z^2}dxdydz$. Using the same change of variables as before, this becomes

$$\begin{aligned} & L\int_{\phi=0}^{2\pi}\left(\int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\cos\theta} r\cos\theta\cdot r\cdot r^2\sin\theta dr\right)d\theta\right)d\phi \\ = \; & L\int_{\phi=0}^{2\pi}\left(\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\left(\left.\frac{r^5}{5}\right|_0^{\cos\theta}\right)d\theta\right)d\phi = \frac{2\pi L}{5}\int_0^{\pi/2}\cos^6\theta\sin\theta d\theta. \end{aligned}$$

The same change of variable $t = -\cos\theta$ gives us

$$\frac{2\pi L}{5}\int_{-1}^{0} t^6 dt = \frac{2\pi L}{5}\cdot\frac{1}{7} = \frac{2\pi L}{35}.$$

We now see, from a standard formula, that

$$\overline{z} = \frac{\iiint_V zL\sqrt{x^2+y^2+z^2}dxdydz}{\iiint_V L\sqrt{x^2+y^2+z^2}dxdydz} = \frac{\frac{2\pi L}{35}}{\frac{\pi L}{10}} = \frac{4}{7}.$$

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4. Let D be the set bounded by $x=-4$, $y=2$ and $y=x$. These three lines intersect each other to form a triangle. So D is the set $\{(x,y) : x \geq -4, y \leq 2, y \geq x\}$ which is clearly a triangle whose vertices are the points $A=(-4,-4)$, $B=(2,2)$ and $C=(-4,2)$.

I am sure that you do not need my help to draw a picture of this triangle.

Here is the solution for the case where $f(x,y)$ is the function $e^{x-y}(x^2-2y^2)$. In other versions the function was $e^{x-y}(x^2-2y^2)$ multiplied by some positive or negative constant. You can easily make the required changes for your version. Of course if the constant is negative, then we have to interchange the points where the maximum and minimum are attained.

Let us first see if the function $f(x,y)$ has critical points in the interior of the set D. The partial derivatives of f are $f'_x(x,y) = e^{x-y}(x^2-2y^2) + e^{x-y}\cdot 2x$ and $f'_y(x,y) = -e^{x-y}(x^2-2y^2) - 4ye^{x-y}$. Since $e^t \neq 0$ for all t, we see that the point (x,y) is a critical point if $x^2-2y^2+2x=0$ and also $x^2-2y^2+4y=0$. Subtracting these two equations gives us that $2y=x$. Substituting this in the first equation gives $4y^2-2y^2+4y=0$, i.e., $2y^2+4y=0$. So $y^2+2y=0$. This means that $y=0$ or $y=-2$. Thus we obtain two points, i.e., $(0,0)$ and $(-4,-2)$. Both of these points lie on the boundary of D. Since they are not in the interior of D we can forget about them. If they are "suspected points" for our problem we will meet them again anyway when we check all points on the boundary in other ways, as we must do in any case.

The boundary of D consists simply of three straight line segments. We could check on these segments using the method of Lagrange multipliers, separately on each segment, but it is easier to simply substitute and reduce the problem to consideration of functions of one variable.

(It would be COMPLETELY WRONG to consider the check of the boundary as a problem in Lagrange multipliers with three simultaneous constraints (ilootsim). I do hope that no one did this.)

The extremum of f on the "north-east" line segment from A to B is the extremum of the function $f(t,t) = e^{t-t}(t^2-2t^2) = -t^2$ on the interval $[-4,2]$. Clearly the suspected points on $[-4,2]$ are at the endpoint values $t=-4$ and $t=2$ and also at $t=0$ where the derivative of $-t^2$ vanishes. So we put the points $(-4,-4)$, $(2,2)$ and $(0,0)$ on our list of "suspected points" in D.*(See! we **did** meet the point $(0,0)$ again!)*

Next we check the behaviour of f on the horizontal line segment BC. The extremum of f on this line is the extremum of the function $f(t,2) = e^{t-2}\left(t^2-8\right)$ on the interval $[-4,2]$. This function has derivative $e^{t-2}(t^2-8)+2te^{t-2}$. This vanishes when $t^2-8+2t=0$, i.e., when $t = \frac{-2\pm\sqrt{4+32}}{2} = \frac{-2\pm\sqrt{36}}{2} = -1\pm 3$. So the two points $t=-4$ and $t=2$ where the derivative vanishes are exactly the two endpoints of the interval which we have to check anyway.

Finally we check the behaviour of f on the vertical line segment CA. The extremum of f on this line is the extremum of the function $f(-4,t) = e^{-4-t}(16-2t^2)$ on the interval - $-4 \leq t \leq 2$. Its derivative is

$-e^{-4-t}(16-2t^2)-4te^{4-t}$. It vanishes when $16-2t^2+4t=0$, i.e., $t^2-2t-8=0$. (This is NOT quite the same equation as we met just above!) So the "suspected" values are $t=\frac{2\pm\sqrt{4+32}}{2}=1\pm\sqrt{9}$, i.e., $t=4$ and $t=-2$. We ignore the point $t=4$ because $(-4,4)$ is outside D. I.e., we have to consider the point $(-4,-2)$ (we meet it again!) as well as the endpoints A and C.

So the maximum and the minimum of f are attained either at $(0,0)$ or $(-4,-2)$ or at one of the three vertices. At $(0,0)$ we have $f(0,0)=0$. Also $f(-4,-2)=e^{-2}(16-8)=8e^{-2}$. At A, $f(-4,-4)=16-32=-16$, At B, $f(2,2)=4-8=-4$, and at C, $f(-4,2)=e^{-6}(4-8)=-4e^{-6}$. So $f(-4,-2)=8e^{-2}$ is the maximum value and $f(-4,-4)=-16$ is the minimum value.

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5. (It is convenient to use the notation $[\alpha,\beta,\gamma]$ for the vector $\alpha\widehat{\mathbf{i}}+\beta\widehat{\mathbf{j}}+\gamma\widehat{\mathbf{k}}$.)

In this question the lines L_1 and L_2 are given respectively by the equations

$$\frac{x-(a+3)}{3}=\frac{y+2-b}{-2}=z-1-c$$

and

$$x+2-a=\frac{z+4-c}{2}, y=b.$$

where a, b and c are constants which take different values in different versions. You can easily calculate what a and b and c have to be in your version.

It follows that the parametric equations for L_1 are $x=3t+a+3$, $y=-2t+b-2$ and $z=t+1+c$.

Also the parametric equations for L_2 are $x=t+a-2$, $y=b$ and $z=2t+c-4$.

ALEF. First we have to check whether L_1 and L_2 have a common point. In other words, we have to check whether there are there two numbers s and t such that

$$(3s+a+3,-2s+b-2,s+1+c)=(t+a-2,b,2t+c-4). \tag{2}$$

(Why is it not correct to simply look for a value of t such that $(3t+a+3,-2t+b-2,t+1+c)=(t+a-2,b,2t+c-4)$?)

The equation (2) is of course the same as three equations in the two unknowns s and t.

$$\begin{aligned}3s+a+3&=t+a-2\\-2s+b-2&=b\\s+1+c&=2t+c-4.\end{aligned}$$

The second of these equations gives $s=-1$ and then the first equation gives $t=2$. So we have to check whether or not these values of s and t also satisfy the third equation. If $s=-1$ and $t=2$ then the left side of the third equation equals $-c$ and the right side also equals $-c$. So indeed the two lines meet at the point $(-3+a+3,2+b-2,-1+1+c)=(a,b,c)$.

There is also another way to solve part ALEF. We can try to solve the system of four linear equations in the three unknowns x, y and z which are the two canonical equations for L_1 taken together with the two canonical equations for L_2. Hopefully three of those equations should be enough to give a solution for x, y and z. Then we simply have to check if this solution also satisfies the fourth equation.

BET. From the parametric equations above we can see that L_1 is parallel to the vector $[3,-2,1]$ and L_2 is parallel to the vector $[1,0,2]$. Since L is perpendicular to both L_1 and L_2 it must be parallel to the vector product of these two vectors, $[3,-2,1]\times[1,0,2]=[-4,-5,2]$. Since L_1 and L_2 are not parallel to each other (how do you know this?) they cannot meet at any other points apart from (a,b,c). So L has to pass through (a,b,c). From all this it follows that L must be given by the parametric equations

$$x=a+4t,\ y=b+5t,\ z=c-2t.$$

Of course this is not the unique answer. There are also other parametric equations which give the same line. It is easy to prove that your answer is correct if and only if it is of the form

$$x=(a+4q)+4pt,\ y=(b+5q)+5pt,\ z=(c-2q)-2pt.$$

for some pair of constants p and q with $p\neq 0$.

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6. Please note that everywhere in this solution $\vec{F}$ always means the field $\frac{-y\hat{\mathbf{i}}+x\hat{\mathbf{k}}}{x^2+y^2}$ and no other field.

We have $\vec{F}(x,y)=P(x,y)\hat{\mathbf{i}}+Q(x,y)\hat{\mathbf{j}}$ where $P(x,y)=\frac{-y}{x^2+y^2}$ and $Q(x,y)=\frac{x}{x^2+y^2}$. A routine calculation shows that, at every point (x,y) except the point $(0,0)$, the functions P and Q have continuous derivatives which satisfy $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$. This means, by Green's theorem, that $\int_C \vec{F}\cdot d\vec{r}=0$ for many closed curves C. But which curves?

For $\int_C \vec{F}\cdot d\vec{r}$ to be zero, the curve C has of course not to pass through the point $(0,0)$. It must also be the boundary of a set D which does not contain $(0,0)$. Finally C must have a parametric representation $\vec{r}(t)=x(t)\hat{\mathbf{i}}+y(t)\hat{\mathbf{j}}$, $a\le t\le b$, where $x(t)$ and $y(t)$ have continuous derivatives at every point $t\in[a,b]$. Or it can be a finite union of curves with this property. In fact in this question we will only have to use curves which are made up of finitely many straight line segments and semicircles. Let us summarize what we have just said.

FACT 1. Let C be a closed curve in $\mathbb{R}^2$ which is made up of a finite number of straight line segments and semicircles, which does not pass through the point $(0,0)$, and also does not surround (makeef) the point $(0,0)$. Then $\int_C \vec{F}\cdot d\vec{r}=0$.

Here are two more properties that will be useful:

FACT 2. Let L be a line segment lying on the x axis but not passing through $(0,0)$, then $\int_L \vec{F}\cdot d\vec{r}=0$.

FACT 3. Let C be a circle and let S be a semicircle of any positive radius, centred[1] at $(0,0)$. Choose the anticlockwise direction of integration on C and also on S. Then $\int_C \vec{F}\cdot d\vec{r}=2\pi$ and $\int_S \vec{F}\cdot d\vec{r}=\pi$.

We can prove Facts 2 and 3 by doing direct calculations of the line integrals. For Fact 2, the curve L must be of the form $L=\{(t,0):a\le t\le b\}$ where $[a,b]$ is some closed interval which does not contain 0. Using the parametric representation $\vec{r}(t)=t\hat{\mathbf{i}}:a\le t\le b$ we see that $\int_L \vec{F}\cdot d\vec{r}=\int_a^b \frac{-0\hat{\mathbf{i}}+t\hat{\mathbf{j}}}{0+t^2}\cdot\hat{\mathbf{i}}dt=\int_a^b 0dt=0$.

(Of course if we choose to change the direction of L so that it starts at $(b,0)$ and ends at $(a,0)$ then we get that $\int_C \vec{F}\cdot d\vec{r}=-0$.)

For Fact 2 we suppose that Σ is either S or C, and that it has radius a at $(0,0)$. Then Σ has the parametric representation $\vec{r}(t)=a\cos t\hat{\mathbf{i}}+a\sin t\hat{\mathbf{j}}$ with $\alpha\le t\le\beta$, for suitable constants α and β such that $\beta-\alpha$ is either π or 2π. It follows that

$$\begin{aligned}\int_\Sigma \vec{F}\cdot d\vec{r} &= \int_\alpha^\beta \frac{-a\sin t\hat{\mathbf{i}}+a\cos t\hat{\mathbf{j}}}{a^2\cos^2 t+a^2\sin^2 t}\cdot\left(-a\sin t\hat{\mathbf{i}}+a\cos t\hat{\mathbf{j}}\right)dt=\int_\alpha^\beta \frac{a^2\sin^2 t+a^2\cos^2 t}{a^2}dt\\ &= \int_\alpha^\beta 1dt=\beta-\alpha\end{aligned}$$

which gives the required result.

Now we have almost all the tools that we need to answer this question. Here is the answer for the version that is posted on the internet. In other versions D_1 and D_2 and D_3 are the same sets, but they appear in a different order.

We first claim that $\vec{F}$ is NOT conservative (meshamer) on the set D_1. This is because there exists at least one closed curve Σ in D_1 with the property that $\int_\Sigma \vec{F}\cdot d\vec{r}\ne 0$. This would be very easy to see, using Fact 2, if we could draw a complete circle C completely contained in D_1 which is centred at $(0,0)$. But it is not quite sure that we can do this. However it is obvious that we can draw a closed rectangular curve Σ consisting of four straight line segments (shown in dark blue in Figure 1)

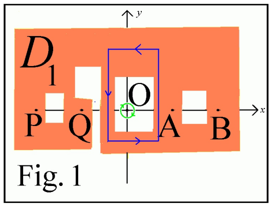

Fig. 1

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[1] Here I used the spellings "centre" and "centred" which are standard in Australia and the United Kingdom and several other countries. Of course in the U.S.A. the spelling is "center" and "centered".

which is completely contained in D_1. We will now show that $\int_\Sigma \vec{F}\cdot d\vec{r} = 2\pi$ if we choose the anticlockwise direction on Σ. To do this we completely forget about the set D_1. We choose a circle C (shown in green) centred at $(0,0)$ which has a very small (but positive!) radius, sufficiently small to be completely surrounded by the rectangular curve Σ. Now let Ω be the region in $\mathbb{R}^2$ which is a rectangle with a circular hole in it and whose boundary consists exactly of the two curves Σ and C. Since the components of $\vec{F}$ both have continuous partial derivatives at every point of Ω, we can apply Green's theorem to give that

$$\iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \int_\Sigma \vec{F}\cdot d\vec{r} + \int_C \vec{F}\cdot d\vec{r}. \tag{3}$$

We have already chosen the direction of integration on Σ correctly for this application of Green's theorem. Since Ω has to be on our left side when we move along the circle C we have to move along C in a clockwise direction, and so, by FACT 3, we have $\int_C \vec{F}\cdot d\vec{r} = -2\pi$. The double integral in (3) is of course 0 and so we obtain that indeed $\int_\Sigma \vec{F}\cdot d\vec{r} = 2\pi \neq 0$ and so $\vec{F}$ cannot be conservative on D_1.

In contrary to this, $\vec{F}$ is a conservative field in D_2 and also in D_3. This is because both of the sets D_2 and D_3 are ***simply connected***, i.e., if C is any closed curve contained completely in D_j for $j = 2$ or 3, then every point of the region surrounded by C is also in D_j. So, again by Green's theorem, we can deduce that $\int_C \vec{F}\cdot d\vec{r} = 0$ for every such closed curve C. (Of course here we are using the fact that the region surrounded by C does not include the point $(0,0)$, and so all conditions required for Green's theorem for this particular field $\vec{F}$ and this particular region are satisfied.) This shows that $\vec{F}$ is indeed conservative in D_j for $j = 2$ and for $j = 3$.

Now we come to the most interesting part of the question.

From the general theory of conservative fields we know that in each of the cases $j = 2$ and $j = 3$ there has to be some function $\psi : D_j \to \mathbb{R}$ such that $\vec{F} = \vec{\nabla}\psi$ at every point of D_j. Since D_2 and D_3 are different sets there is no guarantee that the same function ψ will work for both sets. Indeed we will see that we really have to use two different functions for those two different sets.

In the question we are asked to find $\phi : D_j \to \mathbb{R}$ such that $\vec{F} = \vec{\nabla}\phi$ on D_j and also $\phi(1,0) = 0$. This is easy to do, if we have already found some function ψ such that $\vec{F} = \vec{\nabla}\psi$. We simply choose $\phi(x,y) = \psi(x,y) - \psi(0,0)$.

But how do we find ϕ or ψ? If we work like robots and solve the differential equations $\frac{\partial\phi}{\partial x} = \frac{-y}{x^2+y^2}$ and $\frac{\partial\phi}{\partial y} = \frac{x}{x^2+y^2}$ we will get that $\phi(x,y) = \arctan\frac{y}{x} + const.$ and the condition $\phi(1,0) = 0$ tells us that the constant has to be 0. This solution would be very nice if our sets D_2 and D_3 did not contain any points on the y axis. But both these sets DO contain points on the y axis, and at those points $\arctan\frac{y}{x}$ is not defined. Well you might say, that doesn't matter. We will simply take the limit of $\arctan\frac{y}{x}$ at those points. Sorry, that will not work, because the limit from one side of the y axis is different from the limit from the other side. As you cross over the y axis $\arctan\frac{y}{x}$ jumps by π or by $-\pi$.

There is a more complicated way to write down formulæ for $\phi(x,y)$ on all of D_j by patching together several different functions of the form $\arctan\frac{y}{x} + c$, i.e., with a suitable jump in the value of the constant c each time the set D_j crosses the y axis. This is rather tricky to do, so it is lucky that we were not required to give any formulæ for ϕ. We were only asked to find the values of $\phi(2,0)$, $\phi(-1,0)$ and $\phi(-2,0)$, and there is a way to do this without writing down general formulæ.

Hopefully you remember at least the first part of the proof of the fact that if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C in some set D, then there exists some function ϕ in D such that $\vec{F} = \vec{\nabla}\phi$ at all points D. It goes like this: First we choose some point p in D and decide that ϕ is zero at that point. Then, for every other point q in D we define $\phi(q)$ to equal $\int_{C_{pq}} \vec{F}\cdot d\vec{r}$ where C_{pq} is some curve in D from p to q. This definition makes sense because ***every*** curve in D from p to q will give us the same answer. (Then the proof concludes with a special argument to show that the function ϕ obtained in this way really does satisfy $\vec{F} = \vec{\nabla}\phi$. But we do not have to worry about that just now.)

Obviously the natural thing for us here is to take $p = (1,0)$. So we know that:

FACT 4. For $j = 2$, and also for $j = 3$ the value of the potential function ϕ for D_j at any point (x_0, y_0) in D_j equals the integral $\int_C \vec{F}\cdot d\vec{r}$ where C is some curve **lying completely in** D_j which goes from $(1,0)$ to that point (x_0, y_0).

(Even if you have forgotten the details of that proof you could also deduce Fact 4 in another way, by using the fact that some function ϕ *satisfying* $\vec{F} = \vec{\nabla}\phi$ *in* D_j *exists. Then combine this fact with the formula* $\phi(q) - \phi(p) = \int_C \vec{\nabla}\phi \cdot d\vec{r}$ *which holds under suitable conditions for any curve* C *from the point* p *to the point* q*.)*

Suppose now that ϕ is the potential function for $\vec{F}$ defined for the set D_2.

Look at the curves shown in Figure 2,

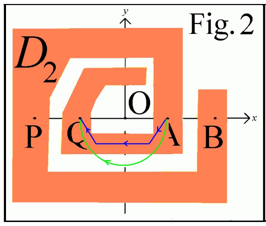
Fig. 2

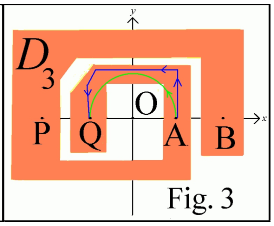

Fig. 3

namely a green semicircle and the dark blue curve in D_2 consisting of three line segments. By Fact 4, $\phi(-1,0)$ equals the integral of $\vec{F}$ along the dark blue curve. By Fact 1 this is equal to the integral of $\vec{F}$ along the green semicircle. By Fact 3 this last integral equals $-\pi$. So $\phi(-1,0) = -\pi$.

Now look at Figure 4.

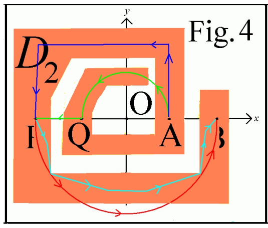
Fig. 4

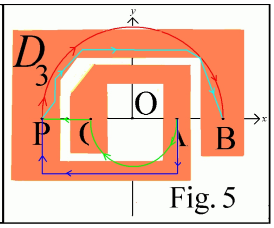

Fig. 5

By Fact 4 we have that $\phi(-2,0)$ is the integral along the dark blue curve. By Fact 1, this equals the integral along the green curve consisting of a semicircle and a line segment on the x axis. By Facts 2 and 3 the integral along the green curve equals $\pi + 0$. So $\phi(-2,0) = \pi$.

Next, still using Figure 4, we claim that $\phi(2,0)$ equals the line integral along the curve which consists of the dark blue curve that we already considered together with the light blue curve. (This combined curve is completely contained in D_2). By Fact 1, the integral along the light blue curve equals the integral along the red semicircle which in turn equals π. So $\phi(2,0) = \pi + \pi = 2\pi$.

Finally we will consider the case where ϕ is the potential function for $\vec{F}$ defined for the set D_3. We will use analogous arguments to calculate $\phi(-1,0)$, $\phi(2,0)$ and $\phi(-2,0)$, this time via the curves shown in Figures 3 and 5. As promised before we will see that ϕ really is a different function.

First $\phi(-1,0)$ has to equal the line integral along the dark blue curve in Figure 3, which has to equal the integral along the green semicircle. So $\phi(-1,0) = \pi$. For the rest of our discussion we will only be considering curves which are all shown in Figure 5. We have that $\phi(-2,0)$ equals the integral along the dark blue curve, which equals the integral along the green curve, which is $-\pi+0$. So $\phi(-2,0) = -\pi$. The integral along the light blue curve in equals the integral along the red semicircle which is $-\pi$. It follows that $\phi(2,0) = -\pi - \pi = -2\pi$.

Further comments about this question can be found below in an appendix.

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7. ALEF There are two basic things that you are expected to know and remember about the "mixed" second derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. If you know them then it is not difficult to answer part ALEF.

- **If both** $\frac{\partial^2 f}{\partial x \partial y}$ **and** $\frac{\partial^2 f}{\partial y \partial x}$ **are continuous at some point** (x_0, y_0), *(which means that they also have to exist in some neighbourhood of* (x_0, y_0), **then they satisfy** $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$**.**

This is a theorem of H. A. Schwarz. (Similar results were announced earlier by Euler and by A. C. Clairaut, but it seems that their proofs were not correct.) This result enables you to immediately see that one of the answers in the given list is wrong.

- **There exists a function** f **which satisfies** $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \neq \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ **at some point** (x_0, y_0)**.**

I.e., both of the derivatives exist at that point but they are not equal. *(See the appendix of this document for more details, including an additional maybe surprising counterexample.)*

The existence of this function is enough to show that the other two explicit answers in the given list are also wrong. *(The answer which you were asked to delete during the exam is also wrong, but you would need another more elaborate example (see the appendix) to show this.)*

So the correct answer here (in all versions of the exam in fact) is **E,** namely that all the other answers in the list are wrong.

BET. By definition, the rotor or curl of the vector field is given by

$$\vec{\nabla}\times\left(P\widehat{\mathbf{i}}+Q\widehat{\mathbf{j}}+R\widehat{\mathbf{k}}\right)=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\widehat{\mathbf{i}}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\widehat{\mathbf{j}}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\widehat{\mathbf{k}}. \tag{4}$$

(This is the official definition, but of course we can use the unofficial "science fiction" sort of ""determinant(!?!?)"" $\begin{vmatrix}\widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$ *to help us remember it. Of course this is not a real determinant, it is just a trick to help us remember (4).)*

It follows that the divergence of this vector field is

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\\
=\;&\frac{\partial^2 R}{\partial x\partial y}-\frac{\partial^2 Q}{\partial x\partial z}+\frac{\partial^2 P}{\partial y\partial z}-\frac{\partial^2 R}{\partial y\partial x}+\frac{\partial^2 Q}{\partial z\partial x}-\frac{\partial^2 P}{\partial z\partial y}\\
=\;&\left(\frac{\partial^2 P}{\partial y\partial z}-\frac{\partial^2 P}{\partial z\partial y}\right)+\left(\frac{\partial^2 Q}{\partial z\partial x}-\frac{\partial^2 Q}{\partial x\partial z}\right)+\left(\frac{\partial^2 R}{\partial x\partial y}-\frac{\partial^2 R}{\partial y\partial x}\right) \qquad (5)
\end{aligned}$$

We have to find conditions for this expression to be 0 at some point (x_0, y_0, z_0). There are all sorts of special and trivial conditions which would make this happen, for example if P, Q and R are all identically 0. But the natural condition to consider should be suggested by the results that you considered in Part ALEF. By the theorem of Schwarz, if the derivatives $\frac{\partial^2 R}{\partial x\partial y}$ and $\frac{\partial^2 R}{\partial y\partial x}$ both exist and are both continuous functions at (x_0, y_0, z_0) then they are equal at that point and the third term in the expression (5) vanishes. In fact we only need $\frac{\partial^2 R}{\partial x\partial y}(x, y, z_0)$ and $\frac{\partial^2 R}{\partial y\partial x}(x, y, z_0)$ to be continuous functions of (x, y) at (x_0, y_0) for that particular fixed value of z_0, but this is more complicated and messy to formulate.

So here is one reasonable version of a theorem.

Theorem. *Suppose that all the derivatives* $\frac{\partial^2 P}{\partial y\partial z}$, $\frac{\partial^2 P}{\partial z\partial y}$, $\frac{\partial^2 Q}{\partial z\partial x}$, $\frac{\partial^2 Q}{\partial x\partial z}$, $\frac{\partial^2 R}{\partial x\partial y}$ *and* $\frac{\partial^2 R}{\partial y\partial x}$ *exist and are continuous at the point* (x_0, y_0, z_0). *Then* $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right)=0$ *holds at* (x_0, y_0, z_0).

Proof. We have already written a large part of the proof, in our motivating discussion. But I will repeat at least part of it here. The first step is to write (4) and then continue with the three lines of (5).

The final three steps are to apply Schwarz' theorem separately to each of the three bracketed terms in the last line of (5). The quick explanation is to say that since all the mixed derivatives of second order are continuous at (x_0, y_0, z_0), the order of derivation does not matter and so Schwarz' theorem gives us that $\frac{\partial^2 P}{\partial y\partial z}=\frac{\partial^2 P}{\partial z\partial y}$ and $\frac{\partial^2 Q}{\partial z\partial x}=\frac{\partial^2 Q}{\partial x\partial z}$ and $\frac{\partial^2 R}{\partial x\partial y}=\frac{\partial^2 R}{\partial y\partial x}$ at (x_0, y_0, z_0). This means that the last line in (5) is zero at (x_0, y_0, z_0) and completes the proof.

But there is also a more precise version of this argument. We do not expect you to mention the sorts of things that I will write now, but any student who does will get some bonus points. The result of Schwarz' theorem (which used to answer part ALEF) was given for functions of **two** variables. But here we are dealing with functions of **three** variables. How do we translate a result about functions of two variables into one which can be applied to functions of three variables? The answer is that, each time we apply Schwarz' theorem, we are holding one of those three variables constant, so we are effectively dealing only with a function of two variables.

As an example, let me show more precisely how we get that $\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x} = 0$ at (x_0, y_0, z_0). The other two expressions $\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 P}{\partial z \partial y}$ and $\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 Q}{\partial x \partial z}$ can of course be treated in the same way, but with a swapping around of the roles of the variables x, y and z. We will use the function of two variables defined by $f(x, y) = R(x, y, z_0)$. Remember that z_0 is a constant. From the definitions we have $\frac{\partial f}{\partial x}(x, y) = \frac{\partial R}{\partial x}(x, y, z_0)$ where in the first partial derivative we are holding y constant, and in the second one we are holding both y and z constant. This formula holds at each point (x, y, z_0) where $\frac{\partial R}{\partial x}(x, y, z_0)$ exists. Similarly we have $\frac{\partial f}{\partial y}(x, y) = \frac{\partial R}{\partial y}(x, y, z_0)$ and so we continue to deduce that $\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial x} \frac{\partial R}{\partial y}(x, y, z_0)$ and $\frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial y} \frac{\partial R}{\partial x}(x, y, z_0)$.

Suppose that $u(x, y, z)$ is any function of three variables which is continuous at the point (x_0, y_0, z_0). Then it follows almost immediately from the definition of continuity that the function of two variables $v(x, y)$ given by the formula $v(x, y) = u(x, y, z_0)$ is defined in some neighbourhood of (x_0, y_0) and is continuous at (x_0, y_0). We now apply this argument to the function $u(x, y, z) = \frac{\partial}{\partial x} \frac{\partial R}{\partial y}(x, y, z)$, which is given to be continuous at (x_0, y_0, z_0) to deduce that $\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x, y)$ is continuous at (x_0, y_0). For exactly analogous reasons we also get that $\frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x, y)$ is continuous at (x_0, y_0). These two continuity conditions are exactly what we need to apply Schwarz' theorem to give us that $\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0)$, and this is exactly the same as $\frac{\partial^2 R}{\partial x \partial y}(x_0, y_0, z_0) - \frac{\partial^2 R}{\partial y \partial x}(x_0, y_0, z_0) = 0$.

Of course I cannot exclude the possibility that you learned this course from a book or from lectures where this generalized version of Schwarz' theorem for functions of three or more variables was stated and proved explicitly. In general it is more than ok for you to state and use this generalized version of the theorem.

GIMEL. Let us use the notation $\vec{H}(x, y, z)$ for the vector field $\vec{H}(x, y, z) = 5x^2yz\widehat{\mathbf{i}} + 24\sin^2 z^3\widehat{\mathbf{j}} + 7xe^{y^2}\widehat{\mathbf{k}}$. I really would not like to have to calculate the rotor of the vector field $\vec{H}$, and I would like even less to have to calculate a surface integral where this rotor appears. Fortunately we do not have to. The components of this field are simply products and compositions of functions with continuous derivatives of all orders, and so all their partial derivatives of first order (and also of all other orders) are continuous at every point (x, y, z) in $\mathbb{R}^3$. So, by the theorem of part BET, the divergence of the rotor of this field is zero. So the divergence of the vector field $\vec{F}(x, y, z) = ax\widehat{\mathbf{i}} + (by + cz)\widehat{\mathbf{j}} + \vec{\nabla} \times \vec{H}$ equals $a + b + 0$.

In other versions of the exam the $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ components of this field are slightly different, but the divergence is always some constant depending on the constants a, b and/or c. In all versions, all components of the vector field have continuous first order partial derivatives, as is required for the divergence theorem.

The spherical surface S is the boundary of a sphere V of radius c . (The radius is different in other versions.) By the Gauss divergence theorem, since the normal on S is taken to point outwards from V, we have that $\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} dxdydz$ and this second integral is the volume of V multiplied by the constant value of $\vec{\nabla} \cdot \vec{F}$. In this version of the exam, this integral equals $(a + b)\frac{4\pi c^3}{3}$.

Recall that we can apply the divergence theorem here because, as already mentioned, the components of the vector field $\vec{F}$ have continuous first order partial derivatives and also because the set V is simultaneously an x-simple , y-simple and z-simple set. (It would suffice if V was the union of a finite collection of non overlapping sets which are each x, y and z-simple.)

DALET. In general, the divergence theorem needs the components of the vector field $\vec{F}$ to have continuous first order partial derivatives at all points of the region V. It can fail if this condition fails even at only one point of V. (To see an example where this happens consider the famous field $\vec{F} = \frac{x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$ in the case where V is a sphere centred at $(0, 0, 0)$.) From the information we are given, our particular field here, namely $\vec{F}(x, y, z) = ax\widehat{\mathbf{i}} + (by + cz)\widehat{\mathbf{j}} + \vec{\nabla} \times \vec{G}$, does not satisfy the conditions of the Gauss divergence theorem. So we cannot directly use the methods that worked for us in part GIMEL.

But there is another way to calculate this integral, which will in fact give us the same answer $(a + b)\frac{4\pi c^3}{3} = \frac{4}{3}(a + b)\pi c^3$ as we got in part GIMEL. Surface integrals, like other integrals are linear operations (can you see why?) and so we have

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S \left(ax\widehat{\mathbf{i}} + (by + cz)\widehat{\mathbf{j}} + \vec{\nabla} \times \vec{G} \right) \cdot d\vec{S} \\ &= \iint_S \left(ax\widehat{\mathbf{i}} + (by + cz)\widehat{\mathbf{j}} \right) \cdot d\vec{S} + \iint_S \vec{\nabla} \times \vec{G} \cdot d\vec{S}. \end{aligned}$$

For the first of these two integrals $\iint_S \left(ax\widehat{\mathbf{i}} + (by+cz)\widehat{\mathbf{j}}\right)\cdot d\vec{S}$ it **is** correct to use the Gauss divergence theorem, and, exactly as in part GIMEL, this integral equals $\frac{4}{3}(a+b)\,\pi c^3$. Now we will show that $\iint_S \vec{\nabla}\times\vec{G}\cdot d\vec{S} = 0$. We will do this, davka, by using Stokes' theorem. We should point out that, despite the bad behaviour of $\vec{G}$ at $(0,0,0)$, this vector field does have components with continuous first order partial derivatives in some open set containing S. This is one of the necessary conditions for applying Stokes' theorem.

Normally when we apply Stokes theorem the surface S has to have a curve along its edge (katzeh), and we calculate a line integral along that edge. But here our surface S is closed. So it does not seem to have an edge. We can overcome this difficulty in various ways. Our way will be to split S into two hemispherical surfaces, $S_+ = \left\{(x,y,z) : x^2+y^2+z^2 = c^2, z \geq 0\right\}$ and $S_- = \left\{(x,y,z) : x^2+y^2+z^2 = c^2, z \leq 0\right\}$. The "edge" of S_+ is the curve C_+ on the "equator" of the sphere, i.e., $\left\{(x,y,0) : x^2+y^2 = c^2\right\}$. According to Stokes' theorem we have $\int_{C_+} \vec{G}\cdot d\vec{r} = \iint_{S_+} \vec{\nabla}\times\vec{G}\cdot d\vec{S}$, where, if the normal on S_+ is pointing away from $(0,0,0)$, we have to choose the direction on C_+ so that appears to be anti-clockwise when we look at C_+ from above, i.e., from the positive side of the z axis.

We can also apply Stokes theorem to the surface S_- and here we get $\int_{C_-} \vec{G}\cdot d\vec{r} = \iint_{S_-} \vec{\nabla}\times\vec{G}\cdot d\vec{S}$ where here C_- is exactly the same curve as C_+ but with the opposite direction, and again on S_- we take the normal pointing away from $(0,0,0)$. Because of the opposite directions of C_+ and C_- and because of Stokes' theorem we have

$$0 = \int_{C_+} \vec{G}\cdot d\vec{r} + \int_{C_-} \vec{G}\cdot d\vec{r} = \iint_{S_+} \vec{\nabla}\times\vec{G}\cdot d\vec{S} + \iint_{S_-} \vec{\nabla}\times\vec{G}\cdot d\vec{S} = \iint_{S} \vec{\nabla}\times\vec{G}\cdot d\vec{S}.$$

This completes our proof that the integral in part DALET really equals $\frac{4}{3}(a+b)\,\pi c^3$.

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Appendix. Further comments about some of the questions of this examination.

Question 6. Note that although the field $\vec{F} = \frac{-y\widehat{\mathbf{i}}+x\widehat{\mathbf{k}}}{x^2+y^2}$ is a conservative field in D_2 and also in D_3 it is NOT conservative in the set $D_2 \cup D_3$ (because this set contains a closed curve which passes around $(0,0)$). So the property of being conservative is not something like continuity or differentiability which is checked separately at each point.

Those of you who go on to study the course "Complex functions" (funktsiyot merukavot) may meet some things which are quite similar to this exercise if and when you have to define different versions (different "branches" or "anafim") of the complex logarithm function $\operatorname{Log} z$ on different simply connected subsets of the complex plane.

A word about "Fact 3": In fact something more general that Fact 3 is true and is easy to check by direct calculation: If L is any line segment which does not have $(0,0)$ as an endpoint or interior point, and which is on a line which passes through $(0,0)$, then $\int_L \frac{-y\widehat{\mathbf{i}}+x\widehat{\mathbf{k}}}{x^2+y^2}\cdot d\vec{r} = 0$. For example L might be on the y-axis. This opens up some other alternative ways of calculating the potential function ϕ for the sets D_2 and D_3.

Since some students have meanwhile asked me about this, I will now give you more details about an alternative (more difficult) way of answering Question 6, which was mentioned briefly in the solution above.

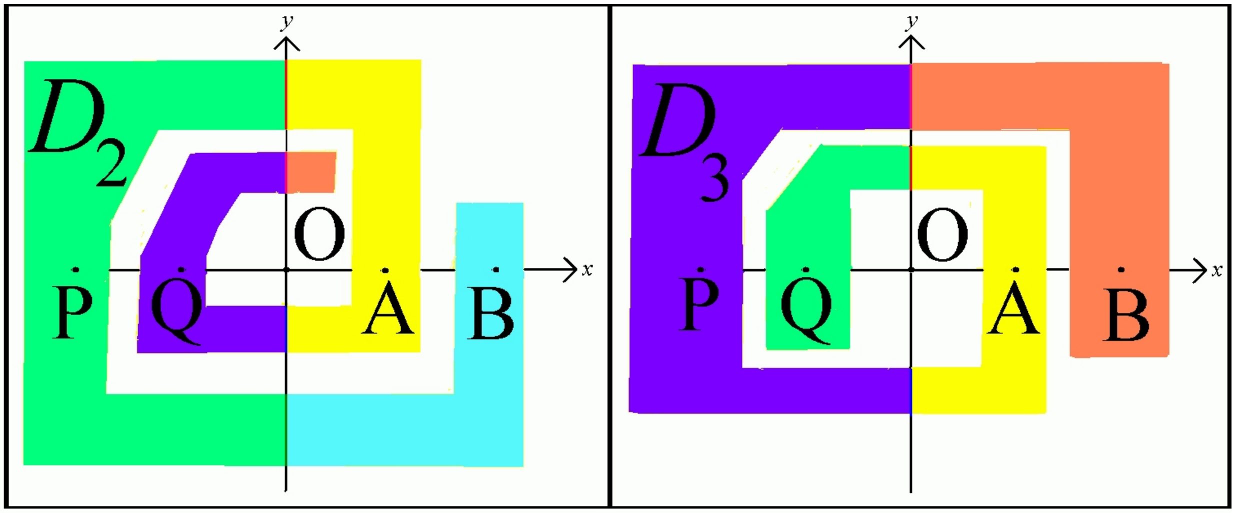
This way gives us the function $\phi(x,y)$, not only at the three special points that we were asked about, but also at *all* other points of the set D_2 or the set D_3.

It works by "patching together" different formulæ using the arctan function to completely describe the potential function $\phi(x,y)$ on each of the sets D_2 and D_3.

You will see that we really do get two different functions on these two different sets. (We already saw this when we used the line integral method.)

Because this method is complicated, we will make things easier here by using a coloured picture. First let us recall that the "official" definition of $\arctan t$ is such that it takes values in the interval $(-\pi/2, \pi/2)$ and it satisfies $\lim_{t\to+\infty}\arctan t = \pi/2$ and $\lim_{t\to-\infty}\arctan t = -\pi/2$.

In the picture below you can see that a certain part of the set D_2 and a certain part of the set D_3 are each shown in yellow 🟨 . For all points in this yellow region we set $\phi(x,y) = \arctan\frac{y}{x}$. In particular, this is consistent with the requirement that $\phi(1,0) = 0$. Then certain parts of D_2 and D_3 are shown in green 🟩 . In both green regions we set $\phi(x,y) = \arctan\frac{y}{x} + \pi$. The interface between the green and yellow regions is a line segment on the y-axis, which, in both sets, is shown in red | . On this line segment we set $\phi(x,y) = \phi(0,y) = \pi/2$. There is also a purple region 🟪 in both of the sets. In both of these regions we set $\phi(x,y) = \arctan\frac{y}{x} - \pi$. On the vertical blue line segment | which is the interface between the yellow and purple regions, we set $\phi(x,y) = \phi(0,y) = -\pi/2$.



At this stage we can already see that the function ϕ **must** be different for different sets. We know that $\arctan\frac{y}{x} = 0$ at both of the points P and Q. In the case of D_2 the point P is in the green region and the point Q is in the purple region, and vice versa in the case of D_3.

But we have still not completely defined ϕ. In the orange region 🟧 we have to set $\phi(x,y) = \arctan\frac{y}{x} - 2\pi$, and on the light purple vertical line segment | which is at the interface of the purple and orange regions we set $\phi(x,y) = \phi(0,y) = -3\pi/2$.

The last step is to consider the light blue region 🟦 . This only appears in the set D_2. Here we set $\phi(x,y) = \arctan\frac{y}{x} + 2\pi$ and on the dark green vertical line segment | which is the interface between the green and light blue regions we set $\phi(x,y) = \phi(0,y) = 3\pi/2$. Again we see a clear difference in the function ϕ for the set D_2 and for the set D_3. The point $(2,0)$ is in the light blue region for D_2 and in the orange region for D_3.

It is not hard to check that the two versions of the function ϕ that we have constructed here, for D_2 and for D_3, are both continuous. But we also need to know that these functions each satisfy $\vec{\nabla}\phi = \vec{F}$ at every point of the relevant set (D_2 or D_3). This is an obvious calculation at most points. But at points on the y axis in D_j we have to think a bit more. There is no problem for the y coordinate, since obviously $\frac{\partial\phi}{\partial y} = 0$ and also the $\widehat{\mathbf{j}}$ component of $\vec{F}$ is 0 at those points. We could calculate $\frac{\partial\phi}{\partial x}$ at these points with the help of the facts that $\lim_{t\to+\infty}\frac{\arctan t - \pi/2}{1/t} = \lim_{t\to\infty}\frac{1/(1+t^2)}{-1/t^2} = -1$ and also $\lim_{t\to-\infty}\frac{\arctan t + \pi/2}{1/t} = \lim_{t\to-\infty}\frac{1/(1+t^2)}{-1/t^2} = -1$. Another way to deal with this calculation is to first show that for each point $(0,y_0)$ in D_j which lies on the y axis, there is a neighbourhood of the point in which $\phi(x,y) = c - \arctan\frac{x}{y}$. The value of the constant c will be different in different cases. It depends on whether we are dealing with D_2 or with D_3, and also it depends on which of the different line segments our point lies. But the value of c will not influence our calculation of $\vec{\nabla}\phi$ at this point, and indeed we will get that $\vec{\nabla}\phi(0,y_0) = \vec{F}(0,y_0)$ as required.

Question 7. For anyone who is curious, here are two quite exotic functions which show that the formula $f''_{xy}(x,y) = f''_{yx}(x,y)$ really can fail sometimes. The first function is well known and appears in textbooks. It is

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases} . \tag{6}$$

With a small effort, sometimes using the formula for the derivative of a quotient (manah) and sometimes using the original definition of partial derivatives, you can show that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ and that $\frac{\partial f}{\partial x}(x,y) = \frac{x^4y+3x^2y^3-y^5}{(x^2+y^2)^2}$ for all $(x,y) \neq (0,0)$ and consequently that $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$. By similar calculations, or by simply appealing to symmetry, we get $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$.

It is not hard to see that, for this function, $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are both continuous at every point, including* $(0,0)$*. So, when a function has continuous first derivatives or is differentiable, even at every point, this gives no guarantee that its mixed second order partial derivatives will be equal everywhere.*

Now, here is a second function which is even crazier than the first. Let c be some constant and let $f(x,y) = \begin{cases} x^c y & , \quad |x| > |y| \\ 0 & , \quad |x| \leq |y| \end{cases}$. Note first that $f(x,0) = 0$ for all $x \in \mathbb{R}$ which implies that $\frac{\partial f}{\partial x}(0,0) = 0$. For every fixed $y \neq 0$ we see that $f(x,y) = 0$ for all x in the interval $[-|y|, |y|]$. Consequently we have $\frac{\partial f}{\partial x}(0,y) = 0$ for every $y \in \mathbb{R}$. We deduce that $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,y) = 0$ for all $y \in \mathbb{R}$, and, in particular $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = 0$. Now we will calculate $\frac{\partial f}{\partial y}(x,0)$ for every $x \in \mathbb{R}$. First, if $x = 0$ we see that $|x| \leq |y|$ for all $y \in \mathbb{R}$ so $f(0,y) = 0$ for all y and so $\frac{\partial f}{\partial y}(0,y) = 0$, and, in particular, $\frac{\partial f}{\partial y}(0,0) = 0$. Then, for each fixed $x \neq 0$ we see that $f(x,y) = x^c y$ for each y in the open interval $(-|x|, |x|)$. This means that $\frac{\partial f}{\partial y}(x,y) = x^c$ for each y in that interval, and, in particular $\frac{\partial f}{\partial y}(x,0) = x^c$. Finally, we are ready to calculate $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)$. By definition this equals

$$\lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h \to 0} \frac{h^c - 0}{h} = \lim_{h \to 0} h^{c-1}.$$

If we choose $c > 1$ then this limit equals 0 and we have the usual situation where the two mixed derivatives are equal. But, if $c = 1$, then we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = 1 \neq 0 = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$. Even more outrageously, if we choose $c < 1$ for example $c = 0$, then the limit $\lim_{h \to 0} h^{c-1}$ does not exist. We cannot even say it is ∞ because h^{-1} tends to $+\infty$ from the right and to $-\infty$ from the left. So the partial derivative $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)$ simply does not exist, even though the partial derivative $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$ exists.

If anyone wants to know more about the formula $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ I can show you my copy of an article which gives other conditions for it to hold, and further historical details and references.

In part DALET of this question, the fact that $\vec{G}(0,0,0) = -3\widehat{\mathbf{k}}$ is not relevant. Note that even if we were given that all the components $\vec{G}$ also have continuous first order derivatives at $(0,0,0)$ just like we know they have at every other point, then we still could not apply the divergence theorem. If we do not have the additional condition that the **second** derivatives of these components also exist, then we cannot even be sure that the divergence of $\vec{\nabla} \times \vec{G}$ exists. If the divergence does not even exist, how can we hope to apply the divergence theorem?

The “vector” $\vec{\nabla} = \frac{\partial}{\partial x}\widehat{\mathbf{i}} + \frac{\partial}{\partial y}\widehat{\mathbf{j}} + \frac{\partial}{\partial z}\widehat{\mathbf{k}}$ does not have all the usual properties of an ordinary vector, and the “determinant” $\begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ and the “vector product” $\vec{\nabla} \times \vec{F}$ which we use as helpful ways of remembering the formula for the rotor of a vector field do not have all the usual properties of ordinary determinants and vector products. These are beautiful and imaginative notations but they are only notations to help us remember formulæ. We cannot use them to prove results. If we try, then some strange things will happen. For example, if in part BET of this question you try to claim that $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F}$ is 0 because it is the scalar triple product of three vectors where two of them are parallel, how does this explain the fact that for some choices of $\vec{F}$ you can in fact get $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} \neq 0$? (This happens, for example, at $(x,y,z) = (0,0,0)$ when $\vec{F}(x,y,z) = f(x,y)\widehat{\mathbf{k}}$ and $f(x,y)$ is defined by (6).) This would be like saying that, for usual vectors $\vec{A}$ and $\vec{B}$, the scalar triple product $\vec{A} \cdot \vec{A} \times \vec{B}$ is zero for some choices of the vector $\vec{B}$ but is not zero for other choices of that same vector $\vec{B}$.

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