Some clarifications about directional derivatives (Nigzerot Mekhuvanot).

Version 2. 15/2/2007

Thank you Shachar for your interesting questions and for noticing the mistake that I will mention below. Thank you also Daniel for correcting some misprints in the expression $\frac{g(x_0+h\nu_1,y_0+h\nu_2)-g(x_0,y_0)}{h}$ *in Version 1.*

Let f be a function of three variables defined on a set which contains the point (x_0, y_0, z_0). Let $\widehat{n} = [n_1, n_2, n_3]$ be a three dimensional unit vector. Then the directional derivative of f at (x_0, y_0, z_0) in the direction of $\widehat{n}$ is defined to be the limit

$$\lim_{h\to 0} \frac{f(x_0 + hn_1, y_0 + hn_2, z_0 + hn_3) - f(x_0, y_0, z_0)}{h} \tag{1}$$

if this limit exists.

In this document I want to stress, as you should have understood from lectures and from books, that this limit is a usual limit, i.e. a two sided limit. It is NOT a one sided limit.

Of course, for it to exist, we also need the function f to be defined at all the points $(x_0+hn_1, y_0+hn_2, z_0+hn_3)$ for sufficiently small positive AND negative values of h.

Please write the corresponding definitions, including the formula

$$\lim_{h\to 0} \frac{g(x_0 + h\nu_1, y_0 + h\nu_2) - g(x_0, y_0)}{h}$$

for the directional derivative of a function of two variables $g(x, y)$ at the point (x_0, y_0) in the direction of the unit vector $\widehat{\nu} = [\nu_1, \nu_2]$.

Examples: If $g(x, y) = \sqrt{x^2 + y^2}$ and $(x_0, y_0) = (0, 0)$ then the ONE SIDED limits $\lim_{h\nearrow 0} \frac{g(x_0+h\nu_1,y_0+h\nu_2)-g(x_0,y_0)}{h}$ and $\lim_{h\searrow 0} \frac{g(x_0+h\nu_1,y_0+h\nu\nu_2)-g(x_0,y_0)}{h}$ both exist. They equal -1 and 1 respectively for each choice of $\widehat{\nu}$. However this function does NOT have a directional derivative at $(0, 0)$ for ANY choice of $\widehat{\nu}$.

(I suggest that you draw the graph of g to see more clearly what is happening here.)

Similarly, for EVERY choice of the unit vector $\widehat{n}$, the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ does not have a directional derivative at $(0, 0, 0)$ in the direction of $\widehat{n}$. The limit (1) does not exist, although the corresponding limits from the left and from the right do exist.

Some corrections: In the Hedva 2m exam of July 2001, in part BET of question 1, the students were asked to study the function

$$f(x, y, z) = \begin{cases} \frac{x^3y^2z}{(x^2+y^2+z^2)^\alpha} & , \quad (x, y, z) \neq (0, 0, 0) \\ 0 & , \quad (x, y, z) = (0, 0, 0). \end{cases}$$

where α is some constant. In part 2 of this question, they were asked to find the condition on α which is equivalent to the condition that the directional derivative of f at $(0, 0, 0)$ exists for every choice of direction $\widehat{n} = [n_1, n_2, n_3]$.

If we look at the fraction appearing in (1), we see that in this case

$$\begin{aligned} \frac{f(x_0 + hn_1, y_0 + hn_2, z_0 + hn_3) - f(x_0, y_0, z_0)}{h} &= \frac{f(hn_1, hn_2, hn_3) - 0}{h} \\ &= \frac{1}{h} \cdot \frac{h^3n_1^3h^2n_2^2hn_3}{(h^2n_1^2 + h^2n_2^2 + h^2n_3^3)^\alpha}. \end{aligned}$$

Since $\widehat{n}$ is a unit vector, this equals

$$\frac{1}{h} \cdot \frac{h^6n_1^3n_2^2n_3}{(h^2)^\alpha} = \frac{h^5}{(h^2)^\alpha}.n_1^3n_2^2n_3$$

We want to know whether the limit as h tends to 0 of this expression exists for EVERY choice of the unit vector $\widehat{n}$. This is of course equivalent to the existence of the limit $\lim_{h\to 0} \frac{h^5}{(h^2)^\alpha}$.

Since h^2 is always non negative positive, the expression $(h^2)^\alpha$ is always well defined even when α is not an integer, and it equals $|h|^{2\alpha}$. So we have to decide for what values of α does the limit

$$\lim_{h\to 0} \frac{h^5}{|h|^{2\alpha}} \tag{2}$$

exist. If $5 > 2\alpha$ then this limit exists and equals 0. If $5 < 2\alpha$ then the expression $\frac{h^5}{|h|^{2\alpha}}$ becomes unbounded as h tends to 0 and the limit cannot exist.

Finally, if $5 = 2\alpha$, then $\frac{h^5}{|h|^{2\alpha}} = \left(\frac{h}{|h|}\right)^5$. This equals 1 for all $h > 0$ and it equals -1 for all $h < 0$. So the TWO SIDED limit (2) does not exist in this case. So the answer to part 2 in the exam that you can see on the Hedva 2m website (`test - 2000/spr_finsol1`), and also on page 48 of the hoveret published by ASAT is not quite right. It should be changed to

$$\alpha < \frac{5}{2}.$$

Now, exactly the same question (with k instead of α) appears in Mathnet, Gilayon Hazara 2, Question 11, part BET. *(Again, thanks Shachar for also noticing this.)*

At this stage when I chose the wrong answer $\alpha \leq \frac{5}{2}$ I was given 100 points.

The Mathnet people tell me that this will be fixed very soon.