A MORE DETAILED VERSION OF THE MATERIAL AT THE END OF MY LECTURE TODAY (WEDNESDAY 17/12/08.

Suppose we take a line segment of length r and place it in the xy plane so that it makes an angle θ with the x axis and and we also place one end of it at $(0,0)$. Then, by simple trigonometry, the other end of the line segment will be at the point (x,y) where

$$x = r\cos\theta \text{ and } y = r\sin\theta. \tag{1}$$

[You might say that perhaps $x = r\cos(-\theta)$ and $y = r\sin(-\theta)$ but we will exclude that possibility by deciding that we choose θ in the range $[0,2\pi)$ and we first place our line segment on the x- axis and then we rotate it ANTI-CLOCKWISE through an angle θ.]

For example, if $r = \sqrt{2}$ and $\theta = \pi/4$ then (1) gives us that $x = 1$ and $y = 1$.

The same point which has xy coordinates $(x,y) = (1,1)$ has "polar coordinates" $(r,\theta) = \left(\sqrt{2}, \pi/4\right)$.

The formulae (1) give us a way to get the xy coordinates of any point, whenever we know its polar coordinates r and θ.

But we want to be able to go in the opposite direction. If we already know the xy coordinates, we want to be able to find r and θ such that $x = r\cos\theta$ and $y = r\sin\theta$.

OK, this is not so hard. You and I know Pythagoras' theorem and some trigonometry, so we can easily see that we can find r and θ from the formulae

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \arctan\frac{y}{x}, \tag{2}$$

at least for $x > 0$. But for now let's pretend we have forgotten the things we needed to find these formulae.

Suppose that all we know is that the numbers $x = 1$, $y = 1$, $r = \sqrt{2}$ and $\theta = \pi/4$ satisfy the two equations of (1). We want to ask these questions:

- Suppose that we have some x and y such that the point (x,y) is "very close" to $(1,1)$. Do there exist some r and θ very close to $\sqrt{2}$ and $\pi/4$ so that the equations (1) are satisfies for these new values of x and y?
- Are these r and θ uniquely determined by the given x and y? In other words, are there functions $R(x,y)$ and $\Theta(x,y)$ defined for all (x,y) close to (1,1) so that $r = R(x,y)$ and $\Theta(x,y)$ satisfy the equations (1)?
- If these functions $R(x,y)$ and $\Theta(x,y)$ exist, even if we do not know a formula for them, can we perhaps calculate their partial derivatives, $\frac{\partial R}{\partial x}$, $\frac{\partial R}{\partial y}$, $\frac{\partial \Theta}{\partial x}$, $\frac{\partial \Theta}{\partial y}$, at least at the point $(x,y) = (1,1)$?

Now I want to state a general theorem which will tell us that the answer to all these questions in yes. We will also be able to use it in other more complicated situations where we cannot take advantage of Pythagoras' theorem and trigonometry.

THEOREM 1. (The implicit function theorem for systems of two equations with two independent variables.)

Let G be an open set in $\mathbb{R}^4$. Let $f_1 : G \to \mathbb{R}$ and $f_2 : G \to \mathbb{R}$ be two functions in $C^1(G)$.

Let $p^* = (x_1^*, x_2^*, y_1^*, y_2^*)$ be a point in G. Let c_1 and c_2 be the numbers $c_1 = f_1(p^*)$ and $c_2 = f_2(p^*)$.

Let E be the set of all points $(x_1, x_2, y_1, y_2) \in G$ which are solutions of the two equations $f_1(x_1, x_2, y_1, y_2) = c_1$ and $f_2(x_1, x_2, y_1, y_2) = c_2$.

(Let me stop and interrupt for a minute and make some comments. We know that there is at least one point in E, namely the point $p^* = (x_1^*, x_2^*, y_1^*, y_2^*)$. We hope to show that there are other points in E which are near to p^*. We hope to show that for every x_1 and x_2 sufficiently close to x_1^* and x_2^* respectively, we can find some y_1 and y_2 close to y_1^* and y_2^* such that (x_1, x_2, y_1, y_2) is also a solution of our system of two equations, i.e., a point in E. To do this we shall need to impose one more condition, corresponding to the condition $\frac{\partial f}{\partial y} \neq 0$ which appears in simpler versions of this theorem.)

Suppose that the four derivatives $\frac{\partial f_1}{\partial y_1}$, $\frac{\partial f_1}{\partial y_2}$, $\frac{\partial f_2}{\partial y_1}$ and $\frac{\partial f_2}{\partial y_2}$ satisfy the condition

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{vmatrix} \neq 0 \text{ at the point } p^* \tag{3}$$

Then there exists a number $r > 0$ and functions ϕ_1 and ϕ_2 of two variables defined and in C^1 on the open disk $D_r = \left\{(x_1, x_2) : (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 < r^2\right\}$ such that the point $(x_1, x_2, \phi_1(x_1, x_2), \phi_2(x_1, x_2))$ is in the set E whenever $(x_1, x_2) \in D_r$. Furthermore, every point (x_1, x_2, y_1, y_2) in E which is also in the "ball"

$$B(p^*, r) = \left\{(x_1, x_2, y_1, y_2) : (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 + (y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 < r^2\right\}$$

has to be of the special form $(x_1, x_2, \phi_1(x_1, x_2), \phi_2(x_1, x_2))$.

That is the theorem.

Now I want to do 3 things:

1. I want to show you that this theorem can be applied to the above question about polar coordinates and points near the point $(1, 1, \sqrt{2}, \pi/4)$.

2. I want to show you how to calculate the partial derivatives of the functions ϕ_1 and ϕ_2 at least at the point (x_1^*, x_2^*), even when we do not know a formula for these functions.

3. I want to suggest some ideas behind the proof of this theorem. The exact proof is much too difficult and long for us to discuss here.

Applying this theorem to the above question about polar coordinates and points near the point $(1, 1, \sqrt{2}, \pi/4)$.

We will write x_1 and x_2 in place of x and y and we will write y_1 and y_2 in place of r and θ.

We choose f_1 and f_2 to be the functions $f_1(x_1, x_2, y_1, y_2) = x_1 - y_1 \cos y_2$ and $f_2(x_1, x_2, y_1, y_2) = x_2 - y_1 \sin y_2$. Then we take $c_1 = c_2 = 0$. So in this case the set E in the statement of Theorem 1 is exactly the set of points where f_1 and f_2 are both 0. It is the same as the set of points (x, y, r, θ) which satisfy (1).

We see that this set contains the point $\left(1, 1, \sqrt{2}, \pi/4\right)$. Now let us see that the condition (3) holds. At the point $\left(1, 1, \sqrt{2}, \pi/4\right)$ we have

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} -\cos y_2 & y_1 \sin y_2 \\ -\sin y_2 & -y_1 \cos y_2 \end{vmatrix} = y_1 \cos^2 y_2 + y_1 \sin^2 y_2 = y_1 = \sqrt{2} \neq 0$$

as required. So all the conditions of the theorem hold. So the functions $\phi_1(x_1, x_2)$ and $\phi_2(x_1, x_2)$ which the theorem tells us must exist, are the functions $R(x_1, x_2)$ and $\Theta(x_1, x_2)$ or $R(x, y)$ and $\Theta(x, y)$ which we hoped to find.

These functions must satisfy the two equations

$$x - R(x, y) \cos \Theta(x, y) = 0 \text{ and } y - R(x, y) \sin \Theta(x, y) = 0. \tag{4}$$

If we differentiate both of these equations partially with respect to x we get two equations: $1 - \frac{\partial R}{\partial x}(x, y) \cos \Theta(x, y) + R(x, y) \sin \Theta(x, y) \frac{\partial \Theta(x,y)}{\partial x} = 0$ and $0 - \frac{\partial R}{\partial x}(x, y) \sin \Theta(x, y) - R(x, y) \cos \Theta(x, y) \frac{\partial \Theta(x,y)}{\partial x} = 0$. We know that when $(x, y) = (1, 1)$ we have $R(1, 1) = \sqrt{2}$ and $\Theta(1, 1) = \pi/4$. So $\sin \Theta(1, 1) = \cos \Theta(1, 1) = 1/\sqrt{2}$. So when we substitute $x = 1$ and $y = 1$ the two equations above become $1 - \frac{1}{\sqrt{2}} \frac{\partial R}{\partial x}(1, 1) + \frac{\partial \Theta}{\partial x}(1, 1) = 0$ and $-\frac{1}{\sqrt{2}} \frac{\partial R}{\partial x}(1, 1) - \frac{\partial \Theta}{\partial x}(1, 1) = 0$. These are two linear equations which we can solve to find the values of the two unknowns $\frac{\partial R}{\partial x}(1, 1)$ and $\frac{\partial \Theta}{\partial x}(1, 1)$.

Exercise 1: Solve this system of two equations to find these two values $\frac{\partial R}{\partial x}(1, 1)$ and $\frac{\partial \Theta}{\partial x}(1, 1)$. Then check that you get the same values as if you calculate them directly using our "secret" formulae (2).

We can now calculate $\frac{\partial R}{\partial y}(1, 1)$ and $\frac{\partial \Theta}{\partial y}(1, 1)$ in exactly the same way. The first step is to differentiate both equations of (4) partially with respect to y. Then we substitute $(x, y) = (1, 1)$ and get a new pair of linear equations to solve.

How can we calculate the partial derivatives of the functions ϕ_1 and ϕ_2 in Theorem 1, at least at the point (x_1^*, x_2^*), even though we do not know a formula for these functions?

This is exactly what we did in Exercise 1, but only in the special case where $f_1(x_1, x_2, y_1, y_2) = x_1 - y_1 \cos y_2$ and $f_2(x_1, x_2, y_1, y_2) = x_2 - y_1 \sin y_2$. Let us now see that the same ideas will also work in general.

Theorem 1 promises us that we have those two functions ϕ_1 and ϕ_2 which satisfy

$$f_1\left(x_1, x_2, \phi_1(x_1, x_2), \phi_2(x_1, x_2)\right) = c_1 \text{ and } f_2\left(x_1, x_2, \phi_1(x_1, x_2), \phi_2(x_1, x_2)\right) = c_2 \tag{5}$$

for all (x_1, x_2) near (x_1^*, x_2^*). Let us calculate $\frac{\partial \phi_1}{\partial x_1}$ and $\frac{\partial \phi_2}{\partial x_1}$ at (x_1^*, x_2^*). (The same method can be used later for $\frac{\partial \phi_1}{\partial x_2}$ and $\frac{\partial \phi_2}{\partial x_2}$.) We want to take the partial derivatives with respect to x_1 of both sides of the two equations (5). We will do this with the help of the chain rule. But first let us decide what notation we want to use for the various partial derivatives of f_1 and f_2. There are several possibilities. We could for example use the notation $\frac{\partial f_j}{\partial x_k}$ and $\frac{\partial f_j}{\partial y_k}$ for $j = 1, 2$ and $k = 1, 2$. Or we could use the notation $(f_j)'_k$ for $j = 1, 2$ and $k = 1, 2, 3, 4$. Or we could give new names to the functions, define $u(x_1, x_2, y_1, y_2) = f_1(x_1, x_2, y_1, y_2)$ and $v(x_1, x_2, y_1, y_2) = f_2(x_1, x_2, y_1, y_2)$. Then it would be comfortable to use the notation u'_k and v'_k for $k = 1, 2, 3, 4$. Well I chose the second option. Also, let me use the symbol Q as an abbreviation for the point $(x_1, x_2, \phi_1(x_1, x_2), \phi_2(x_1, x_2))$. So now I will differentiate the first equation of (5). The chain rule will give me that

$$(f_1)'_1(Q)\cdot\frac{\partial x_1}{\partial x_1}+(f_1)'_2(Q)\cdot\frac{\partial x_2}{\partial x_1}+(f_1)'_3(Q)\cdot\frac{\partial \phi_1}{\partial x_1}(x_1,x_2)+(f_1)'_4(Q)\cdot\frac{\partial \phi_2}{\partial x_1}(x_1,x_2)=0.$$

Of course $\frac{\partial x_1}{\partial x_1} = 0$ and $\frac{\partial x_2}{\partial x_1} = 0$. So we obtain that

$$(f_1)'_3(Q)\cdot\frac{\partial \phi_1}{\partial x_1}(x_1,x_2)+(f_1)'_4(Q)\cdot\frac{\partial \phi_2}{\partial x_1}(x_1,x_2)=-(f_1)'_1(Q). \tag{6}$$

(Here in (6) and also later (in (7), (8) and (9)) I have coloured the expressions which are unknowns for us. The expressions in black in those same equation are things that we know or can easily calculate directly, at least when we choose $(x_1, x_2) = (x_1^*, x_2^*)$.) In exactly the same way, when we differentiate the second equation, we obtain

$$(f_2)'_3(Q)\cdot\frac{\partial \phi_1}{\partial x_1}(x_1,x_2)+(f_2)'_4(Q)\cdot\frac{\partial \phi_2}{\partial x_1}(x_1,x_2)=-(f_2)'_1(Q). \tag{7}$$

We can rewrite these two equations as a single matrix equation.

$$\begin{pmatrix} (f_1)'_3(Q) & (f_1)'_4(Q) \\ (f_2)'_3(Q) & (f_2)'_4(Q) \end{pmatrix}\begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(x_1,x_2) \\ \frac{\partial \phi_2}{\partial x_1}(x_1,x_2) \end{pmatrix} = -\begin{pmatrix} (f_1)'_1(Q) \\ (f_2)'_1(Q) \end{pmatrix}. \tag{8}$$

If we substitute $(x_1, x_2) = (x_1^*, x_2^*)$ then Q becomes the point $p^* = (x_1^*, x_2^*, y_1^*, y_2^*)$. (Why?) So then the determinant of the 2×2 matrix in the above equation is exactly the determinant which appears in (3). Since that determinant is not zero, it is possible to invert the matrix and use that inverted matrix to obtain the values of $\frac{\partial \phi_1}{\partial x_1}(x_1^*, x_2^*)$ and $\frac{\partial \phi_2}{\partial x_1}(x_1^*, x_2^*)$.

Exactly the same 2×2 matrix will appear when we calculate $\frac{\partial \phi_1}{\partial x_2}(x_1^*, x_2^*)$ and $\frac{\partial \phi_2}{\partial x_2}(x_1^*, x_2^*)$. In that case the relevant equation is

$$\begin{pmatrix} (f_1)'_3(Q) & (f_1)'_4(Q) \\ (f_2)'_3(Q) & (f_2)'_4(Q) \end{pmatrix}\begin{pmatrix} \frac{\partial \phi_1}{\partial x_2}(x_1,x_2) \\ \frac{\partial \phi_2}{\partial x_2}(x_1,x_2) \end{pmatrix} = -\begin{pmatrix} (f_1)'_2(Q) \\ (f_2)'_2(Q) \end{pmatrix}. \tag{9}$$

Some ideas behind the proof of Theorem 1.

Exercise 2: Suppose that the functions f_1 and f_2 have the special form

$$f_1(x_1, x_2, y_1, y_2) = U_1(x_1, x_2) + V_1(x_1, x_2)y_1 + W_1(x_1, x_2)y_2 \tag{10}$$

and

$$f_2(x_1, x_2, y_1, y_2) = U_2(x_1, x_2) + V_2(x_1, x_2)y_1 + W_2(x_1, x_2)y_2 \tag{11}$$

where all the six functions of two variables U_1, V_1, W_1, U_2, V_2 and W_2 are in $C^1\left(\mathbb{R}^2\right)$. Use your knowledge about matrices to give a complete proof of Theorem 1 for these particular kinds of functions. In that case you can even write explicit formulae for the two functions $\phi_1(x_1, x_2)$ and $\phi_2(x_1, x_2)$. You can also write an exact condition which defines the set of points (x_1, x_2) for which ϕ_1 and ϕ_2. (For an easier version of this exercise you can suppose that the six functions U_1, V_1, W_1, U_2, V_2 and W_2 are constants.)

Now that you have done this exercise I can wave my hands and try to convince you that Theorem 1 is reasonable. This is NOT a proof. Let me start: Suppose that f_1 and f_2 are both differentiable functions of (y_1, y_2) for each fixed choice of (x_1, x_2). Then for all (y_1, y_2) very close to some point (y_1^*, y_2^*) the functions f_1 and f_2 are very close to functions of the special form (10) and (11) respectively. (Here the particular functions U_1, V_1, W_1, U_2, V_2 and W_2 will depend on our choice of the point (y_1^*, y_2^*).) So it would not be too surprising if these functions f_1 and f_2 had behaviour similar to the sort of behaviour that you saw in Exercise 2. I repeat, this is NOT a proof.

The idea of giving this "explanation" came to me because of a very interesting discussion with some students after today's lecture. Thanks very much to those students.