A "FLOW DIAGRAM" FOR CALCULATING LIMITS OF FUNCTIONS (OF SEVERAL VARIABLES).

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In many ways it is silly to try to describe a sophisticated intellectual activity by a simple and childish "flow diagram" but I will try anyway.

The main point I want to make here, is that the methods for showing that a limit **exists** and the methods for showing that a limit **does not exist** are usually quite **DIFFERENT**. Over the years I have seen many students make mistakes because they used the wrong methods to try to justify their guesses.

The main flow diagram is on page 3.

First here are some other remarks.

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OTHER METHODS AND WARNINGS!!!

The danger of using polar coordinates:

It can sometimes be useful to use polar coordinates, in particular to show that some limit does not exist. E.g. when we consider the double limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \tag{1}$$

we can study the behaviour of the function $g(r,\theta) = f(x_0 + r\cos\theta, y_0 + r\sin\theta)$. IF the limit (1) exists and equals some number L, then we must have

$$\lim_{r\to 0} g(r,\theta) = L \text{ for every constant } \theta \in [0, 2\pi].$$

So if this second condition fails we know that the limit (1) does not exist. The danger here is that some people forget that if this second condition holds it does NOT imply that the limit (1) exists. For a relevant counter-example here, choose $(x_0, y_0) = (0,0)$ and consider the function f defined by $f(x,y) = \frac{x^2 y}{x^4 + y^2}$ for all $(x,y) \neq (0,0)$.

Well yes, there IS another way to use polar coordinates to sometimes show that a limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists. If you decide to use it, please make sure that you know exactly what you are doing, exactly what is the condition that you have to check. See Appendix 2 for an example of how you might "histabekh".

The danger of using repeated limits:

Repeated limits usually cause more trouble than they help. I sometimes wish that repeated limits had never been invented. Sometimes I think their existence should be kept a secret from all students. But if you ask, I will tell you. But then I suggest you simply forget about them.

If the function f of two variables is defined in some neighbourhood of the point (x_0, y_0) then its two repeated limits at that point are

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x,y) \right) \quad \text{and} \quad \lim_{y \to y_0} \left(\lim_{x \to x_0} f(x,y) \right) . \tag{2}$$

As the notation indicates, the idea, for example, in the first of these limits, is to try to calculate the limit inside the brackets for each constant value of x very close to x_0. This should give a function depending only on x. Then the final step is to calculate the limit of this new function as x tends to x_0. The calculation of the second of these two limits is exactly the same, but with the roles of x and y interchanged.

The only *slightly* useful thing I know about repeated limits is this:

If both the repeated limits in (2) exist and their values are different, then the "double limit" $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ *does not exist.*

The danger with using this result carelessly is that its converse is wrong:
(•) If both repeated limits exist and are equal, this does NOT imply that the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(You can see this by considering the same function $f(x,y) = \frac{x^2 y}{x^4+y^2}$ that was already mentioned above.)

Here is another perhaps surprising fact which further suggests that repeated limits are not so useful or interesting:
(••) The existence of the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does NOT imply that both of the repeated limits exist.

(This last statement corrects a wrong statement in an earlier version of these notes. It is quite possible that similar wrong statements may appear in other material prepared for this course and other courses.)

To summarize, repeated limits can sometimes help prove that a double limit does not exist, but never that it does exist.

See Appendix 1 at the end of this document for some more details about the statement (••).

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Now here is the main

"FLOW DIAGRAM".

(I apologize that I don't know how to draw the arrows in it yet.)

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Step 1: Try to guess if the limit $\lim_{p \to p_*} f(p)$ exists.

What can help you guess? Many things,

e.g. comparing the powers in the numerator and the denominator, and rough versions of the things mentioned in Steps 2A and 2B.

If you guessed YES go to 2A. If you guessed NO then go to 2B.

Step 2A: Try various ways to justify that the limit exists and if possible find its value, for example:

a. Express f as a some sort of combination (e.g. sum, product, quotients, composition with a continuous function of one variable) of simpler functions for which you know the limits and use the relevant theorems for such combinations.

b. Estimate f above and below by simpler functions which you know have the same limit at p_* and use the "sandwich theorem".

c. Other methods? Various tricks with changes of variables and substitutions. (Please tell/remind me about other methods so I can add them here.)

Succeeded? Go to next problem, or celebrate.

If you did not succeed, go back to Step 1.

Step 2B: Try various ways to prove that the limit does not exist, for example:

a. Show that f is unbounded near p_*

b. Show that f tends to different limits along different curves or which pass through p_*.

c. Describe the level curves or level sets of f and show that there are points from two different level sets which are both arbitrarily close to p_*.

d. Find a sequence of points p_k which converges to p_* with $p_k \neq p_*$ such that $\lim_{k \to \infty} f(p_k)$ does not exist.

e. Find two sequences p_k and q_k of points which both converge to p_* but such that $\lim_{k \to \infty} f(p_k) \neq \lim_{k \to \infty} f(q_k)$.

(f. Again, if you know some other good methods, please tell me so I can add them here.)

Succeeded? Go to next problem, or celebrate.

If you did not succeed, go back to Step 1.

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APPENDIX 1

Here, only for those who want to understand these things more deeply, is how to construct a function f which proves the fact (••). As is quite usual, we will choose $(x_0, y_0) = (0, 0)$. We will also choose our function f to be given by the formula $f(x, y) = (x^2 + y^2)g(x, y)$ where we still have to explain how to choose $g(x, y)$. First we will decide that g will have to satisfy

$$|g(x, y)| \leq 1 \text{ for all } (x, y) \in \mathbb{R}^2. \tag{3}$$

It is easy to check that this condition on g is already enough to guarantee that the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and equals 0. (In fact it also guarantees that f is differentiable at $(0, 0)$.) But now we will choose g so that it "jumps about so wildly" that the limit $\lim_{x\to 0} f(x, y)$ does not exist for any constant $y \neq 0$ and the limit $\lim_{y\to 0} f(x, y)$ does not exist for any constant $x \neq 0$. Since the calculation of these limits are the first steps when we try to calculate the repeated limits in (2), this implies that neither of those repeated limits exist for our particular choice of the function f.

Here is one possible choice for such a "wildly jumping" g which satisfies (3). First set $g(x, y) = 0$ whenever $x = 0$ or $y = 0$. Then, for each (x, y) such that both x and y are non zero, set $g(x, y) = \sin \frac{1}{xy}$. It is easy to check that, for each constant $a \neq 0$, the limit $\lim_{t\to 0}(t^2 + a^2) \sin \frac{1}{at}$ does not exist. If we choose t to be x and y to be a, or vice versa, this shows that, for our $f(x, y) = (x^2 + y^2)g(x, y))$, the limits mentioned above indeed do not exist.

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APPENDIX 2.

I am very grateful to a student who wrote to me after reading a previous version of this document.

שלום

קראתי את הקובץ בנוגע לגבולות אך לא הבנתי מה הסכנה בשימוש בקואורדינטות
פולריות. האם הכוונה שרק באשר מגיעים למשהו מהצורה f(r)g(theta) והפונקציה
שתלויה בצ שואפת לאפס באשר צ שואף לאפס אז ניתן לומר שהגבול הוא 0?

Here is my answer.

Thanks for your message. In fact your question is very helpful because it shows that it can be dangerous to work automatically without thinking too much about what you are doing. Here below is an example which proves that your statement is not correct.

KTBH

Michael Cwikel

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$u(x,y) = \begin{cases} \frac{|x^3|+|x|y^2}{|y|} & , \quad y \neq 0 \\ 0 & , \quad y = 0 \end{cases} .$$

Then

$$u(r\cos\theta, r\sin\theta) = \begin{cases} 0 & , \quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\ \frac{r^3|\cos\theta|}{r|\sin\theta|} & , \quad \theta \in (0,\pi), \theta \in (\pi, 2\pi) \end{cases} .$$

So we can write $u(r\cos\theta, r\sin\theta) = f(r)g(\theta)$ where $f(r) = r^2$ and $g(\theta) = \begin{cases} 0 & , \quad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\ 1/\left|\tan\theta\right| & , \quad \theta \in (0,\pi), \theta \in (\pi, 2\pi) \end{cases}$.

Obviously $\lim_{r\to 0} f(r) = 0$. But $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist, at least not as a finite number. Worse than that, u is not even a bounded function near $(0,0)$. To see this consider the sequence of numbers $u(x_k, y_k)$ for $k \in \mathbb{N}$, where each $(x_k, y_k) = \left(1/k, 1/k^4\right)$. On the one hand $\lim_{k\to\infty}(x_k, y_k)$ exists and equals $(0,0)$. However $\lim_{k\to\infty} u\left(x_k, y_k\right) = +\infty$. So there is absolutely no hope for us to show that $\lim_{(x,y)\to(0,0)} u(x,y) = 0$.

Although your statement is unfortunately wrong, there is a way to convert it into a correct statement. But I am not sure that I want to give the details about that just now.

In general when you apply any theorem to reach any conclusion, please be careful to apply the correct version of that theorem and to check that all the conditions that you need to be true really are true.

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