HEDVA 2M
SOLUTION TO THE EXAMINATION OF 20/2/2006.

Version 2, 28/3/2009

The small changes between Version 1.1 and Version 1 are only in style. Version 1.2 inserts a forgotten minus sign in 2BET and includes some additional comments about Question 7. Version 1.21 adds a remark at the beginning of the solution of Question 7 and slightly rewords a remark in the introduction to the solution of Question 2. Version 1.22 adds a remark at the end of the solution of Question 3. Version 2 corrects a small misprint in the solution of Question 7 part Dalet. THANKS ARIE for that. How did we all miss it for three years?

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Certain parts of this solution are explained in rather more detail than we would expect from most students. We also sometimes give several different ways of answering questions, whereas we only expected one way from you.

A word to those of you who have some difficulty reading technical material in English. I could apologize for not writing a Hebrew solution. But it is in your very best interests to learn to read this sort of material in English NOW. Next year you will be older and it will be harder for you to learn. It is impossible, or almost impossible to have a reasonable career in engineering or science without being able to read English well.

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1. In this question the sets V_1 and V are of the forms

$$V_1 = \left\{(x, y, z) : x^2 + y^2 \le a^2, 0 \le z \le b\sqrt{x^2 + y^2}, x \ge 0\right\}$$

and $V = \left\{(x, y, z) : x^2 + (y-2)^2 \le a^2, 0 \le z \le b\sqrt{x^2 + (y-2)^2}, x \ge 0\right\}$, where a and b are positive numbers which have different values in different versions of the exam.

ALEF. Use the formula for change of variables where the new variables are cylindrical coordinates. The set $W = \{(r, \phi, z) : (r\cos\phi, r\sin\phi, z) \in V_1, r \ge 0, 0 \le \phi \le 2\pi\}$ equals

$$\{(r, \phi, z) : 0 \le r \le a, -\pi/2 \le \phi \le \pi/2, 0 \le z \le br\}. \tag{0.1}$$

(The condition $x \ge 0$ is equivalent to $-\pi/2 \le \phi \le \pi/2$.)

So, remembering that the relevant Jacobian is r, we have

$$\iiint_{V_1} f(z)dxdydz = \iiint_W f(z)rdrd\phi dz = \iint_P \left(\int_0^{br} f(z)dz\right) rdrd\phi$$

where $P = \{(r, \phi) : 0 \le r \le a, -\pi/2 \le \phi \le \pi/2\}$. But we need to change the order of integration in the repeated integral so that the **last** integration is with respect to z. We observe that we can also write

$$W = \{(r, \phi, z) : -\pi/2 \le \phi \le \pi/2, 0 \le z \le ab, z/b \le r \le a\}. \tag{0.2}$$

We can see this by checking algebraically that the three inequalities appearing in (0.1) hold if and only if the three inequalities in (0.2) hold. Another way of seeing this is to reason more geometrically, using the fact that the set V_1 is obtained by rotating the triangle in the xz plane $\{(x, z) : 0 \le x \le a, 0 \le z \le bx\}$ around the z axis, but only a half rotation, in the region where $x \ge 0$. This same triangle can also be expressed as $\{(x, z) : 0 \le z \le ab, z/b \le x \le a\}$.

In view of (0.2) we have

$$\iiint_W f(z)rdrd\phi dz = \iint_Q \left(\int_{z/b}^a f(z)rdr\right) d\phi dz$$

where Q is the rectangle $\{(\phi, z) : -\pi/2 \le \phi \le \pi/2, 0 \le z \le ab\}$. So

$$\begin{aligned}\iint_Q \left(\int_{z/b}^a f(z)rdr\right) d\phi dz &= \iint_Q f(z) \left.\frac{r^2}{2}\right|_{z/b}^a d\phi dz = \iint_Q f(z)\left(\frac{a^2}{2} - \frac{z^2}{2b^2}\right) d\phi dz = \pi\int_0^{ab}\left(\frac{a^2}{2} - \frac{z^2}{2b^2}\right) f(z)dz \\ &= \int_0^{ab}\left(\frac{\pi a^2}{2} - \frac{\pi z^2}{2b^2}\right) f(z)dz.\end{aligned}$$

BET. The set V is very similar to the set V_1. The only difference is that y has been replaced everywhere by $y-2$. So we can obtain V from V_1 by moving every point (x, y, z) in V_1 to the point $(x, y+2, z)$. Here we have two possible natural choices of change of variable. One possibility is to use "shifted" cylindrical coordinates, (u, v, w) where $x = u\cos v$, $y = 2 + u\sin v$ and $z = w$. The Jacobian for the map from (u, v, w) to (x, y, z) is u and the integral $\iiint_V xf(z)dxdydz$ will equal the triple integral of $xf(z)u = u^2\cos v f(w)$ on the same set W

as was used in part ALEF, (except that now we are calling our variables u and v and w instead of r and ϕ and z). A second possibility is to use the simple affine change of variables $u = x, v = y - 2, w = z$ which maps V in xyz space onto the set V_1 of part ALEF, but now in uvw space. This map, and also its inverse map both have Jacobian equal to 1. So in this case we have $\iiint_V xf(z)dxdydz = \iiint_{V_1} uf(w)dudvdw$. We can if we wish change the names of the variables in this last integral and write it as $\iiint_{V_1} xf(z)dxdydz$. Then, using the same change of variables as in part ALEF, this integral in turn must equal $\iiint_W r\cos\phi \cdot f(z)rdrd\phi dz = \iint_Q \left(\int_{z/b}^a r\cos\phi \cdot f(z)rdr\right) d\phi dz$ where Q is as defined above. Except for the fact that the variables have different names, this is exactly the same integral as we get from using shifted cylindrical coordinates. Let us rewrite it as a suitable repeated integral so that we can convert it to the required form. It is equal to

$$\begin{aligned}
\iint_Q \left(\int_{z/b}^a r\cos\phi \cdot f(z)rdr\right) d\phi dz &= \iint_Q \cos\phi \cdot f(z) \left.\frac{r^3}{3}\right|_{z/b}^a d\phi dz = \iint_Q \cos\phi \cdot f(z)\left(\frac{a^3}{3} - \frac{z^3}{3b^3}\right) d\phi dz \\
&= \int_0^{ab} \left(\int_{-\pi/2}^{\pi/2} \cos\phi \cdot \left(\frac{a^3}{3} - \frac{z^3}{3b^3}\right) f(z)d\phi\right) dz \\
&= \int_{-\pi/2}^{\pi/2} \cos\phi d\phi \cdot \int_0^{ab} \left(\frac{a^3}{3} - \frac{z^3}{3b^3}\right) f(z)dz \\
&= 2\int_0^{ab} \left(\frac{a^3}{3} - \frac{z^3}{3b^3}\right) f(z)dz = \int_0^{ab} \left(\frac{2a^3}{3} - \frac{2z^3}{3b^3}\right) f(z)dz.
\end{aligned}$$

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2. In this question, the function $f(x,y)$ satisfies $\vec{\nabla} f\Big|_{(-3,6)} = a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}}$ and $\frac{\partial^3}{\partial x^3} f(-3,6) = M$ where a, b and M are constants with different values in different versions. In all versions we have $M \neq 0$.

ALEF. Suppose that $f(x,y) = ax + by + c$ for all (x,y) and for some constant c, where a and b are the same constants as in the formula for $\vec{\nabla} f\Big|_{(-3,6)}$. Then, for all (x,y) in $\mathbb{R}^2$ we have $\frac{\partial f}{\partial x} = a$ and so $\frac{\partial^2 f}{\partial x^2} = 0$ and so $\frac{\partial^3 f}{\partial x^3} = 0$. Since $M \neq 0$ this means that the answer to part ALEF has to be NO.

BET. In this calculation you have to be careful not to be confused by the fact that the variables x and y and z appear in unexpected places. Note for example that the two derivatives $\frac{\partial f}{\partial x}(x^2 - y^2, xyz)$ and $\frac{d}{dx} f(x^2 - y^2, xyz)$ have quite different meanings, even though the notation for them looks almost the same. One way to avoid or reduce confusion is to give other names to the variables in the definition of g. So, for example, it is quite correct to say that $g(u,v,w) = f(u^2 - v^2, uvw)$ for all $(u,v,w) \in \mathbb{R}^3$.

The continuity of the first derivatives of f is sufficient to ensure that f is differentiable at every point in $\mathbb{R}^2$, including the point $(-3,6)$. Because of this differentiability, we can apply the chain rule to obtain that

$$\begin{aligned}
\frac{\partial g}{\partial u}(u,v,w) &= g_1'(u,v,w) = \frac{\partial f}{\partial x}(u^2 - v^2, uvw)\frac{\partial\left(u^2 - v^2\right)}{\partial u} + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)\frac{\partial\left(uvw\right)}{\partial u} \\
&= \frac{\partial f}{\partial x}(u^2 - v^2, uvw)2u + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)vw.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial g}{\partial v}(u,v,w) &= g_2'(u,v,w) = \frac{\partial f}{\partial x}(u^2 - v^2, uvw)\frac{\partial\left(u^2 - v^2\right)}{\partial v} + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)\frac{\partial\left(uvw\right)}{\partial v} \\
&= -\frac{\partial f}{\partial x}(u^2 - v^2, uvw)2v + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)uw
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g}{\partial w}(u,v,w) &= g_3'(u,v,w) = \frac{\partial f}{\partial x}(u^2 - v^2, uvw)\frac{\partial\left(u^2 - v^2\right)}{\partial w} + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)\frac{\partial\left(uvw\right)}{\partial w} \\
&= 0 + \frac{\partial f}{\partial y}(u^2 - v^2, uvw)uv.
\end{aligned}$$

In particular, at the point $(u, v, w) = (1, 2, 3)$ we have $(u^2 - v^2, uvw) = (-3, 6)$ and so, substituting in the above formulæ gives us that

$$g_1'(1,2,3) = \frac{\partial f}{\partial x}(-3,6) \times 2 + \frac{\partial f}{\partial y}(-3,6) \times 6 = 2a + 6b$$

and

$$g_2'(1,2,3) = -\frac{\partial f}{\partial x}(-3,6) \times 4 + \frac{\partial f}{\partial y}(-3,6) \times 3 = -4a + 3b$$

and

$$g_3'(1,2,3) = \frac{\partial f}{\partial y}(-3,6) \times 2 = 2b.$$

It follows that

$$\begin{aligned} \vec{\nabla} g(1,2,3) &= g_1'(1,2,3)\widehat{\mathbf{i}} + g_2'(1,2,3)\widehat{\mathbf{j}} + g_3'(1,2,3)\widehat{\mathbf{k}} \\ &= (2a+6b)\widehat{\mathbf{i}} + (-4a+3b)\widehat{\mathbf{j}} + 2b\widehat{\mathbf{k}}. \end{aligned}$$

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3. The function $f(x, y) = xy$ has a critical point at the point $(0, 0)$ which is not an internal point of the set $D = \{(x, y) : x \geq 0, y \geq 0, x^{1/2} + 2y^{1/3} = 1\}$. The boundary of D consists of three curves. Two of them are portions of the x and y axes and f equals 0 identically on both of them.

So now we look for the extremum of f on the third boundary curve of D which is the set $C = \{(x, y) : x \geq 0, y \geq 0, x^{1/2} + 2y^{1/3} = 1\}$. Clearly C is in the first quadrant of the plane and meets the x axis at $(1, 0)$ and the y axis at $(0, 1/8)$. These are the two endpoints of C and f vanishes at both of them.

We can try using the method of Lagrange multipliers. It seems natural to choose $g(x, y) = x^{1/2} + 2y^{1/3}$. But $\frac{\partial g}{\partial x}$ does not exist at points where $x = 0$ and $\frac{\partial g}{\partial y}$ does not exist at points where $y = 0$. In fact it is not hard to show that the fact that the Lagrange method still works, because $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are both continuous at all points of C which are not end points. We will explain why this is correct, at least approximately, in a remark at the end of this solution.

The Lagrange method tells us that the extremum of f on C is attained, either (i) at one of the end points $(1, 0)$ or $(0, 1/8)$ or (ii) at a point on C where $\vec{\nabla} g = \vec{0}$ or (iii) at a point on C where, for some value of λ, we have $\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$ and $\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$. It is clear that $\vec{\nabla} g = \frac{1}{2\sqrt{x}}\widehat{\mathbf{i}} + \frac{2}{3y^{2/3}}\widehat{\mathbf{j}}$ does not equal $\vec{0}$ at any point of C. So we have to solve the system of three equations $y + \frac{\lambda}{2}x^{-1/2} = 0$, $x + \frac{2\lambda}{3}y^{-2/3} = 0$ and $x^{1/2} + 2y^{1/3} = 1$. It seems a good idea to first eliminate λ, since anyway we do not need to know its value. The first equation gives $\lambda = -2yx^{1/2}$ and the second gives $\lambda = -\frac{3xy^{2/3}}{2}$. So $2yx^{1/2} = \frac{3xy^{2/3}}{2}$ which gives $\frac{4}{3}y^{1/3} = x^{1/2}$. (We can assume that x and y are non zero since we are considering the points where they are zero, i.e. the endpoints of C separately.) Substituting in the third equation gives us that $\frac{4y^{1/3}}{3} + 2y^{1/3} = 1$. So $\frac{10y^{1/3}}{3} = 1$ which implies $2y^{1/3} = 3/5$ and so $x^{1/2} + 3/5 = 1$. We deduce that $x = (2/5)^2 = 4/25$ and $y = (3/10)^3 = 27/1000$. Since $f(4/25, 27/1000) = 27/(25 \times 250) = 27/6250$ and $f(x, y) = 0$ at all the other suspected points we have mentioned, it follows that the minimum value of f on the curve C and indeed on the set D is 0 and the maximum value is 27/6250

Here is a trick for doing the same calculations in a perhaps simpler way. Let us make a change of variables, $u = x^{1/2}$ and $v = y^{1/3}$ for all points (x, y) in the first quadrant of the plane $x \geq 0, y \geq 0$. As the point (x, y) moves along the curve C, from $(0, 1/8)$ to $(1, 0)$ the point (u, v) moves along the curve (in fact the line segment) $u + 2v = 1$ from $(0, 1/2)$ to $(1, 0)$. Let us call this line segment Γ. The values taken by the function $f(x, y) = xy$ on the curve C are exactly the values taken by the function $w(u, v) = u^2v^3$ on the line segment Γ. So we are looking for the maxium of the function w on the line segment Γ. There are (at least) three ways to do this.

(i) We can look for the extremum of the function $\phi(u) = w(u, \frac{1-u}{2}) = u^2\left(\frac{1-u}{2}\right)^3$ on the interval $[0, 1]$.

Or (ii) we can look for the extremum of the function $\eta(v) = w(1 - 2v, v) = (1 - 2v)^2v^3$ on the interval $[0, 1/2]$.

Or (iii) we can apply the Lagrange method, and look for solutions of the system of three equations $2uv^3 + \lambda = 0$, $3u^2v^2 + 2\lambda = 0$, and $u + 2v = 1$.

We will only give the details for method (iii). This time it is clear that all the relevant functions have continuous derivatives at all points of the uv plane. Also the gradient of the function $u + 2v$ is non zero at every point. We see immediately that $4uv^3 = 3u^2v^2$ so $4v = 3u$ and so $u = 4v/3$. This gives $1 = u + 2v = 4v/3 + 2v = 10v/3$. So $v = 3/10$ and $u = 2/5$. The function $w(u, v) = 0$ at the two end points of Γ and it equals $(2/5)^2(3/10)^3$ at the point $(2/5, 3/10)$ which we have just found. So we reach the same conclusion as before.

Here is yet another possible approach. Perhaps, for this particular problem, it is the easiest one. It is easy to see that the equation $x^{1/2}+2y^{1/3} = 1$ defines y as a function of x (and it also in fact defines x and a function of y). And in this case we can give an explicit formula for that function, namely $y(x) = \left(\frac{1-x^{1/2}}{2}\right)^3 = \frac{1}{8}\left(1-\sqrt{x}\right)^3$. So the curve C is simply the graph of this function on the interval $[0, 1]$. To find the maximum and minimum value of $f(x, y)$ on C is exactly the same as finding the maximum and minimum values of the function $\phi(x) = f(x, y(x)) = \frac{x}{8}\left(1-\sqrt{x}\right)^3$ on the interval $[0, 1]$. Since ϕ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ its extremum is attained either at $x = 0$ or $x = 1$ or at a point x in $(0, 1)$ where $\phi'(x) = 0$. Now $\phi'(x) = \frac{1}{8}\left(1-\sqrt{x}\right)^3 + \frac{3x}{8}\left(1-\sqrt{x}\right)^2 \cdot \frac{-1}{2\sqrt{x}}$. If $\phi'(x) = 0$ then either $x = 1$ or $1-\sqrt{x}-\frac{3}{2}\sqrt{x} = 0$. So $\sqrt{x} = 2/5$ which gives $x = 4/25$ and the rest of the argument is as in the other versions.

Remark: Some students, after finding that $(0, 0)$ *is a critical point of* $f(x, y)$ *decided to check the second derivatives of* f *at* $(0, 0)$ *and discovered, correctly, that* $(0, 0)$ *is a saddle point of* f *. But, in this kind of question, doing this check with second derivatives is irrelevant and can even be misleading. You might expect that since* $(0, 0)$ *is a saddle point then the minimum of* f *cannot be attained there. But this is wrong. Davka,* 0 *IS the minimum of* f *on* D*. There is no contradiction because we are working with respect to the set* D *which contains only some of the points on just one side of* $(0, 0)$.

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4. It might be helpful to first try to draw the set V. We start by considering the set of all points (x, y, z) which satisfy $x^2 + y^2 \leq 1$. This is an infinitely long cylinder of radius 1 whose axis is the z axis. To get V we have to consider the part of this cylinder which lies under the surface $z = h(x, y)$ and above the surface $z = g(x, y)$.

So the "floor" of V is the set $\{(x, y, z) : x^2 + y^2 \leq 1, z = g(x, y)\}$ and the "ceiling" of V is the set $\{(x, y, z) : x^2 + y^2 \leq 1, z = h(x, y)\}$. Then there is the vertical "wall" of V which is a part of the cylindrical surface defined by the condition $x^2 + y^2 = 1$. In fact it has to be the set $\{(x, y, z) : x^2 + y^2 = 1,\ g(x, y) \leq z \leq h(x, y)\}$.

ALEF. All points (x, y, z) (or position vectors) which are given by the formula $\vec{r}(\theta, z) = \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}$ satisfy $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$ so the surface S_3 cannot be the "floor" or "ceiling" of V. It has to be the "wall", i.e. $S_3 = \{(x, y, z) : x^2 + y^2 = 1,\ g(x, y) \leq z \leq h(x, y)\}$. For EVERY real value of θ the point of the form $(x, y, z) = (\cos\theta, \sin\theta, z)$ belongs to the set S_3 if and only if

$$g(\cos\theta, \sin\theta) \leq z \leq h(\cos\theta, \sin\theta) \tag{0.3}$$

If we want to reach all points of S_3 we have to allow θ to take every value on the interval $[0, 2\pi)$ or on some other half open interval of length 2π. (If we choose a longer interval, then the parametric representation will not be one to one, which may cause problems e.g. if we try to calculate the area of S_3.) So a reasonable choice for D is the set $D = \{(\theta, z) : 0 \leq \theta < 2\pi, g(\cos\theta, \sin\theta) \leq z \leq h(\cos\theta, \sin\theta)\}$. It would also be acceptable to take $0 \leq \theta \leq 2\pi$. In that case the parametric representation is not quite one to one, but the points which we count twice are on a set which is too small to cause any problems in calculation of surface integrals. It is also acceptable to replace the endpoints 0 and 2π by $-\pi$ and π respectively, or by any pair of points c and $c + 2\pi$ for some constant c.

BET. Intuitively it is clear that S_3 is made up of lines parallel to the z axis, i.e. vertical lines. So, at each point of S_3, the vector perpendicular ("nitsav") to S_3 has to point in some horizontal direction, i.e. its $\widehat{\mathbf{k}}$ component is 0. To show this more precisely we use the given parametric representation $\vec{r}(\theta, z) = \cos\theta\widehat{\mathbf{i}}+\sin\theta\widehat{\mathbf{j}}+ z\widehat{\mathbf{k}}$ for S_3. It satisfies $\frac{\partial\vec{r}}{\partial\theta} = -\sin\theta\widehat{\mathbf{i}}+\cos\theta\widehat{\mathbf{j}}$ and $\frac{\partial\vec{r}}{\partial z} = \widehat{\mathbf{k}}$. So $\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial z} = -\sin\theta(\widehat{\mathbf{i}}\times\widehat{\mathbf{k}})+\cos\theta(\widehat{\mathbf{j}}\times\widehat{\mathbf{k}}) = \cos\theta\widehat{\mathbf{i}}+\sin\theta\widehat{\mathbf{j}}$.

So, the formula for calculating $\iint_{S_3} \vec{F} \cdot d\vec{S}$ is

$$\iint_D \vec{F}(\cos\theta, \sin\theta, z) \cdot \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial z} d\theta dz = \iint_D f(\cos\theta, \sin\theta, z)\widehat{\mathbf{k}} \cdot \left(\cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}\right) d\theta dz.$$

Since the integrand $f(\cos\theta, \sin\theta, z)\widehat{\mathbf{k}} \cdot \left(\cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}\right)$ equals 0 at every point $(\theta, z) \in D$, the integral must also equal 0.

GIMEL. The proof which you are asked to give in this section is the first step in one of the standard proofs of the general form of the Gauss' divergence theorem. Well in fact it is a special case, where we have an explicit formula for the "wall" of the set V. We have to show that

$$\iiint_V \vec{\nabla} \cdot \vec{F} dxdydz = \iint_S \vec{F} \cdot d\vec{S} \tag{0.4}$$

where S is the surface which is the boundary of V with normal vector pointing outwards from V.

In our case we have $\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}f(x,y,z) = \frac{\partial}{\partial z}f(x,y,z)$.

Because V is a "z-simple" set we can use the usual formula for calculating triple integrals on such sets and obtain that

$$\iiint_V \vec{\nabla} \cdot \vec{F} dxdydz = \iiint_V \frac{\partial}{\partial z} f(x,y,z) dxdydz = \iint_\Omega \left(\int_{z=g(x,y)}^{h(x,y)} \frac{\partial}{\partial z} f(x,y,z) dz \right) dx \tag{0.5}$$

where $\Omega = \{(x,y) : x^2 + y^2 \leq 1\}$. By the Newton-Leibniz formula we have $\int_a^b u'(z)dz = u(b) - u(a)$ whenever u has a continuous derivative u' on some open interval containing $[a,b]$. For each constant x and y we can choose $u(z) = f(x,y,z)$, $a = g(x,y)$ and $b = h(x,y)$. Then $u'(z) = \frac{\partial}{\partial z}f(x,y,z)$ and the Newton-Leibniz formula gives us that $\int_{z=g(x,y)}^{h(x,y)} \frac{\partial}{\partial z} f(x,y,z)dz = f(x,y,h(x,y)) - f(x,y,g(x,y))$. So (0.5) gives us

$$\iiint_V \vec{\nabla} \cdot \vec{F} dxdydz = \iint_\Omega \left(f(x,y,h(x,y)) - f(x,y,g(x,y))\right) dxdy.$$

Now we have to show that the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ equals this same expression. We have already observed that S consists of three sets S_1, S_2 and S_3, and so $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S}$. We have already shown that $\iint_{S_3} \vec{F} \cdot d\vec{S} = 0$ for the "vertical wall" S_3. So it remains to calculate $\iint_{S_1} \vec{F} \cdot d\vec{S}$ and $\iint_{S_2} \vec{F} \cdot d\vec{S}$. Without loss of generality, we can suppose that S_1 is the "floor" $\{(x,y,z) : x^2 + y^2 \leq 1, z = g(x,y)\}$ and S_2 is the "ceiling" $\{(x,y,z) : x^2 + y^2 \leq 1, z = h(x,y)\}$. A natural parametric representation fo S_2 is thus

$$\vec{r}(x,y) = x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + h(x,y)\widehat{\mathbf{k}}, (x,y) \in \Omega$$

where, as before, Ω is the unit disk in the xy plane. For this representation we have $\frac{\partial \vec{r}}{\partial x} = \widehat{\mathbf{i}} + \frac{\partial h}{\partial x}\widehat{\mathbf{k}}$ and $\frac{\partial \vec{r}}{\partial y} = \widehat{\mathbf{j}} + \frac{\partial h}{\partial y}\widehat{\mathbf{k}}$ and so $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$ is unofficially equal to the "determinant" $\begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ 1 & 0 & \frac{\partial h}{\partial x} \\ 0 & 1 & \frac{\partial h}{\partial y} \end{vmatrix}$ and so, more officially, we have $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = -\frac{\partial h}{\partial x}\widehat{\mathbf{i}} - \frac{\partial h}{\partial y}\widehat{\mathbf{j}} + \widehat{\mathbf{k}}$. Since this vector has positive $\widehat{\mathbf{k}}$ component, it is pointing upwards, and so out of V as required. Using the formula for calculating surface integrals we obtain that

$$\begin{aligned} \iint_{S_2} \vec{F} \cdot d\vec{S} &= \iint_\Omega \vec{F}(x,y,h(x,y)) \cdot \left(-\frac{\partial h}{\partial x}\widehat{\mathbf{i}} - \frac{\partial h}{\partial y}\widehat{\mathbf{j}} + \widehat{\mathbf{k}} \right) dxdy \\ &= \iint_\Omega f(x,y,h(x,y))\widehat{\mathbf{k}} \cdot \left(-\frac{\partial h}{\partial x}\widehat{\mathbf{i}} - \frac{\partial h}{\partial y}\widehat{\mathbf{j}} + \widehat{\mathbf{k}} \right) dxdy \\ & \iint_\Omega f(x,y,h(x,y)) dxdy. \end{aligned}$$

The calculation of $\iint_{S_1} \vec{F} \cdot d\vec{S}$ is exactly analogous, with the function $g(x,y)$ replacing $h(x,y)$ in the parametric representation and in the resulting integral. However since S_1 is the "floor" of V the normal vector on S_1 must point downwards, i.e. its $\widehat{\mathbf{k}}$ component must be negative. Since the vector $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = -\frac{\partial g}{\partial x}\widehat{\mathbf{i}} - \frac{\partial g}{\partial y}\widehat{\mathbf{j}} + \widehat{\mathbf{k}}$ points upwards we have to multiply it by -1 (or simply choose a different order of variables and consider the vector $\frac{\partial \vec{r}}{\partial y} \times \frac{\partial \vec{r}}{\partial x}$ instead of $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$). It follows that $\iint_{S_1} \vec{F} \cdot d\vec{S} = - \iint_\Omega f(x,y,g(x,y))dxdy$. Combining the preceding two calculations we see that

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}\\ &= -\iint_\Omega f(x,y,g(x,y))dxdy + \iint_\Omega f(x,y,h(x,y))dxdy\\ &= \iint_\Omega f(x,y,h(x,y)) - f(x,y,g(x,y))dxdy\end{aligned}$$

and we have already shown that this last integral equals $\iiint_V \vec{\nabla}\cdot\vec{F}dxdydz$. This proves the formula (0.4) in this case and completes our proof.

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5. ALEF. Here is the version of the implicit function theorem which is relevant for the equation $F(x,y,z) = 0$.

Theorem. *Suppose that the function F of three variables is defined in some neighbourhood N of the point (x_0,y_0,z_0) and all its partial derivatives of first order exist and are continuous in this neighbourhood. Suppose further that $\frac{\partial F}{\partial z}(x_0,y_0,z_0) \neq 0$.*

Then there exists a function of two variables $Z(x,y)$ defined in some (two dimensional) neighbourhood M of the point (x_0,y_0) which has continuous partial derivatives in that neighbourhood, and there exists another (three dimensional) neighbourhood N_1 of (x_0,y_0,z_0) (possibly smaller than N) such that, for every point (x,y,z) in N_1, we have $(x,y)\in M$ and

$$F(x,y,z) = 0 \text{ if and only if } z = Z(x,y).$$

In particular, $Z(x_0,y_0) = z_0$.

The partial derivatives of Z at (x_0,y_0) are given by the formulæ

$$\frac{\partial Z}{\partial x}(x_0,y_0) = -\frac{\frac{\partial F}{\partial x}(x_0,y_0,z_0)}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)} \text{ and } \frac{\partial Z}{\partial y}(x_0,y_0) = -\frac{\frac{\partial F}{\partial y}(x_0,y_0,z_0)}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)} . \qquad (0.6)$$

Of course their are variants of this theorem for the cases where $\frac{\partial F}{\partial x}(x_0,y_0,z_0) \neq 0$ or $\frac{\partial F}{\partial y}(x_0,y_0,z_0) \neq 0$ and the equation $F(x,y,z)$ defines x as a function of y and z, or y and a function of x and z. These would also be acceptable answers for part ALEF, but they would be less relevant for the next part of the question.

BET. We choose $F(x,y,z)$ to be the function $F(x,y,z) = z^3+xy+xz+yz^2-4$. Then we have $F(1,1,1) = 4-4 = 0$. We also have $\frac{\partial F}{\partial x} = y+z$, $\frac{\partial F}{\partial y} = x+z^2$ and $\frac{\partial F}{\partial z} = 3z^2+x+2yz$. These are all polynomials and therefore they are continuous functions at all points of $\mathbb{R}^3$. In particular $\frac{\partial F}{\partial z}(1,1,1) = 3+1+2 = 6 \neq 0$. So all the conditions of the theorem stated above are fulfilled, and the equation $z^3+xy+xz+yz^2 = 4$, which is equivalent to the equation $F(x,y,z) = 0$, defines a function $Z(x,y)$ in some neighbourhood of $(1,1)$ with all the properties stated in the theorem. In particular $Z(1,1) = 1$. Furthermore, by substituting $x_0 = y_0 = z_0 = 1$ in the equations (0.6) we obtain that $\frac{\partial Z}{\partial x}(1,1) = -\frac{1+1}{6} = -1/3$ and also $\frac{\partial Z}{\partial y}(1,1) = -\frac{1+1}{6} = -1/3$.

In principle it could happen that the same equation $z^3+xy+xz+yz^2-4 = 0$ also defines another different function $Z(x,y)$ in the neighbourhood of $(1,1)$. This would be possible if there are other real numbers z such that $F(1,1,z) = 0$ and $\frac{\partial F}{\partial z}(1,1,z) \neq 0$. We did not expect you to check this possibility in the exam, but in fact it is not very hard to see that there are no such numbers. Let us see this now: We are looking at the equation $F(1,1,z) = 0$, which is exactly $z^3+z^2+z-3 = 0$. Obviously $z = 1$ is one solution, so the polynomial z^3+z^2+z-3 equals $(z-1)P(z)$ for some polynomial of degree 2. This means that $z^3+z^2+z-3 = (z-1)(az^2+bz+c)$ for some constants a, b and c. So $z^3+z^2+z-3 = az^3+bz^2+cz-az^2-bz-c$. It follows that $a = 1$, $b = 2$ and $c = 3$. Any $z \neq 1$ such that $F(1,1,z) = 0$ must satisfy $z^2+2z+3 = 0$. But this equation has no real solutions, only complex ones.

GIMEL. There are two natural ways to find the tangent plane to S at $(1,1,1)$.

(i) We have seen that the part of S which is contained in a small neighbourhood of $(1,1,1)$ is the graph of a function $Z(x,y)$ which is differentiable (because its derivatives are continuous, by the implicit function theorem).

So we can use the usual formula for the tangent plane of such a function, namely

$$z = Z(1,1) + \frac{\partial Z}{\partial x}(1,1)(x-1) + \frac{\partial Z}{\partial y}(1,1)(y-1).$$

Since $Z(1,1) = 1$ (by the implicit function theorem) we substitute from the calculations in BET to obtain the equation $z = 1 - (x-1)/3 - (y-1)/3$. This can be rewritten as $x + y + 3z = 5$.

(ii) Alternatively we can use the fact that if F is a function of three variables with continuous partial derivatives and $F(x_0, y_0, z_0) = c$ and $\vec{\nabla} F(x_0, y_0, z_0) \neq \vec{0}$, then the part of the set $S = \{(x,y,z) : F(x,y,z) = c\}$ in some small neighbourhood of (x_0, y_0, z_0) is a surface (graph of a function of two variables, which may be x, y, or y, z or z, x) and at (x_0, y_0, z_0) the tangent plane to this surface is perpendicular to $\vec{\nabla} F(x_0, y_0, z_0)$. (In fact the precise proof of this uses the implicit function theorem in a version identical or quite similar to the one mentioned above.)

Applying this in our case, we obtain that $\vec{\nabla} F(1,1,1) = 2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 6\widehat{\mathbf{k}}$. So our tangent plane is the plane which contains the point $(1,1,1)$ and is perpendicular to $2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 6\widehat{\mathbf{k}}$. This has the equation $2(x-1) + 2(y-1) + 6(z-1) = 0$ which is the same as $2x + 2y + 6z = 10$. If we divide by 2 this becomes the same equation as we found by the previous method.

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6. To help you understand the solution of this question, please look at the picture at http://www.math.technion.ac.il/~mcwikel/h2m/pict206.jpg

In general, the intersection of a plane with the surface of a sphere is always a circle. If the plane passes through the centre of the sphere (as happens in this question) then of course the radius of the circle is the same as the radius of the sphere.

The circle L (shown in black in the picture) is the intersection of the plane $z = 4x/3$ with the spherical surface $x^2 + y^2 + z^2 = a^2$ of radius a, where the positive number a has different values in different versions of the examination. The fact that the direction of integration along L has positive $\widehat{\mathbf{k}}$ component at the point $(0, a, 0)$ means that the direction of integration is as shown by the red arrow, i.e. it is anticlockwise if you look at L from the side of the POSITIVE x axis.

The easiest way to solve this problem seems to be via Stokes' theorem. The circle L is the edge of the disk S which is shown in yellow in the picture. The direction of integration along L as shown by the red arrow, corresponds to choosing $\vec{N}$ the normal vector to S to point in the direction as shown in blue in the picture. This means that the $\widehat{\mathbf{k}}$ component of $\vec{N}$ has to be negative. Since S is contained in the plane $z = 4x/3$ or $4x - 3z = 0$, the vector $\vec{N}$ must be parallel to $4\widehat{\mathbf{i}} - 3\widehat{\mathbf{k}}$, and in fact it must also have the same sense (or "megama") as $4\widehat{\mathbf{i}} - 3\widehat{\mathbf{k}}$ (since -3 is a negative number!!).

The curl of the vector field $(z-y)\widehat{\mathbf{i}} + (x-z)\widehat{\mathbf{j}} + (y-x)\widehat{\mathbf{k}}$ is the constant vector field $2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}$.

So, by Stokes' theorem the line integral that we were asked to calculate equals the surface integral $\iint_S (2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}) \cdot d\vec{S}$.

There are various different ways to find a parametric representation $\{\vec{r}(u,v), (u,v) \in D\}$ for S. But we can also do this calculation without even writing down any explicit representation. No matter how we choose the representation $\vec{r}(u,v)$, we know that $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ must have the same direction and sense as the vector $4\widehat{\mathbf{i}} - 3\widehat{\mathbf{k}}$. So we have $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| \frac{4\widehat{\mathbf{i}} - 3\widehat{\mathbf{k}}}{\|4\widehat{\mathbf{i}} - 3\widehat{\mathbf{k}}\|} = \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| \left(\frac{4}{5}\widehat{\mathbf{i}} - \frac{3}{5}\widehat{\mathbf{k}}\right)$. When we substitute this in the formula for calculating surface integrals we obtain

$$\begin{aligned}
\iint_S (2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}) \cdot d\vec{S} &= \iint_D (2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}) \cdot \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| \left(\frac{4}{5}\widehat{\mathbf{i}} - \frac{3}{5}\widehat{\mathbf{k}}\right) dudv \\
&= \iint_D \left(\frac{8}{5} - \frac{6}{5}\right) \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| dudv \\
&= \frac{2}{5} \iint_D \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| dudv.
\end{aligned}$$

Now, no matter how we choose $\{\vec{r}(u,v),\ (u,v)\in D\}$, this last integral $\iint_D \left\|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\| dudv$ gives the area of S. Since S is a disk of radius a its area is πa^2. It follows that $\iint_S(2\widehat{\mathbf{i}}+2\widehat{\mathbf{j}}+2\widehat{\mathbf{k}})\cdot d\vec{S}=\frac{2\pi a^2}{5}$, which is also the value of the line integral that we had to calculate.

If you want an explicit parametric representation for S, here are two natural ones to choose.

(i) $\vec{r}(x,y)=x\widehat{\mathbf{i}}+y\widehat{\mathbf{j}}+\frac{4z}{3}\widehat{\mathbf{k}}$, $(x,y)\in D$ where D is the projection of S onto the xy plane. From the picture it is not hard to see that D must be an ellipse whose semi axes have lengths a in the y direction and $3a/5$ in the x direction. So $D=\left\{(x,y):\frac{25x^2}{9a^2}+\frac{y^2}{a^2}\le 1\right\}$.

(ii) Another way to get a parametric representation for the disk S is to first look for two vectors $\vec{A}$ and $\vec{B}$ which lie in S and have length a (the radius of S) and which are perpendicular to each other. Again, from the picture, we can see that the two vectors $\vec{A}=a\widehat{\mathbf{j}}$ and $\vec{B}=\frac{3a}{5}\widehat{\mathbf{i}}+\frac{4a}{5}\widehat{\mathbf{k}}$ have all these properties. Since the centre of S is at $(0,0,0)$, we can get a parametric representation for S by taking $\vec{r}(\rho,\phi)=\rho\cos\phi\vec{A}+\rho\sin\phi\vec{B}$ with $(\rho,\phi)\in D$, where this time $D=\{(\rho,\phi):0\le\rho\le 1, 0\le\phi\le 2\pi\}$.

The same vectors $\vec{A}$ and $\vec{B}$ can also be used to get a parametric representation for the curve L. We take $\vec{r}(\phi)=\cos\phi\vec{A}+\sin\phi\vec{B}$, where $0\le\phi\le 2\pi$, and this representation has the direction of the red arrow.

So we can also calculate the line integral directly:

But first we have to express $\vec{r}(\phi)$ in terms of its $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ and $\widehat{\mathbf{k}}$ components.

$\vec{r}(\phi)=\cos\phi\vec{A}+\sin\phi\vec{B}=\frac{3a}{5}\sin\phi\widehat{\mathbf{i}}+a\cos\phi\widehat{\mathbf{j}}+\frac{4a}{5}\sin\phi\widehat{\mathbf{k}}$.

We also need the derivative, $\vec{r}'(\phi)=\frac{3a}{5}\cos\phi\widehat{\mathbf{i}}-a\sin\phi\widehat{\mathbf{j}}+\frac{4a}{5}\cos\phi\widehat{\mathbf{k}}$. Then

$$\int_L\left((z-y)\widehat{\mathbf{i}}+(x-z)\widehat{\mathbf{j}}+(y-x)\widehat{\mathbf{k}}\right)\cdot d\vec{r}$$
$$=\int_0^{2\pi}\left((\frac{4a}{5}\sin\phi-a\cos\phi)\widehat{\mathbf{i}}+(\frac{3a}{5}\sin\phi-\frac{4a}{5}\sin\phi)\widehat{\mathbf{j}}+(a\cos\phi-\frac{3a}{5}\sin\phi)\widehat{\mathbf{k}}\right)\cdot\vec{r}'(\phi)d\phi$$

The integrand here equals

$$\left((\frac{4a}{5}\sin\phi-a\cos\phi)\widehat{\mathbf{i}}-\frac{a}{5}\sin\phi\widehat{\mathbf{j}}+(a\cos\phi-\frac{3a}{5}\sin\phi)\widehat{\mathbf{k}}\right)\cdot\left(\frac{3a}{5}\cos\phi\widehat{\mathbf{i}}-a\sin\phi\widehat{\mathbf{j}}+\frac{4a}{5}\cos\phi\widehat{\mathbf{k}}\right)$$
$$=-\frac{3a^2}{5}\cos^2\phi+\frac{a^2}{5}\sin^2\phi+\frac{4a^2}{5}\cos^2\phi+C\cos\phi\sin\phi$$
$$=\frac{a^2}{5}+C\cos\phi\sin\phi$$

where C is some constant whose exact value is irrelevant for us, because $\int_0^{2\pi}\cos\phi\sin\phi d\phi=0$. When we integrate from 0 to 2π we obtain the same answer $\frac{2\pi a^2}{5}$ as we obtained before.

If you want to see a solution of this problem in Hebrew, then I mention that the case $a=10$ can be found in part III of the hoveret for Matematika 2Khet. It was in an exam given on 21/9/1983. It is exactly Question 7 on page 388 (number in a circle at the top of the page (or 220HEY at the bottom of the page.). The solution of this question appears a few pages later, on page 393 of the same hoveret. As explained there, depending on which direction you choose on L, the integral equals 40π or -40π. The same calculation shows that it equals $\pm 2\pi a^2/5$ for any choice of the positive number a. In fact we also asked a very similar question on 4/2/1981. See Question 3 on page 287 and its solution on page 291 of the same hoveret.

Sorry if you are confused by the various page numbering systems for the hoveret. Let me try to help you. To get to the pages mentioned above, open the pdf file from the address

http://www.math.technion.ac.il/courses/Math2H/Hoveret3b.pdf

using Acrobat reader. Then press Shift+Ctrl+N and you will be asked for a number. Type the number 121 to get to 388, or 126 to get to 393. (Similarly, type 20 to get 287, or 24 to get 291.)

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7. We had a special reason for introducing this question. Some of the teachers in this course have had contact with some engineers working in industry, and have observed that when they need to use various forms of Taylor's theorem from time to time, they sometimes have a very poor understanding of what this theorem means, and are unable to make reliable estimates of how much error is introduced when it is used.

ALEF. For $g(x,y)=\sqrt{x^2+y^2}$ we have $g'_x(x,y)=x/\sqrt{x^2+y^2}$, $g'_y(x,y)=y/\sqrt{x^2+y^2}$ and so $g''_{xx}(x,y)=\frac{\sqrt{x^2+y^2}-x^2/\sqrt{x^2+y^2}}{x^2+y^2}=\frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}}=\frac{y^2}{(x^2+y^2)^{3/2}}$. Similarly $g''_{yy}(x,y)=\frac{\sqrt{x^2+y^2}-y^2/\sqrt{x^2+y^2}}{x^2+y^2}=\frac{x^2}{(x^2+y^2)^{3/2}}$. Then we also have $g''_{xy}(x,y)=\frac{-xy}{(x^2+y^2)^{3/2}}=g''_{yx}(x,y)$.

BET. In this section the point (x,y) is always in S_1. So we have $x^2\le 9$ and $y^2\le 16$ which implies that $\sqrt{x^2+y^2}\le\sqrt{9+16}=5$. We also have $x^2\ge 7$ and $y^2\ge 9$ so $\sqrt{x^2+y^2}\ge\sqrt{7+9}=4$. So we have shown that $4\le g(x,y)\le 5$ as required. It follows that $0<\frac{1}{\sqrt{x^2+y^2}}\le\frac{1}{4}$. We also have $y^2\le 16$ and $x^2\le 9$ and $xy\le 12$, i.e. $-xy\ge -12$.

So, using the formulæ obtained for the derivatives of g we see that $0\le g_{xx}(x,y)\le\frac{16}{4^3}=\frac{1}{4}$. Similar calculations show that $0\le g_{yy}(x,y)\le\frac{9}{4^3}=\frac{9}{64}$. Finally, using the above estimate for $-xy$, we have $-\frac{12}{4^3}=-\frac{3}{16}\le g''_{xy}(x,y)\le 0$.

Many of you used a wrong argument here. You assumed, for example, that the maximum value of $\frac{x^2}{(x^2+y^2)^{3/2}}$ *on* S_1 *is equal to (or perhaps less than) the maximum value of* x^2 *on* S_1 *divided by the maximum value of* $(x^2+y^2)^{3/2}$ *on* S_1*. The correct version of this is that maximum value of* $\frac{x^2}{(x^2+y^2)^{3/2}}$ *on* S_1 *is less than or equal to the maximum value of* x^2 *on* S_1 *divided by the* ***MINIMUM*** *value of* $(x^2+y^2)^{3/2}$ *on* S_1*.*

In general if $0\le a\le A$ *and* $0<b\le B$ *it does NOT follow that* $a/b\le A/B$*. Instead we have* $a/B\le A/b$*.*

GIMEL. According to Taylor's theorem, we have

$$f(x_0+h,y_0+k)=f(x_0,y_0)+f'_x(x_0,y_0)h+f'_y(x_0,y_0)k+\frac{1}{2}\left(f''_{xx}(Q)h^2+2f''_{xy}(Q)hk+f''_{yy}(Q)k\right) \tag{0.7}$$

where $Q=(x_0+\theta h,y_0+\theta k)$ for some $\theta\in(0,1)$. So $a_1=f(x_0,y_0)$, $a_2=f'_x(x_0,y_0)$, $a_3=f'_y(x_0,y_0)$, $b_1=\frac{1}{2}f''_{xx}(Q)$, $b_2=f''_{xy}(Q)$ and $b_3=\frac{1}{2}f''_{yy}(Q)$.

Many of you did not bother to say at which point we have to take the various derivatives of f *in these formulæ. Not specifying which point makes your answer meaningless. Many of you thought that in the formulæ for* b_1*,* b_2*,* b_3*, the derivatives are calculated at the point* (x_0,y_0) *which is of course wrong in general.*

Remark. *You were not explicitly asked to say which conditions* f *has to satisfy for the formula (0.7) to be true. It is sufficient to require that all the derivatives of* f *of order* 1 *and* 2 *exist and are continuous in an open set containing the line segment from* (x_0,y_0) *to* (x_0+h,y_0+k)*. But more usually we require that open set to be a disk or square centred at* (x_0,y_0) *so we can obtain the formula (0.7) for many different values of* h *and* k *(depending on the size of the disk or square.)*

DALET. Now we have set $f(x,y)=g(x,y)$. So $B=f(3-0.1,4-0.1)$. In this case Taylor's formula, i.e. the formula (0.7) becomes

$$f(3-0.1,4-0.1)=f(3,4)-0.1f'_x(3,4)-0.1f'_y(3,4)+\frac{1}{2}\left(f''_{xx}(Q)(0.1)^2+2f''_{xy}(Q)(0.1)^2+f''_{yy}(Q)(0.1)^2\right) \tag{0.8}$$

where Q is some point on the line between $(3,4)$ and $(2.9,3.9)$. From ALEF we have $f'_x(3,4)=3/5$ and $f'_y(3,4)=4/5$. So (0.8) gives us that

$$B=5-\frac{3/5}{10}-\frac{4/5}{10}+\frac{1}{2}\left(f''_{xx}(Q)(0.1)^2+2f''_{xy}(Q)(0.1)^2+f''_{yy}(Q)(0.1)^2\right). \tag{0.9}$$

We note that $5-\frac{3/5}{10}-\frac{4/5}{10}=5-\frac{14}{100}=4.86$. We also have

$$\begin{aligned}&\frac{1}{2}\left(f''_{xx}(Q)(0.1)^2+2f''_{xy}(Q)(0.1)^2+f''_{yy}(Q)(0.1)^2\right)\\ =\ &\frac{0.01}{2}\left(f''_{xx}(Q)+2f''_{xy}(Q)+f''_{yy}(Q)\right)\\ =\ &0.005\left(f''_{xx}(Q)+2f''_{xy}(Q)+f''_{yy}(Q)\right).\end{aligned}$$

The point Q is somewhere on the line segment from $(3,4)$ to $(2.9,3.9)$. Since $\sqrt{7}<2.7<2.9$ it is clear that Q lies in the rectangle S_1. So the expression $f''_{xx}(Q)+2f''_{xy}(Q)+f''_{yy}(Q)$ is bounded above by $\frac{1}{4}+0+\frac{9}{64}=\frac{25}{64}$

and it is bounded below by $0 - 2 \times \frac{3}{16} + 0 = -\frac{3}{8}$. Combining these estimates, we see that

$$4.86 - \frac{3}{8} \times 0.005 \leq B \leq 4.86 + \frac{25}{64} \times 0.005$$

as required.

In this last part many of you did not bother to check that the point Q is in S_1. If Q is not in S_1 there is no guarantee that you can use the estimates from part BET. Many of you did not seem to understand the difference between writing an exact formula for B and writing an approximate formula for B. Some of you simply assumed that, instead of (0.9) you could write that B is EXACTLY equal to $5 - \frac{3/5}{10} - \frac{4/5}{10} + \frac{1}{2}\left(f''_{xx}(3,4)(0.1)^2 + 2f''_{xy}(3,4)(0.1)^2 + f''_{yy}(3,4)(0.1)^2\right)$. This is not true. Taylor's theorem is a tool for calculating approximate values of expressions which are difficult to calculate exactly. Even when Taylor's theorem is used carefully it usually does NOT give you an exact value. If your goal is to find an approximate value of an expression and you know in advance how much error you can tolerate in your calculation, Taylor's theorem often, but not always, tells you how, by using sufficiently many terms, to get an approximate value with an error no bigger than the amount you can tolerate. You cannot hope to know the error exactly. That would be equivalent to knowing the value of the expression exactly.

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