HEDVA 2M
SOLUTION TO THE MID TERM TEST OF 10/1/2007.

Version 4. 17/9/2007

(No one is infallible. There is always a small possibility of a mistake here.

If there *is* a mistake in this solution, then the first student to send me an e-mail reporting it will get a (modest) tsiyun magen.)

We may perhaps write a new version of this solution later, even if there are no mistakes, if we see after reading your solutions that some things need to be discussed in more detail.

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Please note that this solution is written in a lot of detail, with extra discussion of some points. It also sometimes describes more than one way of solving some of the problems.

So it is considerably longer and sometimes more elaborate than the solutions which we would expect students to write during a test.

It is designed to also help students in future semesters prepare for their tests.

If it bothers you that it is written in English, please stop and ask yourself, how are you going to manage in your future professional career if you cannot read technical explanations in English? Please make the effort NOW to get used to reading technical English. Later it will be harder for you.

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1. In this question the function $f(x,y)$ is of the form $f(x,y) = \alpha x^3 + \beta xy + \gamma y^3$ where α, β and γ are constants which take different values in different versions.

ALEF. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function with continuous partial derivatives at every point. Suppose that at the point (x_0, y_0) we have $\vec{\nabla} f(x_0,y_0) \neq \vec{0}$ and we set $c_0 = f(x_0,y_0)$. Then we know (for example via the implicit function theorem) that for some positive (maybe very small) number r the part of the set $E = \left\{(x,y) \in \mathbb{R}^2 : f(x,y) = c_0\right\}$ inside the disk $\left\{(x,y) : (x-x_0)^2 + (y-y_0)^2 < r^2\right\}$ is part of a curve (given either by y as a function of x or x as a function of y) and at (x_0,y_0) the tangent to this curve is perpendicular to the vector $\vec{\nabla} f(x_0,y_0)$.

In our particular case, where $f(x,y) = \alpha x^3 + \beta xy + \gamma y^3 + d$ and $(x_0,y_0) = (-1,2)$ all these conditions hold, and we have $\vec{\nabla} f(x,y) = \left(3\alpha x^2 + \beta y\right)\widehat{\mathbf{i}} + (\beta x + 3\gamma y^2)\widehat{\mathbf{j}}$ for all $(x,y) \in \mathbb{R}^2$. We deduce that, at the point $(-1,2)$ the vector $\vec{\nabla} f(-1,2) = (3\alpha + 2\beta)\widehat{\mathbf{i}} + (12\gamma - \beta)\widehat{\mathbf{j}}$ is perpendicular to the level curve of f which passes through $(-1,2)$.

BET. Since f is differentiable at $(-1,2)$ its directional derivative at $(-1,2)$ in the direction of any unit vector $\widehat{\nu} = \nu_1\widehat{\mathbf{i}} + \nu_2\widehat{\mathbf{j}}$ is given by the formula $D_{\widehat{\nu}} f(-1,2) = \vec{\nabla} f(-1,2) \cdot \widehat{\nu}$. Since $\vec{\nabla} f(-1,2) \cdot \widehat{\nu} = \left\|\vec{\nabla} f(-1,2)\right\| \|\widehat{\nu}\| \cos\theta = \left\|\vec{\nabla} f(-1,2)\right\| \cos\theta$ where θ is the angle between $\widehat{\nu}$ and $\vec{\nabla} f(-1,2)$, we see that the biggest possible value of this vector is obtained when we choose $\widehat{\nu} = \frac{1}{\|\vec{\nabla} f(-1,2)\|}\vec{\nabla} f(-1,2)$ so that $\theta = 0$. So that biggest possible value equals $\vec{\nabla} f(-1,2) \cdot \frac{1}{\|\vec{\nabla} f(-1,2)\|}\vec{\nabla} f(-1,2) = \frac{\|\vec{\nabla} f(-1,2)\|^2}{\|\vec{\nabla} f(-1,2)\|} = \left\|\vec{\nabla} f(-1,2)\right\| = \sqrt{(3\alpha + 2\beta)^2 + (12\gamma - \beta)^2}$.

GIMEL. Let us first decide what will be the horizontal component (harekhiv ha-ofki) of the initial motion of the ball. Of course the ball will start moving in the direction where the slope downwards on the surface $z = f(x,y)$ is greatest. This is the direction, defined by a two dimensional unit vector $\widehat{\nu} = \nu_1\widehat{\mathbf{i}} + \nu_2\widehat{\mathbf{j}}$, for which the directional derivative $D_{\widehat{\nu}} f(-1,2)$ is "smallest", i.e. most negative. By almost the same arguments as in part BET, we see that the unit vector $\widehat{\nu}$ for which $D_{\widehat{\nu}} f(-1,2)$ is most negative is the vector which makes the angle θ equal to π. This is the vector $\widehat{\nu} = -\frac{1}{\|\vec{\nabla} f(-1,2)\|}\vec{\nabla} f(-1,2)$.

So the HORIZONTAL COMPONENT of the direction of the ball will be in the direction of the vector

$$-\frac{1}{\left\|\vec{\nabla}f(-1,2)\right\|}\vec{\nabla}f(-1,2)=\frac{-(3\alpha+2\beta)\widehat{\mathbf{i}}-(12\gamma-\beta)\widehat{\mathbf{j}}}{\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}}. \tag{1}$$

This is the initial direction that you would see if you plotted the motion of the ball on a map (i.e. in the xy plane). But of course, as it moves horizontally the ball also moves downwards. Similarly to what we discovered in part BET, we see that the slope ("shipua") of the path of the ball at the beginning of its motion must equal $-\left\|\vec{\nabla}f(-1,2)\right\|$. The answer to part GIMEL is a three dimensional vector $\vec{b}$ whose horizontal component has the same direction and sense as the vector in (1) but has a slope of $-\left\|\vec{\nabla}f(-1,2)\right\| = -\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}$ with respect to the xy plane. Since the horizontal vector in (1) has length 1, we can get $\vec{b}$ from it simply by adding the vector $-\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}\widehat{\mathbf{k}}$. So one correct answer to part GIMEL would be

$$\vec{b}=\frac{-(3\alpha+2\beta)\widehat{\mathbf{i}}-(12\gamma-\beta)\widehat{\mathbf{j}}}{\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}}-\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}\widehat{\mathbf{k}}.$$

Since we only care about the direction and sense of $\vec{b}$ and not about its length, we can multiply this vector by the positive scalar $\sqrt{(3\alpha+2\beta)^2+(12\gamma-\beta)^2}$ to get another correct answer which looks simpler, namely

$$\vec{b}=-(3\alpha+2\beta)\widehat{\mathbf{i}}-(12\gamma-\beta)\widehat{\mathbf{j}}-\left((3\alpha+2\beta)^2+(12\gamma-\beta)^2\right)\widehat{\mathbf{k}}.$$

Of course you have to substitute the values of α, β and γ which appeared in your version of the question to get the correct numerical answer to your version.

REMARK: Some students have a misunderstanding about these kinds of problems. They consider the function $g(x,y,z)=f(x,y)-z$, and presume that the little ball will start moving in the direction of $\vec{\nabla}g$ at the point $p_0=(-1,2,f(-1,2))$. But in fact this vector $\vec{\nabla}g(p_0)$ is perpendicular to the surface $g(x,y,z)=0$ at p_0. This surface is of course simply the surface $z=f(x,y)$, i.e. the graph of f. This means that the ball will move initially in a direction which is, davka, **perpendicular** to $\vec{\nabla}g(p_0)$. If you wrote the vector $\vec{\nabla}g(p_0)$, i.e., the vector $(3\alpha+2\beta)\widehat{\mathbf{i}}+(12\gamma-\beta)\widehat{\mathbf{j}}-\widehat{\mathbf{k}}$, or the opposite vector $-(3\alpha+2\beta)\widehat{\mathbf{i}}-(12\gamma-\beta)\widehat{\mathbf{j}}+\widehat{\mathbf{k}}$ as your answer to part GIMEL, then I am afraid that we will have to give you almost no points for this incorrect answer, even though the $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ components have the correct ratio between them, and so you may think that your answer is at least partly right. No, sorry, such an answer is based on a complete misunderstanding.

So, what about looking for a vector that is perpendicular to $\vec{\nabla}g(p_0)$? The problem here is that we are in $\mathbb{R}^3$, not $\mathbb{R}^2$, so there are infinitely many different directions which are all perpendicular to $\vec{\nabla}g(p_0)$. Which one should you choose? Well, from the first step of the solution, we know that the vector we are looking for has to have horizontal component with the same direction and the same sense as $-\vec{\nabla}f(-1,2)$. So $\vec{b}=\lambda\left(-\vec{\nabla}f(-1,2)+c\widehat{\mathbf{k}}\right)$ where λ is a positive number (any positive number is OK) and c must be negative. Now, if we wish, we can use the fact that $-\vec{\nabla}f(-1,2)+c\widehat{\mathbf{k}}$ has to be perpendicular to $-\vec{\nabla}f(-1,2)+\widehat{\mathbf{k}}$ to calculate c.

In case you did not realize it yet, the "secret" aim of this exercise is to oblige you to understand the different meanings of gradient and of directional derivatives of functions of two variables, and of functions of three variables, and to be careful not to confuse these different meanings[1].

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[1] Here is another often given exercise which has a similar aim: Suppose that the differentiable function $f(x,y,z)$ is the temperature in a room. An ant and a fly are both in the room, sitting on the table $z=c$ at the point (x_0,y_0,c). They are both cold, and they both want to move in the direction where they will feel the quickest increase in temperature. The fly can fly, but the ant can only walk on the table. What are the best options for each of them?

2. We shall use the notation $[a, b, c]$ for the vector $a\widehat{\mathbf{i}} + b\widehat{\mathbf{j}} + c\widehat{\mathbf{k}}$. I.e., $[a, b, c]$ is the vector from the point $(0, 0, 0)$ to the point (a, b, c).

There were several versions of this question. This is a solution of just one of those versions (the version which we have published on the internet). But if you make some obvious changes (if necessary) you can easily change it to the solution for the version that you were asked to solve.

ALEF. By subtracting coordinates we see that the vector from M_3 to M_2 is $[2, 0, -1]$. Similarly, the vector from M_2 to M_1 is $[2, 2, 1]$.

The vector product of these two vectors is

$[2, 0, -1] \times [2, 2, 1] = [2, -4, 4]$.

This last vector is perpendicular to the plane P, and it is parallel to $[1, -2, 2]$.

So one possible equation for this plane is given by $x - 4 - 2(y + 1) + 2z = 0$ and this is the same as

$x - 2y + 2z = 6$ or $x - 2y + 2z - 6 = 0$.

Of course any equation of the form $cx - 2cy + 2cz = 6c$, where c is some non zero constant, is also a correct answer.

BET. The point $\left(2t, e^{t-1}, 2\sqrt{t} + e^{t-1}\right)$ lies on the plane P if and only if it satisfies the equation that we found in part ALEF. So t has to satisfy $2t - 2e^{t-1} + 4\sqrt{t} + 2e^{t-1} = 6$. Equivalently, $2t + 4\sqrt{t} = 6$, i.e. $t + 2\sqrt{t} = 3$. It is easy to see that $t = 1$ is a solution of this equation. To be completely sure that it is the only solution, set $\sqrt{t} = s$ then we have to solve the equation $s^2 + 2s = 3$, i.e. $s^2 + 2s - 3 = 0$. This is the same as $(s + 3)(s - 1) = 0$. So $s = -3$ or $s = 1$. But $\sqrt{t}$ cannot equal -3. (We understand $\sqrt{t}$ to denote the POSITIVE square root of t.)

Substituting $t = 1$ we see that M_0 has to be the point $(2, 1, 3)$.

GIMEL. In general, if a curve is given by a parametric representation $(x(t), y(t), z(t))$, $a \le t \le b$, where the three functions $x(t)$, $y(t)$ and $z(t)$ are differentiable, then at each point of the curve $(x(t), y(t), z(t))$, the tangent to the curve at that point is the vector $[x'(t), y'(t), z'(t)]$ where we substitute that same value of t. (Different choices of parametrization for the same curve may give different tangent vectors, but at any given point on the curve, all the tangent vectors from all the parametrizations will all be parallel to each other.)

Applying this to our curve Γ, we see that at each point $\left(2t, e^{t-1}, 2\sqrt{t} + e^{t-1}\right)$on Γ, the tangent vector to Γ is $\left[2, e^{t-1}, 1/\sqrt{t} + e^{t-1}\right]$. In particular, at the point M_0, where $t = 1$, this tangent vector is $[2, 1, 2]$.

From the formula $x - 2y + 2z = 6$ for the plane P we see that the vector $[1, -2, 2]$ is perpendicular (=normal="nitsav") to P. The angle θ between this normal vector $[1, -2, 2]$ and the tangent vector $[2, 1, 2]$ satisfies $\|[1, -2, 2]\| \, \|[2, 1, 2]\| \cos\theta = [1, -2, 2] \cdot [2, 1, 2]$. Clearly $\|[1, -2, 2]\| = \|[2, 1, 2]\| = \sqrt{9} = 3$ and the inner product $[1, -2, 2] \cdot [2, 1, 2] = 2 - 2 + 4 = 4$. So $9 \cos\theta = 4$, i.e. $\cos\theta = 4/9$. However the angle θ is NOT the angle that you were asked to find. We will call the angle that you were asked to find α. (In fact α is also the angle between the plane and the tangent vector.) The angle α satisfies $\alpha = \frac{\pi}{2} - \theta$. (To see this, draw a picture showing the plane P, the point M_0 on P, the normal vector to P at M_0 and the tangent vector to Γ at M_0.) So $\cos\theta = \sin\alpha$. We deduce that $\alpha = \arcsin\frac{4}{9}$.

(This answer $\arcsin\frac{4}{9}$ to part GIMEL is correct for ALL versions of this question.)

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3. ALEF. We have been told to assume that there exist differentiable functions $u(x, y)$ and $v(x, y)$ defined for all points (x, y) sufficiently close to $(1, 0)$ such that

(A) $$\cos u(x, y) + \sin v(x, y) = x + y \text{ and } \sin u(x, y) + \cos v(x, y) = x - y$$

for all these points (x, y). In particular we have also been told to assume that these equations hold at the point $(x, y, u, v) = (1, 0, \frac{\pi}{2}, \frac{\pi}{2})$. This means that we must have

$$\text{(B)} \qquad u(1, 0) = \frac{\pi}{2} \text{ and } v(1, 0) = \frac{\pi}{2}.$$

Then we can differentiate the two equations (A) partially with respect to y. (The variable x is held constant during this partial differentiation.) We obtain that $-\sin u(x, y)\frac{\partial u}{\partial y} + \cos v(x, y)\frac{\partial v}{\partial y} = 1$ and $\cos u(x, y)\frac{\partial u}{\partial y} - \sin v(x, y)\frac{\partial v}{\partial y} = -1$. When we substitute $(x, y) = (1, 0)$ and use (B), these two equations become $-\frac{\partial u}{\partial y}(1, 0) + 0 = 1$ and $0 - \frac{\partial v}{\partial y}(1, 0) = -1$. In particular $\frac{\partial v}{\partial y}(1, 0) = 1$.

BET. This time we have been told to assume that there exist differentiable functions $v(u, y)$ and $x(u, y)$ defined for all points (u, y) sufficiently close to $(\frac{\pi}{2}, 0)$ such that

$$\text{(C)} \qquad \cos u + \sin v(u, y) = x(u, y) + y \text{ and } \sin u + \cos v(u, y) = x(u, y) - y$$

for all these points (u, y). In particular we have also been told to assume that these equations hold at the point $(x, y, u, v) = (1, 0, \frac{\pi}{2}, \frac{\pi}{2})$. This means that we must have

$$\text{(D)} \qquad x\,(\pi/2, 0) = 1 \text{ and } v(\pi/2, 0) = \pi/2.$$

(Note that here the notation $v(u, y)$ stands for a function of two variables which is DIFFERENT from the function which we denoted by $v(x, y)$ in part ALEF. So perhaps it would be better to use some different notation, for example $\phi(u, y)$, for this function. See also part GIMEL and DALET for more comments about this.)

Then we can differentiate the two equations (C) partially with respect to y. (This time we will be holding the variable u constant.) We obtain that $0 + \cos v(u, y)\frac{\partial v}{\partial y} = \frac{\partial x}{\partial y}(u, y) + 1$ and $0 - \sin v(u, y)\frac{\partial v}{\partial y} = \frac{\partial x}{\partial y}(u, y) - 1$. When we substitute $(u, y) = (\pi/2, 0)$ and use (D), these two equations become $0 = \frac{\partial x}{\partial y}(\pi/2, 0) + 1$ and $-\frac{\partial v}{\partial y}(\pi/2, 0) = \frac{\partial x}{\partial y}(\pi/2, 0) - 1$. So we obtain that $\frac{\partial x}{\partial y}(\pi/2, 0) = -1$ and $\frac{\partial v}{\partial y}(\pi/2, 0) = 1 - \frac{\partial x}{\partial y}(\pi/2, 0) = 2$.

GIMEL. The assumptions made in ALEF, and also the assumptions made in BET are both correct. This can be shown by using the appropriate version of the implicit function theorem (in two slightly different ways).

Define the functions $F(x, y, u, v)$ and $G(x, y, u, v)$ by $F(x, y, u, v) = \cos u + \sin v - x - y$ and $G(x, y, u, v) = \sin u + \cos v - x + y$. The system of equations (1) considered in parts ALEF and BET can then be rewritten as

$$\text{(E)} \qquad \begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases}.$$

To apply the implicit function theorem to this system of equations, we first need to find one particular point (x_0, y_0, u_0, v_0) in $\mathbb{R}^4$ where this system of equations holds. Of course we can see that the point $P_0 = (x_0, y_0, u_0, v_0) = (1, 0, \frac{\pi}{2}, \frac{\pi}{2})$ is such a point.

Then we have to check that all eight partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$, $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial y}$, $\frac{\partial G}{\partial u}$ and $\frac{\partial G}{\partial v}$ exist and are continuous in some open set D in $\mathbb{R}^4$ which contains (x_0, y_0, u_0, v_0). This is obviously true, since each of these derivatives will be either c or $c \sin u$ or $c \cos u$ or $c \sin v$ or $c \cos v$, for suitable (different) choices of the constant c. We can even take D to be all of $\mathbb{R}^4$.

Now that we know that the above conditions are satisfied, the implicit function theorem tells us that we can, at least in theory, "solve" the system (E) to get u and v as functions of x and y, near P_0, provided (this is only a sufficient condition) that the determinant $\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$ is non zero at P_0. Now this determinant equals $\begin{vmatrix} -\sin u & \cos v \\ \cos u & -\sin v \end{vmatrix} = \sin u \sin v - \cos u \cos v$. At the point P_0 its value is $\sin^2 \frac{\pi}{2} - \cos^2 \frac{\pi}{2} = 1 - 0$. So it is non zero, and we can apply the implicit function theorem in the context of part ALEF.

To apply the same theorem in the context of part BET, i.e., to show that it enables us to "solve" the system (E) to get x and v as functions of u and y, we use almost exactly the same procedure. The only difference is that this time we have to check a different determinant, namely $\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & \cos v \\ -1 & -\sin v \end{vmatrix} = \sin v + \cos v$. At P_0 this equals $\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 \neq 0$, so indeed we can apply the theorem.

We talked about "solving". That is not very precise. But now here is exactly what the theorem tells us. We will state it in the context of part ALEF. The statement in the context of part BET is exactly analogous, but with a permutation of the roles of the different variables x, y, u and v.

Let D *be an open set in* $\mathbb{R}^4$. *Let* $F : D \to \mathbb{R}$ *and* $G : D \to \mathbb{R}$ *be functions all of whose partial derivatives of first order are continuous in* D. *Suppose that there exists a point* $(x_0, y_0, u_0, v_0) \in D$ *such that* $F(x_0, y_0, u_0, v_0) = 0$ *and* $G(x_0, y_0, u_0, v_0) = 0$. *Suppose also that the determinant* $\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$ *is non zero at* (x_0, y_0, u_0, v_0). *Let* Q *be the set of all points* (x, y, u, v) *in* D *which satisfy the equations (E). Then there exist positive numbers* r *and* ρ *and two functions of two variables* $U(x, y)$ *and* $V(x, y)$ *defined for all* (x, y) *such that* $(x - x_0)^2 + (y - y_0)^2 < r^2$ *such that:*

- *For each point* (x, y, u, v) *such that* $(x - x^2)^2 + (y - y_0)^2 + (u - u_0)^2 + (v - v_0)^2 < \rho^2$, *the two equations* $F(x, y, u, v) = 0$ *and* $G(x, y, u, v) = 0$ *hold if and only if* $u = U(x, y)$ *and* $v = V(x, y)$.
- *The partial derivatives* $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial V}{\partial x}$ *and* $\frac{\partial V}{\partial y}$ *all exist and are continuous for all* (x, y) *such that* $(x - x_0)^2 + (y - y_0)^2 < r^2$.

In particular the last condition guarantees that U and V are differentiable, which is another condition mentioned in the statement of part GIMEL.

You may wonder why I chose the notation $U(x, y)$ and $V(x, y)$ here instead of $u(x, y)$ and $v(x, y)$. I think that using a different letter (or differently sized letter) makes the statement of the theorem clearer. If possible, I prefer here to use the letter u and v to denote variables which do not depend on any other variables, and then U and V to denote functions which DO depend on other variables.

DALET. If you worked correctly then you got two different values for the partial derivative $\frac{\partial v}{\partial y}$ in part ALEF and in part BET. There is no contradiction here, because the same letter v represents two DIFFERENT functions obtained by two different processes in part ALEF and in BET.

Apart from anything else, in part ALEF we differentiate with respect to y while keeping x constant. But in part BET we differentiate with respect to y while keeping a DIFFERENT variable constant, namely u. But anyway we are differentiating two different functions. We could have stressed this fact by using different letters in place of v for the two different functions $v(x, y)$ and $v(u, y)$.

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4. ALEF. You are asked to find the maximum and minimum of the function $u(x, y) = ax + by$ on the circle $\Gamma = \{(x, y) : x^2 + y^2 = c^2\}$ where a, b and c are positive constants which take different values in different versions.

When applying the Lagrange method here, it is natural to use the function $g(x, y) = x^2 + y^2 - c^2$ which vanishes at all points of Γ. We have to note that the functions u and g both have continuous derivatives at every point of an open set containing Γ. In fact we can take that open set to be the whole of $\mathbb{R}^2$.

Before we consider the set of equations which arises from Lagrange's method we have to consider two other kinds of points on Γ which do not solve these equations, but could still be points where the maximum or minimum of u on Γ is attained.

- Points on the "edge" ("katzeh") of Γ. In our case the curve Γ is a complete circle, so it has no "edge" points.

• Points (x,y) on Γ where $\vec{\nabla} g(x,y)=\vec{0}$. Here we have $\vec{\nabla} g(x,y)=2x\widehat{\mathbf{i}}+2y\widehat{\mathbf{j}}$ and this vector is non zero at every point except $(0,0)$. Since $(0,0)\notin\Gamma$ we see that there are no points of this kind which we have to consider.

So this leaves only the points (x,y) on Γ which satisfy $\vec{\nabla} u(x,y)+\lambda\vec{\nabla} g(x,y)=\vec{0}$ for some number λ. These points satisfy the the three equations

$$\begin{aligned} a+2\lambda x&=0\\ b+2\lambda y&=0\\ x^2+y^2&=c^2 \end{aligned} \tag{2}$$

Since a and b are both non zero in all version of this question, we see that λ has to be non zero. Then we have $x=-a/2\lambda$ and $y=-b/2\lambda$. From the third equation we now have that $a^2/4\lambda^2+b^2/4\lambda^2=c^2$.

It so happens that $a^2+b^2=c^2$ in all versions of this question. So we have $4\lambda^2=1$, so $2\lambda=\pm 1$. So the only two points (x,y) which satisfy the system of equations (2) are $(x,y)=(a,b)$ and $(x,y)=(-a,-b)$. Since both a and b are positive, we deduce that $u(a,b)=a^2+b^2>u(-a,-b)=-a^2-b^2$. So the maximum and minimum values of u on Γ are a^2+b^2 and $-a^2-b^2$ respectively.

BET. I am going to list four different definitions of differentiability at a point. Each one of them would be acceptable here.

It can of course be proved that they are all equivalent to each other. Please note that none of these definitions talk about the existence of partial derivatives of f at (x_0,y_0).

Definition 1:

There exist numbers A and B and a function $\varepsilon(h,k)$ defined in the set $E=\left\{(h,k):0<h^2+k^2<r^2\right\}$ for some positive number r, such that $\lim_{(h,k)\to(0,0)}\varepsilon(h,k)=0$ and

$$f(x_0+h,y_0+k)=f(x_0,y_0)+Ah+Bk+\sqrt{h^2+k^2}\,\varepsilon(h,k) \tag{3}$$

for all $(h,k)\in E$.

REMARK: It turns out when f is differentiable the numbers A and B and the function $\varepsilon(h,k)$ are all defined UNIQUELY by the condition (3).

Definition 2:

There exist numbers A and B and two function $\alpha(h,k)$ and $\beta(h,k)$ defined in the set $E=\left\{(h,k):0<h^2+k^2<r^2\right\}$ for some positive number r, such that $\lim_{(h,k)\to(0,0)}\alpha(h,k)=0$ and $\lim_{(h,k)\to(0,0)}\beta(h,k)=0$ and

$$f(x_0+h,y_0+k)=f(x_0,y_0)+Ah+Bk+h\alpha(h,k)+k\beta(h,k) \tag{4}$$

for all $(h,k)\in E$.

REMARK: I personally do not like this definition because, in contrast to the first definition, the condition (4) does NOT determine uniquely what the functions $\alpha(h,k)$ and $\beta(h,k)$ have to be.

The next two definitions are less convenient to work with. But they perhaps give you a little bit more insight about the "geometrical" meaning of differentiability.

Definition 3:

There exist numbers A, B and C such that the function $L(x,y)$ defined by $L(x,y)=Ax+By+C$ satisfies $L(x_0,y_0)=f(x_0,y_0)$ and

$$\lim_{(x,y)\to(x_0,y_0)}\frac{L(x,y)-f(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0. \tag{5}$$

Definition 4:

There exists a plane Π which is a tangent plane to the graph of $f(x,y)$ at the point $(x_0,y_0,f(x_0,y_0))$.

This means that, if Π is given by the equation $ax+by+cz=d$, then $c \neq 0$ and the function $L(x,y) := \frac{1}{c}(d-ax-by)$ satisfies $L(x_0,y_0)=f(x_0,y_0)$ and the formula (5).

REMARK. If you only wrote the first sentence of the fourth definition this is not enough. We need you to explain exactly what you mean by a tangent plane. The condition $L(x_0,y_0)=f(x_0,y_0)$ is of course the same as saying that the point $(x_0,y_0,f(x_0,y_0))$ lies on the plane Π.

Even though the conditions in every one of these four definitions imply automatically that f has to be defined in some neighbourhood of (x_0,y_0), it makes things clearer if you state this explicitly, i.e. it is nicer if you begin your definition by saying that f has to be defined in some neighbourhood of (x_0,y_0). But we will not take off points if you did not do this.

Here is the definition which you were explicitly asked NOT to use:

Definition 5:

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at (x_0,y_0) and there exists a function $\varepsilon(h,k)$ defined in the set $E=\left\{(h,k): 0<h^2+k^2<r^2\right\}$ for some positive number r, such that $\lim_{(h,k)\to(0,0)}\varepsilon(h,k)=0$ and

$$f(x_0+h,y_0+k)=f(x_0,y_0)+h\frac{\partial f}{\partial x}(x_0,y_0)+k\frac{\partial f}{\partial y}(x_0,y_0)+\sqrt{h^2+k^2}\varepsilon(h,k) \tag{6}$$

for all $(h,k)\in E$.

If you, davka, wrote this definition, or its analogue with the functions $\alpha(h,k)$ and $\beta(h,k)$ you will only get one point out of four for part BET.

GIMEL

Here are the standard proofs of the continuity of f and of the existence of its partial derivatives of f at (x_0,y_0).

Sorry, I wrote (x_0,y_0) here instead of (a,b). But you can of course make the obvious changes. And we will not take off points if you also wrote (x_0,y_0) instead of (a,b).

Both proofs can be quite short. But since they will be different for different choices of your definition in Part BET, we will have to write a bit more in our solution here.

Continuity:

The idea is to show that $f(x_0+h,y_0+k)=f(x_0,y_0)+\Phi(h,k)$ where $\Phi(h,k)$ is a function which satisfies $\lim_{(h,k)\to(0,0)}\Phi(h,k)=0$.

The function $\Phi(h,k)$ will have different forms, depending on which definition you used for differentiability.

In each case it will be a sum of three or four terms, and each term can be seen to tend to 0 as (h,k) tends to $(0,0)$.

So the sum of these terms also tends to 0 as (h,k) tends to $(0,0)$.

We may have $\Phi(h,k)=Ah+Bk+\sqrt{h^2+k^2}\varepsilon(h,k)$ or $\Phi(h,k)=Ah+Bk+h\alpha(h,k)+k\beta(h,k)$ or $\Phi(h,k)=h\frac{\partial f}{\partial x}(x_0,y_0)+k\frac{\partial f}{\partial y}(x_0,y_0)+\sqrt{h^2+k^2}\varepsilon(h,k)$.

All terms appearing in all these expressions for $\Phi(h,k)$ are products of two functions. At least one of these functions tends to 0 as $(h,k)\to(0,0)$. The other function either tends to 0 or is bounded. So in all cases the products of the two functions tends to 0.

If you used Definition 3 or Definition 4, the approach is a little different. First, the function $L(x,y)$ is obviously continuous everywhere. Then, since

$$L(x,y)-f(x,y)=\frac{L(x,y)-f(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$$

for all $(x,y) \neq (x_0,y_0)$, at least in a small neighbourhood of (x_0,y_0), we see that $L(x,y) - f(x,y)$ is a product of two functions which both tend to 0 as (x,y) tends to (x_0,y_0).

So

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \left(L(x,y) - (L(x,y) - f(x,y))\right) = L(x_0,y_0) - 0 = L(x_0,y_0).$$

Since our definition includes the condition that $L(x_0,y_0) = f(x_0,y_0)$, this shows that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ as required.

Existence of partial derivatives:

Here again the exact form of the proof will depend on which definition you chose for differentiability. Of course if you chose Definition 5 in part BET, then there is nothing to prove, but then, alas, we also have no justification to give you any points for proving nothing.

First we need to use the fact that the condition $\lim_{(h,k)\to(0,0)} \varepsilon(h,k)$ implies that $\lim_{h\to 0} \varepsilon(h,0) = 0$ and also $\lim_{k\to 0} \varepsilon(h,k) = 0$. If you used Definition 2 then you need the analogous fact for the functions $\alpha(h,k)$ and $\beta(h,k)$ in place of $\varepsilon(h,k)$.

Here is the proof if you used Definition 1.

We can substitute $k = 0$ in (4) and obtain that, for all non zero h in $(-r,r)$,

$$f(x_0+h,y_0) = f(x_0,y_0) + Ah + \sqrt{h^2}\varepsilon(h,0) = f(x_0,y_0) + Ah + |h|\,\varepsilon(h,0).$$

Now subtract $f(x_0,y_0)$ from both sides and divide by h. This gives

$$\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} = A + \frac{|h|}{h}\varepsilon(h,0).$$

As h tends to 0 the expression $\frac{|h|}{h}$ is bounded (it equals 1 or -1) and the expression $\varepsilon(h,0)$ tends to 0. So the product of these two expressions tends to 0. This means that $\lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = A + \lim_{h\to 0} \frac{|h|}{h}\varepsilon(h,0) = A + 0$. This shows that $\frac{\partial f}{\partial x}(x_0,y_0)$ exists and equals A.

The proof that $\frac{\partial f}{\partial y}(x_0,y_0)$ exists and equals B is exactly analogous and if, after proving CORRECTLY that $\frac{\partial f}{\partial x}(x_0,y_0)$ exists and equals A, you said that the proof of the existence of $\frac{\partial f}{\partial y}(x_0,y_0)$ is analgous, without providing details, we will accept that.

If you used Definition 2 then the proof is very similar to the above, even slightly simpler, since then you get

$$\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} = A + \alpha(h,0).$$

Again here it will be acceptable to simply claim that the case of the existence of $\frac{\partial f}{\partial y}(x_0,y_0)$ is very similar.

If you used Definition 3 or Definition 4 in part BET, then you will have to work a little harder here. You need a preliminary step to deduce a formula like (3). Here is one way to do it:

We know from the definition that $f(x,y)$ is defined in some disc $\left\{(x,y) : (x-x^2)^2 + (y-y_0)^2 < r^2\right\}$. So $f(x_0+h, y_0+h)$ is defined for all (h,k) such that $h^2+k^2 < r^2$. For all (h,k) such that $0 < h^2+k^2 < r^2$, let us set $x = x_0+h$ and $y = y_0+k$ and define

$$\varepsilon(h,k) = \frac{L(x,y) - f(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = \frac{L(x_0+h,y_0+k) - f(x_0+h,y_0+k)}{\sqrt{h^2+k^2}}.$$

We know, from (5), that

$$\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0. \tag{7}$$

Then, multiplying by $\sqrt{h^2+k^2}$ we have

$$\sqrt{h^2+k^2}\varepsilon(h,k) = L(x_0+h, y_0+k) - f(x_0+h, y_0+k). \tag{8}$$

In both Definitions 3 and 4, $L(x,y)$ is a function of the form $Ax + By + C$ for some constants A, B and C, and we also have that $L(x_0, y_0) = f(x_0, y_0)$. This means that $Ax_0 + By_0 + C = f(x_0, y_0)$ and so $L(x_0, y_0) = A(x - x_0) + B(y - y_0) + f(x_0, y_0)$. Substituting this in (8) gives

$$\sqrt{h^2+k^2}\varepsilon(h,k) = Ah + Bk + f(x_0, y_0) - f(x_0+h, y_0+k).$$

This is exactly the same as (3), and since we also have (7) we have shown that we have all the conditions of Definition 1. So we can finish the proof by using the argument presented above for Definition 1.

DALET. There are many examples of functions with the required properties. Perhaps one of the simplest and best known examples is $f(x,y) = x^{1/3}y^{2/3}$. Others are $f(x,y) = \begin{cases} 0 & , \quad (x,y) = (0,0) \\ \frac{xy^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \end{cases}$ and $f(x,y) = \begin{cases} 0 & , \quad (x,y) = (0,0) \\ \frac{x^3y^2}{x^4+y^4} & , \quad (x,y) \neq (0,0) \end{cases}$.

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