An exercise about double integrals.

Let a and b be positive numbers.

Let E be the set in the plane defined by $E = \left\{(x,y) : 0 \le x \le \pi/2, 0 \le y \le \min\left\{\frac{a}{\cos x}, \frac{b}{\sin x}\right\}\right\}$

Calculate the double integral $\iint_E y dx dy$.

(If you use the funny (but very logical) notation that I like, you can also write this double integral as $\int_{\mathbb{R}^2} g$ where $g : \mathbb{R}^2 \to \mathbb{R}$ is the function defined by $g(x,y) = \begin{cases} y & , \quad (x,y) \in E \\ 0 & , \quad (x,y) \notin E \end{cases}$.)

Hints. • One of the functions $a/\cos x$ and $b/\sin x$ is increasing and the other is decreasing. Where do their graphs intersect? • What is the derivative of $\tan x$? • The change of variable $t = \pi/2 - x$ might be helpful for one step of your calculation.

Remark. I have a special reason for being interested in this exercise.

This result of this calculation can be used for a special, maybe surprising purpose.

Very soon in the lectures we will discuss a way of calculating double integrals via polar coordinates. There will be a special formula for doing this.

If you look at the calculation of the above integral carefully and write θ in place of x and r in place of y, then you will see that it shows that the formula for calculating double integrals via polar coordinates is true for the function $f : \mathbb{R}^2 \to \mathbb{R}$ which is 1 on the rectangle $\{(x,y) : 0 \le x \le a, 0 \le y \le b\}$ and zero at all other points of $\mathbb{R}^2$. Once we know this fact, there are several tricks which enable us to show, without too too much difficulty, that the same formula is true for every Riemann integrable function of two variables.

If you are curious to know more, then you can look at pages 49–51 of the document

`http://www.math.technion.ac.il/~mcwikel/infi2/intgrn-n.ps`

but this is not an easy document to read. It was written for a different more advanced course and uses different definitions and different terminology from what I have been using in this course.