Some exotic examples of repeated integrals

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In this document we will see that we cannot always change the order of integration in repeated integrals, i.e., formulas like

$$\int_a^b \left(\int_a^b f(x,y)dy \right) dx = \int_a^b \left(\int_a^b f(x,y)dx \right) dy \tag{1}$$

or

$$\int_a^b \left(\int_a^x f(x,y)dy \right) dx = \int_a^b \left(\int_y^b f(x,y)dx \right) dy. \tag{2}$$

are sometime false, even in certain cases where all the integrals appearing in them are well defined in the sense of Rieman and are finite.

Of course in most applications of repeated integrals in Hedva 2m we are using these integrals to calculate double integrals of some function $f(x,y)$ for which the double integral exists in the sense of Riemann. For example, you were asked to prove the formula (2) in one of the MathNet exercises, in the case where $f(x,y)$ is continuous on the closed square $\{(x,y) : a \le x \le b, a \le y \le b\}$.

In the exotic examples which we shall consider here, the functions f are unbounded and so do NOT have a double integral in the sense of Riemann.

1. The formula (1) can sometimes be false.

It is quite easy (after someone has told us the secret of how to do it) to construct a function $f(x,y)$ for which (1) does not hold. We will do this in the case where $a = 0$ and $b = 1$. It is easiest if we proceed indirectly. We start by considering the function $g(x,y) = \frac{y}{x^2+y^2}$. We note that $\frac{\partial g}{\partial y} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$ at each point $(x,y) \ne (0,0)$.

In particular, this means that, for each constant $x \ne 0$ and each pair of constants α and β, we have

$$\int_\alpha^\beta \frac{x^2-y^2}{(x^2+y^2)^2} dy = g(x,\beta) - g(x,\alpha) = \frac{\beta}{x^2+\beta^2} - \frac{\alpha}{x^2+\alpha^2}. \tag{3}$$

By an exactly analogous calculation, using the function $\frac{x}{x^2+y^2}$, in fact interchanging the roles of x and y, we see that, for each constant $y \ne 0$ and each α and β,

$$\int_\alpha^\beta \frac{y^2-x^2}{(x^2+y^2)^2} dx = \frac{\beta}{y^2+\beta^2} - \frac{\alpha}{y^2+\alpha^2}. \tag{4}$$

These two simple formulæ will now show us that we cannot always interchange order of integration in repeated integrals.

Let us consider the function $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ on the open square $\{(x,y) : 0 < x < 1, 0 < y < 1\}$. Since f is unbounded as (x,y) approaches $(0,0)$ we cannot define the double integral (in the sense of Riemann) of f on this square. But we can calculate the integral of $f(x,y)$ on $(0,1)$ with respect to x for each constant $y \in (0,1)$ and also the integral of $f(x,y)$ on $(0,1)$ with respect to y for each constant $x \in (0,1)$. By (3) we have

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dy = \frac{1}{x^2+1}. \tag{5}$$

Then, if we multiply (4) by -1 we get

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dx = -\frac{1}{y^2+1}. \tag{6}$$

From (5) we see that the repeated (NOT double) integral $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dy \right) dx$ is well defined and in fact

$$\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dy \right) dx = \int_0^1 \frac{1}{x^2+1} dx = \arctan x|_0^1 = \frac{\pi}{4}.$$

Similarly, from (6), the repeated integral $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx\right) dy$ is well defined and in fact

$$\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{\left(x^2+y^2\right)^2}dx\right) dy = \int_0^1 \frac{-1}{y^2+1}dx = -\arctan y|_0^1 = -\frac{\pi}{4}.$$

So although $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx\right) dy$ and $\int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy\right) dx$ are both well defined, they are not equal.

Remarks: • *As already mentioned, this example does not contradict the results about changing order of integration in the formulæ for calculating double integrals via repeated integrals, because here our function* $\frac{x^2-y^2}{(x^2+y^2)^2}$ *is NOT Riemann integrable on the square* $\{(x,y) : 0 < x < 1, 0 < y < 1\}$.

• *This particular example* $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ *is well known and appears in many books and other sources. You can see it for example in [W]. However in [W] the calculation is done in a more complicated way by using trigonometric substitutions. If we already know what the answer should be, we can avoid those substitutions and complicated calculations by starting with the function* g.

2. The formula (2) can sometimes be false.

Sorry, this section is much more complicated than the previous one. Let us say that it is only for "freakim".

Again we will consider the case where $a = 0$ and $b = 1$.

Here, unfortunately, I could not find a simple counterexample like the one we know for (1).

If you find one, please let me know.

I had to work fairly hard to construct this example. I got the idea of how to do it from an approach which is given in Example 8.9(a) on page 166 of [Ru]. The example there shows that (1) can be false. But it needs some adaptation and adjustment to also show that (2) can be false. The function in [Ru] is continuous at every point of the square S except at $(1,1)$. The function in the example here could be modified to make it continuous at every point of the square S except at $(0,0)$, but that would make the presentation more complicated.

Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ be two sequences of numbers satisfying

(7) $$a_n > 0 \text{ and } r_n > 0 \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} r_n = 0.$$

For each $n \in \mathbb{N}$ let S_n be the square $S_n = \{(x,y) : a_n \le x \le a_n + r_n, 3a_n/4 \le y \le 3a_n/4 + r_n\}$

We need all the squares S_1, S_2, ... to be contained in the triangle $T = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$ and we will achieve this by first requiring that

(8) $$a_1 + r_1 \le 1,$$

which of course will also guarantee that $3a_1/4 + r_1 \le 1$. This means that we have the top right hand vertex of S_1 in the square $S = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$ and so each S_n is contained in S. To further ensure that each S_n is contained in T it will suffice to ensure that the top left hand vertex of S_n, namely the point $(a_n, 3a_n/4 + r_n)$ lies below the line $y = x$. So we will require that $3a_n/4 + r_n \le a_n$, which is equivalent to the condition

(9) $$r_n \le a_n/4 \text{ for each } n \in \mathbb{N}$$

Our next requirement is that, for each n, the square S_{n+1}lies entirely to the left of, and entirely under the square S_n. More precisely, we require that

(10) $$a_{n+1} + r_{n+1} < a_n \text{ and } 3a_{n+1}/4 + r_{n+1} < 3a_n/4 \text{ for each } n \in \mathbb{N}.$$

There are many possible ways to choose the positive sequences $\{a_n\}_{n\in\mathbb{R}}$ and $\{r_n\}_{n\in\mathbb{N}}$ such that all the stated conditions (7), (8), (9) and (10) are fulfilled. For example we can take $a_n = 4^{-n-1}$ and $r_n = 8^{-n-1}$.

Our next step is to define a sequence of rectangles $\{R_n\}_{n\in\mathbb{N}}$ by setting

$$R_n = \{(x,y) : a_n \le x \le a_n + r_n, 3a_{n+1}/4 \le y \le 3a_{n+1}/4 + r_{n+1}\}.$$

Note that every point of R_n lies underneath some points of S_n and to the right of some points in S_{n+1}. Also, if $n \ge 2$, then every point in S_n lies above some points of R_n and to the left of some points in S_{n-1}.

Let us replace the preceding sentence by a more precise statement. Because of the first inequality in (10), the "horizontal" intervals $H_n = [a_n, a_n + r_{n+1}]$ which are the projection of S_n and also of R_n onto the x axis, are disjoint from each other. This means that any line $x = x_0$ which is parallel to the y axis either passes through both S_n and R_n for some value of n, or it does not pass through any of these squares or rectangles.

We are going to define our function f so that it takes the constant value σ_n on each square S_n and the constant value ρ_n on each rectangle R_n. It will be 0 at all points which do not lie in any of the sets S_n or R_n. We will decide, as we proceed with our calculation, what values we want for the numbers σ_n and ρ_n.

Let us now calculate the integral $\int_0^1 f(x,y)dy$ for any constant value x_0 of x. If the line $x = x_0$ does not pass through any of the squares S_n then it also does not pass through any of the rectangles R_n and, in that case, $f(x_0, y) = 0$ for all $y \in [0,1]$ and we get $\int_0^1 f(x,y)dy = 0$. The only other possibility is that the line $x = x_0$ passes through one, and only one of the squares S_n. If it passes through S_n then it also passes through R_n (i.e. we have $x_0 \in H_n$) and then we have

$$\begin{aligned}
\int_0^1 f(x_0,y)dy &= \int_0^{3a_{n+1}/4} f(x_0,y)dy + \int_{3a_{n+1}/4}^{3a_{n+1}/4+r_{n+1}} f(x_0,y)dy + \int_{3a_{n+1}/4+r_{n+1}}^{3a_n/4} f(x_0,y)dy \\
&\quad + \int_{3a_n/4}^{3a_n/4+r_n} f(x_0,y)dy + \int_{3a_n/4+r_n}^{1} f(x_0,y)dy \\
&= \int_0^{3a_{n+1}/4} 0dy + \int_{3a_{n+1}/4}^{3a_{n+1}/4+r_{n+1}} \rho_n dy + \int_{3a_{n+1}/4+r_{n+1}}^{3a_n/4} 0dy + \int_{3a_n/4}^{3a_n/4+r_n} \sigma_n dy + \int_{3a_n/4+r_n}^{1} 0dy \\
&= r_{n+1}\rho_n + r_n\sigma_n.
\end{aligned}$$

Now we are ready to impose our first condition on the numbers σ_n and ρ_n. We will require them to satisfy

(11) $$\rho_n = -\sigma_n r_n / r_{n+1} \text{ for each } n \in \mathbb{N}.$$

This is exactly what we need to obtain that the integral in the previous calculation satisfies $\int_0^1 f(x_0,y)dy = 0$ for each x_0 in each interval H_n and so also for all $x_0 \in [0,1]$.

Since all the squares S_n and rectangles R_n lie below the line $y = x$ the above calculation also shows that $\int_0^x f(x,y)dy = 0$ for all $x \in [0,1]$. So we now know that the repeated integrals $\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx$ and $\int_0^1 \left(\int_0^x f(x,y)dy\right) dx$ both exist and both equal 0.

Finally we shall show that we can choose the numbers σ_n in such a way that the repeated integrals $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy$ and $\int_0^1 \left(\int_y^1 f(x,y)dx\right) dy$ both exist and are equal to each other, but are not equal to 0. Of course once we choose each σ_n this also fixes each ρ_n via the formula (11).

Because of the second inequality in (10) the "vertical" intervals $V_n = [3a_n/4, 3a_n/4 + r_n]$, which are the projections of S_n and also of R_{n-1} onto the y axis, are disjoint from each other. This means that any line $y = y_0$ which is parallel to the x axis must do one of the following three things:

(i) It does not pass through any of the sets S_n and R_n.

(ii) Or it passes through the square S_1.

(iii) Or it passes through the square S_n for some $n \geq 2$, which means that it also passes through the rectangle R_{n-1}.

Let us now calculate the integral $\int_0^1 f(x,y_0)dx$ for each constant y_0 in $[0,1]$. First, since all the sets R_n and S_n lie to the right of the line $y = x$, it follows that this integral must equal $\int_{y_0}^1 f(x,y_0)dx$.

If the line $y = y_0$ has property (i) among the three properties listed just above, i.e. if y_0 does not belong to any of the intervals V_n, then of course $\int_0^1 f(x,y_0)dx = 0$. In the case of property (ii), i.e. if $y_0 \in V_1$, then

$$\int_0^1 f(x,y_0)dx = \int_{a_1}^{a_1+r_1} f(x,y_0)dx = \int_{a_1}^{a_1+r_1} \sigma_1 dx = r_1\sigma_1.$$

In the remaining case (iii) there exists some $n \geq 2$ such that $y_0 \in V_n$, and then

$$\begin{aligned} \int_0^1 f(x,y_0)dx &= \int_0^{a_n} f(x,y_0)dx + \int_{a_n}^{a_n+r_n} f(x,y_0)dx + \int_{a_n+r_n}^{a_{n-1}} f(x,y_0)dx \\ &\quad + \int_{a_{n-1}}^{a_{n-1}+r_{n-1}} f(x,y_0)dx + \int_{a_{n-1}+r_{n-1}}^{1} f(x,y_0)dx \qquad (12) \\ &= \int_0^{a_n} 0dx + \int_{a_n}^{a_n+r_n} \sigma_n dx + \int_{a_n+r_n}^{a_{n-1}} 0dx + \int_{a_{n-1}}^{a_{n-1}+r_{n-1}} \rho_{n-1}dx + \int_{a_{n-1}+r_{n-1}}^{1} 0dx \\ &= r_n\sigma_n + r_{n-1}\rho_{n-1} = r_n\sigma_n - r_{n-1}^2\sigma_{n-1}/r_n. \qquad (13) \end{aligned}$$

Let us now define the numbers σ_n recursively, by setting $\sigma_1 = 1$ and

$$\sigma_n = \frac{r_{n-1}^2}{r_n^2}\sigma_{n-1} \text{ for all } n \geq 2. \qquad (14)$$

In that case we obtain (cf. (13)) that $\int_0^1 f(x,y_0)dx = 0$ whenever $y_0 \in V_n$ for some $n \geq 2$. Altogether we see that $\int_0^1 f(x,y_0)dx = r_1\sigma_1 = r_1$ for all $y_0 \in V_1 = [3a_1/4, 3a_1/4 + r_1]$ and $\int_0^1 f(x,y_0)dx = 0$ for all $y_0 \in [0,1] \setminus V_1$. This means that the function $g(y) = \int_0^1 f(x,y)dx$ is a Riemann integrable function on $[0,1]$ and its integral is given by

$$\int_0^1 g(y)dy = \int_{3a_1/4}^{3a_1/4+r_1} r_1 dy = r_1^2.$$

Thus indeed we have that $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = \int_0^1 \left(\int_y^1 f(x,y)dx\right) dy = r_1^2 > 0$, i.e.,

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = \int_0^1 \left(\int_y^1 f(x,y)dx\right) dy = r_1^2 \neq 0 = \int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = \int_0^1 \left(\int_0^x f(x,y)dy\right) dx.$$

To make this example more concrete, we may return to our explicit choices for the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$, namely $a_n = 4^{-n-1}$ and $r_n = 8^{-n-1}$. Then (14) becomes

$$\sigma_n = \left(\frac{8^{-(n-1)-1}}{8^{-n-1}}\right)^2 \sigma_{n-1} = 64\sigma_{n-1}$$

and, consequently, $\sigma_n = 64^{n-1} = 8^{2n-2}$ for all $n \in \mathbb{N}$.

Note that, since 8^{2n-2} is unbounded as n tends to ∞, the function $f(x,y)$ which we have just constructed is unbounded on the square S and so it is not Riemann integrable on S.

References

[Ru] W. Rudin, Real and Complex Analysis, McGraw Hill, Third Edition.

[W] http://en.wikipedia.org/wiki/A_counterexample_related_to_Fubini%27s_theorem