"כדורים", "קוביות" ו-"מרחקים" ב-$\mathbb{R}^n$.

Version 1, 20/11/06, so far only in English.

1 Introduction

It seems that some of you were confused by the "distances" or "metrics" d_2 and d_∞ and also some other things that I defined in the lecture last Wednesday.
Let me try here to present some of what I said then in a slightly different way.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points in $\mathbb{R}^3$. For each such pair of points we define the following two numbers:

$$d_2\left((x_1, y_1, z_1), (x_2, y_2, z_2)\right) = \sqrt{\sum_{j=1}^{3} (x_j - y_j)^2} \tag{1}$$

and

$$d_{\max}\left((x_1, y_1, z_1), (x_2, y_2, z_2)\right) = \max\left\{|x_j - y_j|, j = 1, 2, 3\right\}. \tag{2}$$

For now I will just call these two things numbers, "stum" numbers. I will not tell you (yet) about the other funny names that mathematicians like to give them. To be sure that there is no misunderstanding about the notation in (2) let me remind you that if $a_1, a_2, ..., a_n$ are any collection of n real numbers, then the symbol $\max\{a_j, j = 1, 2, ..., n\}$ stands for the number b with the properties that $a_j \leq b$ for all $j = 1, 2, ..., n$ and $a_j = b$ for at least one value of j in the range $1, 2, ..., n$.

2 A little discussion about d_2.

We know, from Pythagoras' theorem, that the number $d_2\left((x_1, y_1, z_1), (x_2, y_2, z_2)\right)$ is exactly the usual distance (sometimes called "euclidean distance") between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2). This fact enables us to give a simple geometric description of a special set which we denote by $B_2((x_1, y_1, z_1), r)$.

For each constant $r > 0$ and each constant point (x_1, y_1, z_1) in $\mathbb{R}^3$, the set $B_2((x_1, y_1, z_1), r)$ is defined to consist of all the points (x, y, z) in $\mathbb{R}^3$ which satisfy the condition

$$d_2\left((x_1, y_1, z_1), (x, y, z)\right) < r. \tag{3}$$

The condition (3) is the same as the condition

$$(x - x)^2 + (y - y_1)^2 + (z - z_1)^2 < r^2$$

and the points (x, y, z) which satisfy it are exactly all the points in an "open" ball of radius r whose centre is at (x_1, y_1, z_1). (Americans write "center" but Australians and British people and others write "centre".) Here "ball" means exactly (or almost exactly) the kind of ball that you know since you played with balls as a child, and "open" means that the points on the surface of that ball, i.e. the points of the sphere ("pney kadoor") which satisfy $(x - x)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$, are not included in the set. If we wanted to include this sphere in the set we would have to replace "<" by "≤" in (3). The "closed" ball defined by this modified condition is sometimes denoted by $\overline{B_2((x_1, y_1, z_1), r)}$.

3 A little discussion about $d_{\max}$.

We do not yet know what is the "meaning" of the number $d_{\max}((x_1, y_1, z_1), (x_2, y_2, z_2))$. (Perhaps we will never know? And, anyway, what is the meaning of "meaning"?) But we will now try to see why it might be sometimes useful to work with $d_{\max}((x_1, y_1, z_1), (x_2, y_2, z_2))$.

We want to define a new set in $\mathbb{R}^3$ which we will call $B_{\max}((x_1, y_1, z_1), r)$. It will be an exact analogue of the set $B_2((x_1, y_1, z_1), r)$ that we defined above.

For each constant $r > 0$ and each constant point (x_1, y_1, z_1) in $\mathbb{R}^3$, the set $B_{\max}((x_1, y_1, z_1), r)$ is defined to consist of all the points (x, y, z) in $\mathbb{R}^3$ which satisfy the condition

$$d_{\max}((x_1, y_1, z_1), (x, y, z)) < r. \tag{4}$$

The condition (4) is the same as $\max\{|x - x_1|, |y - y_1|, |z - z_1|\} < r$ and this holds if and only x, y and z satisfy each of the following three conditions,

$$|x - x_1| < r, \ \ |y - y_1| < r \text{ and } \ |z - z_1| < r.$$

These in turn can be rewritten as

$$x_1 - r < x < x_1 + r, \ y_1 - r < y < y_1 + r \text{ and } z_1 - r < z < z_1 + r. \tag{5}$$

The first of these three conditions means that the point (x, y, z) has to lie somewhere between the plane $x = x_1 - r$ and the plane $x = x_1 + r$. These two planes are both perpendicular (nitsav) to the x-axis and they meet the x-axis at the points $(x_1 - r, 0, 0)$ and $(x_1 + r, 0, 0)$ respectively (b'hat'amah). In other words the point (x, y, z) is somewhere inside a certain "strip" of width $2r$ which is perpendicular to the x-axis.

If we look at the second condition of (5), then we see that it means that the point (x, y, z) has to lie in the strip of width $2r$ which is between the two planes $y = y_1 - r$ and $y = y_1 + r$. These two planes are both perpendicular to the y-axis. Similarly, the third condition of (5) means that (x, y, z) has to lie in the strip of width (or thickness) $2r$ between the two horizontal planes $z = z_1 - r$ and $z = z_1 + r$. Combining these three conditions, (please draw some pictures!) we see that the set $B_{\max}((x_1, y_1, z_1), r)$ has to be a cube or "box" whose sides all have the length $2r$ and are each parallel to one of the three coordinate axes. The centre of this box is at the point (x_1, y_1, z_1). This box is "open" in the sense that its boundary (consisting of six square faces) does not belong to it. If we wanted to include these faces in the set we would have to replace "$<$" by "$\leq$" in (4). The "closed" box defined by this modified condition is sometimes denoted by $\overline{B_{\max}((x_1, y_1, z_1), r)}$.

Cubes or boxes are sometimes useful sets to work with. For example small cubes can be easily packed together to make big cubes. (Balls do not have this property). Since cubes can be useful, there is some hope that the numbers $d_{\max}((x_1, y_1, z_1), (x_2, y_2, z_2))$ which can be used to define cubes might also be useful sometimes.

4 Connections between d_2 and $d_{\max}$. Fitting cubes and balls inside each other.

Obviously, for all choices of the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$, we have $|x_1 - x_2| = \sqrt{(x_1 - x_2)} \leq \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = d_2((x_1, y_1, z_1), (x_2, y_2, z_2))$ and in exactly the same way we have that

$|y_1 - y_2| \leq d_2((x_1, y_1, z_1), (x_2, y_2, z_2))$ and $|z_1 - z_2| \leq d_2((x_1, y_1, z_1), (x_2, y_2, z_2))$.

So the maximum of the three numbers $|x_1 - x_2|$, $|y_1 - y_2|$ and $|z_1 - z_2|$ is less than or equal to $d_2((x_1, y_1, z_1), (x_2, y_2, z_2))$. So we have now shown that

$$d_{\max}((x_1, y_1, z_1), (x_2, y_2, z_2)) \le d_2((x_1, y_1, z_1), (x_2, y_2, z_2)) . \qquad (6)$$

Exercise: Use the inequality (6) to show (easily, in one line!) that

$$B_2((x_1, y_1, z_1), r) \subset B_{\max}((x_1, y_1, z_1), r)$$

for each constant point (x_1, y_1, z_1) and each constant positive number r.

The result of this exercise is not a big deal. Probably you have known, since you were a small child playing with blocks and balls, that a ball of radius r will fit inside a box or cube whose (inside) side length is $2r$.

But let me whisper something to you: One reason that we do this and other silly and simple and obvious exercises in $\mathbb{R}^3$ is to prepare ourselves to do other analogous exercises in $\mathbb{R}^4$ and $\mathbb{R}^n$ for $n > 4$. Here our instinct is not so good. At my kindergarten and I suppose also at yours, they did not give us four dimensional balls or cubes to play with.

We were able to put a ball inside a box. But now we want to put a box inside a ball. For this to work it is clear that the diameter of the ball will have to be bigger than the side length of the box. But how much bigger? The first step is to find an inequality which is a sort of reverse of (6).

Let c be the maximum of the three numbers $|x_1 - x_2|$, $|y_1 - y_2|$ and $|z_1 - z_2|$. Then, obviously $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \le c^2 + c^2 + c^2 = 3c^2$. If we take square roots of this inequality we see that $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \le \sqrt{3c^2} = \sqrt{3} \cdot c$. Remembering what c is, we have shown that

$$d_2((x_1, y_1, z_1), (x_2, y_2, z_2)) \le \sqrt{3} d_{\max}((x_1, y_1, z_1), (x_2, y_2, z_2)) . \qquad (7)$$

Exercise: Use the inequality (6) to show (easily, in one line!) that $B_{\max}((x_1, y_1, z_1), r) \subset B_2((x_1, y_1, z_1), \sqrt{3}r)$ for each constant point (x_1, y_1, z_1) and each constant positive number r.

So each cube of side length $2r$ will fit inside a ball of radius $\sqrt{3}r$. (You might like to wonder whether the cube will fit inside a smaller ball.)

5 A big big jump to more abstract things

One of the "secret weapons" of mathematics and of the branches of science and technology that use mathematics is that we take objects or ideas which we know very well, perhap we even know them since our childhood, and we use them to motivate definitions of new, much more general and abstract objects and ideas, which can turn out to be very useful tools, sometimes in very unexpected ways. Perhaps it would be better if we chose new names for these new things, but sometimes we prefer to use old familiar names. For example you probably already did an algebra course in which the word "space" was used with a new meaning, to mean something much more general than the three dimensional space in which we have all been living since our childhood.

So this is a challenge, to take a word that we have heard and used many times, and use it and understand it with a new much more general meaning.

Let me warn you: From here on, several familiar words, in particular "distance" and "ball", are going to be given new meanings, far beyond what you imagined before. This might be confusing at first, but you already managed to make this transition with the word "space". So please try to enjoy the excitement and the challenge of looking at old words and new things in new ways.

Our first step will be to choose a constant number $n \in \mathbb{N}$. If we choose $n = 3$ we will get things which are essentially the same as were described in the previous sections. If we choose $n = 2$ we will get simpler things, for example disks ("igulim") instead of balls, and squares instead of cubes. If we choose $n = 1$, things will be even simpler, intervals. But if we choose $n > 3$ we will get new things that we cannot describe by pictures or by the geometry that we know from three dimensional space. But these are things that we will have to use sometimes. The things that we want to define and describe have to make sense no matter what positive whole number value we have chosen for n.

So we have chosen n. Now we recall that $\mathbb{R}^n$ is the set of all n-tuples of real numbers. Each element $u \in \mathbb{R}^n$ is an ordered set of n real numbers. We use the notation $u = (u_1, u_2, ..., u_n)$ where u_j are all real numbers and u is the n-tuple that they form.
If $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are both elements of $\mathbb{R}^n$, then we define the two numbers $d_2(u, v)$ and $d_{\max}(u, v)$ by the formulæ

$$d_2(u,v) = \sqrt{\sum_{j=1}^{n} (u_j - v_j)^2}$$

and

$$d_{\max}(u,v) = \max\{|u_j - v_j|, j = 1, 2, ..., n\}.$$

Of course the definitions of d_2 and $d_{\max}$ now depend on the particular value of n that we have chosen. Perhaps you might want me to show this by using more complicated notation, for example $d_{2,(n)}(u, v)$ and $d_{\max,(n)}(u, v)$. But I do not want to do this. Please allow me to keep using the simpler notation. If I forget that it depends on n in some crucial place you will have to shout and remind me.

Exercise. (This exercise was already given in the lecture, maybe with slightly different notation). Find numbers α_n and β_n which depend ONLY on n and which have the property (you have to prove this) that

$$d_2(u, v) \le \alpha_n d_{\max}(u, v) \text{ and } d_{\max}(u, v) \le \beta_n d_2(u, v) \text{ for ALL } u \text{ and } v \text{ in } \mathbb{R}^n. \tag{8}$$

(You already know α_3 and β_3 from the calculation that we did above.)

Perhaps you can guess that the preceding exercise is a first step to showing how to fit n-dimensional balls into n-dimensional cubes, and n-dimensional cubes into n-dimensional balls. But first we have to define these new sets. It should not be so hard to guess what their definitions should be from what we already did for $n = 3$.

Definition. Fix $n \in \mathbb{N}$. For each constant point $u \in \mathbb{R}^n$ and each $r > 0$:
(i) Define $B_2(u, r)$ to be the set of all elements $v \in \mathbb{R}^n$ which satisfy $d_2(u, v) < r$.
(ii) Define $\overline{B_2(u, r)}$ to be the set of all elements $v \in \mathbb{R}^n$ which satisfy $d_2(u, v) \le r$.
(iii) Define $B_{\max}(u, r)$ to be the set of all elements $v \in \mathbb{R}^n$ which satisfy $d_{\max}(u, v) < r$.
(iv) Define $\overline{B_{\max}(u, r)}$ to be the set of all elements $v \in \mathbb{R}^n$ which satisfy $d_{\max}(u, v) \le r$.

I will now tell you the names that are sometimes used for the four sets that we have just defined. They might sound horrible or silly at first, but there is a certain logic behind these names. You do not have to try to remember these names. I think you will find that you remember them anyway.

(i) the open n-dimensional euclidean ball of radius r with centre u.

(ii) the closed n-dimensional euclidean ball of radius r with centre u.

(iii) the open n-dimensional cube of side length $2r$ with centre u.

(iv) the closed n-dimensional cube of side length $2r$ with centre u.

Sometimes people say just "ball" or "cube" and forget to say the other things. Then you might get a little confused.

Exercise. Suppose that $n = 4$ and $u = (0, 0, 0, 0)$. Don't get upset if you cannot draw pictures in $\mathbb{R}^4$. But simply use the definition to check which of the following statements are true.

$(1, -1, 1/2, -2/3) \in B_{\max}(u, 1)$,
$(1, -1, 1/2, -2/3) \in \overline{B_{\max}(u, 1)}$,
$(1, -1, 1, -1) \in B_2(u, 1)$,
$(1, -1, 1, -1) \in \overline{B_2(u, 2)}$.

Remark. Some people call the number $d_2(u, v)$ "the n-dimensional euclidean distance between the points u and v in $\mathbb{R}^n$". They also have different names for $d_{\max}(u, v)$. For example they might say "the sup norm distance between u and v" or "the ℓ^∞ distance between u and v." They might also call $B_{\max}(u, r)$ "the ℓ^∞ ball of radius r centred at u". Many people (including me in last Wednesday's lecture) use the notation $d_\infty(u, v)$ instead of $d_{\max}(u, v)$. I gave you a hint in that lecture about why it is natural to write $d_\infty(u, v)$. But you do not have to know about that. In this course you do not have to use or remember any of the names or notation mentioned in this particular remark if you do not like them or do not want to. In later courses you might not have this choice.

Now we are finally ready to try to put "balls" into "cubes" and "cubes" into "balls" in n-dimensions.

*Exercise.*You already found numbers α_n and β_n and proved that they satisfied (8). Use these two numbers to find other numbers $\widetilde{\alpha_n}$ and $\widetilde{\beta_n}$ which also depend ONLY on n and satisfy (you have to prove it)

$$B_2(u, r) \subset B_{\max}(u, \widetilde{\alpha_n} r) \text{ and } B_{\max}(u, r) \subset B_2(u, \widetilde{\beta_n} r) \text{ for every } u \in \mathbb{R}^n \text{ and every } r > 0.$$

Remark. To save writing things twice, let me use the notation $d_\$(x, y)$ to stand for either $d_2(x, y)$ or $d_{\max}(x, y)$. It is a simple exercise to check that, for all choices of n, for all u, v and w in $\mathbb{R}^n$, for $\$ = 2$ and also for $\$ = \max$ we have:

(i) $d_\$(u, v) \geq 0$
(ii) $d_\$(u, v) = 0$ if and only if $u = v$
(iii) $d_\$(u, v) = d_\$(v, u)$

There is also a fourth property, often called "the triangle inequality". But it is sometimes harder to check. Here it is:

(iv) $d_\$(u, v) \leq d_\$(u, w) + d_\$(w, v)$.

When $\$ = 2$ and $n = 1, 2$ or 3 we can easily see that (iv) is true by using geometrical considerations, including properties of real triangles. When $\$ = \max$ it is easy to check (iv) algebraically for all values of n. When $n > 3$ and $\$ = 2$ we need a cleverer argument, which you do not need to know for this course, to prove (iv). It uses an inequality called the Cauchy-Schwartz inequality or Cauchy-Schwartz-Bunyakowsky inequality, which you will meet if you take the course "Fourier series and integral transforms".
