

## Uniform convergence “survival kit” for Fourier series.

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Uniform convergence (hitkansut bmidah shavah) is a very important general notion in mathematical analysis. It is not easy to understand. (I remember from my student days that I had problems with it myself at first.) Later I will remind you of its definition. First let me recall the two main things that you need to know about it for **this** course.

**Theorem 1.** *Suppose that for each  $n \in \mathbb{N}$  the function  $\phi_n : E \rightarrow \mathbb{C}$  are all defined and continuous on the interval  $E \subset \mathbb{R}$ . Suppose also that the series  $\sum_{n=1}^{\infty} \phi_n$  converges uniformly on  $E$  to the function  $S$ . Then  $S$  is continuous on  $E$ .*

**Remark:** When we deal with Fourier series, the functions  $\phi_n$  will be of the form  $\phi_n(x) = c_n e^{inx}$  or  $\phi_n(x) = c_n e^{inx} + c_{-n} e^{-inx}$  or  $\phi_n(x) = a_n \cos nx + b_n \sin nx$ . These are all continuous functions. So if the sum of a Fourier series on some set  $E$  is not continuous on  $E$ , then, by Theorem 1, the convergence of the Fourier series on  $E$  **cannot** be uniform.

**Theorem 2. (The Weierstrass “m-test”).** *Suppose that, for each  $n \in \mathbb{N}$ , the function  $\phi_n : E \rightarrow \mathbb{C}$  are all defined on the interval  $E \subset \mathbb{R}$ . Suppose that there exists a sequence of numbers  $m_n \geq 0$  such that, for each  $n$ ,*

$$|\phi_n(x)| \leq m_n \text{ for all } x \in E.$$

*Suppose also that the series (of non negative numbers)  $\sum_{n=1}^{\infty} m_n$  converges. Then the series  $\sum_{n=1}^{\infty} \phi_n$  converges uniformly on  $E$ .*

**Remark:** We can apply Theorem 2 to Fourier series, for example when we know that the Fourier coefficients of some function  $f$  have the property that  $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$  converges, or that  $\sum_{n=-\infty}^{\infty} |c_n|$  converges. Then, since  $|\sin nx|$ ,  $|\cos nx|$  and  $|e^{inx}|$  are all bounded by 1, we can choose  $m_n = |c_n|$ , or  $m_n = |c_n| + |c_{-n}|$  or  $m_n = |a_n| + |b_n|$ . Then Theorem 2 gives the uniform convergence of the Fourier series of  $f$  on any set  $E$ , even  $E = \mathbb{R}$ . So, among other things, the sum of the Fourier series for a function which has such nice Fourier coefficients is a continuous function on all  $\mathbb{R}$ .

The things I have written so far should be **almost** enough to help you cope with **most** questions in **this course** related to uniform convergence. But I hope you want to know a bit more. Uniform convergence also enables you to do other quite different useful things that you might need to do in other courses.

### Definition of uniform convergence.

First we define uniform convergence for **sequences** (sdarot) of functions, and then later we can use this to give the definition of uniform convergence of **series** (turim).

Let  $E$  be some subset of  $\mathbb{R}$  and suppose for each  $n \in \mathbb{N}$  we are given a function  $f_n : E \rightarrow \mathbb{R}$ . In other words  $\{f_n\}_{n \in \mathbb{N}}$  is a **sequence of complex functions on  $E$** .

*I recommend thinking of  $\{f_n\}_{n \in \mathbb{N}}$  in the following ways. (This may seem silly, but I think it can really help):*

(i) *Imagine that you have a pack of cards. On card number 1 draw the graph of  $f_1$ , on card number 2 draw the graph of  $f_2$ , ... on card number  $n$  draw the graph*

of  $f_n$ . (For this you need each function  $f_n$  be real valued and bounded and  $E$  has to be a bounded set.)

(ii) Now imagine that you have scanned the pictures on all the cards into your computer. You see the graph of  $f_1$  on your screen. Then you press the PgDn key (page down). The picture changes to the graph of  $f_2$ . ... and so on. Each time you press PgDn, the picture changes from the graph of  $f_n$  to the graph of  $f_{n+1}$ . Of course you should set all this up so that the  $x$  and  $y$  axes of all your graphs are always in the same place on your screen, and so that you always use the same scale.

**Definition of uniform convergence for sequences of functions.** Suppose that  $\{f_n\}_{n \in \mathbb{N}}$  is a sequence of complex functions on  $E$ . Suppose that  $f : E \rightarrow \mathbb{C}$  is another complex function on  $E$ . Then we say that  $f_n$  **converges uniformly on  $E$  to  $f$**  if there exists a sequence of **numbers**  $\alpha_n \geq 0$ , such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and such that, for each  $n \in \mathbb{N}$  we have

$$|f_n(x) - f(x)| \leq \alpha_n \text{ for ALL } x \in E.$$

This is not the usual definition in most books<sup>1</sup>, but it is equivalent to that definition and I think it is often more convenient to work with. Please note that the numbers  $\alpha_n$ , like the numbers  $m_n$  in the Weierstrass  $m$ -test are really just numbers, NOT functions. They do NOT depend on  $x$ .

In the case where all the functions are real, uniform convergence means that for each  $n$  the graph of the function  $f_n$  has to lie inside a thin strip, of thickness  $2\alpha_n$  around the graph of the function  $f$ , i.e.

$$f(x) - \alpha_n \leq f_n(x) \leq f(x) + \alpha_n, \text{ for all } x \in E.$$

As  $n$  tends to  $\infty$  this strip gets thinner and thinner. Imagine how this would look on your imaginary computer screen using the setup described above.

Here is the example that every teacher gives to explain uniform convergence:

**Example 1:** Let  $f(x) = 0$  and let  $f_n(x) = x^n$ . Let  $E = [0, c]$  where  $c = 0.99999$ .

(i) Show that  $f_n$  converges uniformly to  $f$ . Hint. You can choose  $\alpha_n = c^n$ .

(ii) Show that the same proof works for  $E = [0, c]$  for every  $c \in (0, 1)$ . Please note: We keep  $c$  CONSTANT while we check this property on  $[0, c]$ .

(iii) Show that the same sequence of functions  $f_n(x) = x^n$  does NOT converge uniformly to  $f(x) = 0$  on  $E = [0, 1)$ .

The result (iii) might surprise you at first. We do have that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for every CONSTANT  $x \in [0, 1)$ . But this "pointwise convergence" is not enough to give uniform convergence.

To get a better feeling for what pointwise convergence means, think about the imaginary computer screen mentioned above. If you want to know if your sequence is converging pointwise at some point  $x_0$ , prepare a piece of cardboard with a very thin vertical slit. Put it on the screen so it hides everything except the vertical line  $x = x_0$ . So then you will only see one point of the graph, at height  $f_n(x_0)$ . Each time you press PgDn this point may move up and/or down. If there is pointwise convergence to the function  $f(x)$  at  $x = x_0$  then this point in the slit will get closer and closer to the point at height  $f(x_0)$ .

Then you might want to repeat this experiment for many different constant values of  $x_0$ .

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<sup>1</sup>The standard definition uses the notion of supremum.

Now, as promised, I will give the definition for series, by using the previous definition for sequences:

**Definition of uniform convergence for series of functions:** *Suppose that  $\{g_n\}_{n \in \mathbb{N}}$  is a sequence of complex functions on  $E$ . Suppose that  $S : E \rightarrow \mathbb{C}$  is another complex function on  $E$ . Then we say that the **series**  $\sum_{n=1}^{\infty} g_n$  **converges uniformly on  $E$  to  $S$**  if the **sequence**  $\{f_n\}_{n \in \mathbb{N}}$  converges uniformly to  $S$ , where each  $f_n$  is defined by  $f_n = \sum_{k=1}^n g_k$ .*

**Example 2:** Suppose we want to consider the case where  $g_n(x) = x^n$ . Then  $f_n(x) = x + x^2 + x^3 + \dots + x^n$ . Fortunately we know how to find another formula for  $f_n$ , i.e.,

$$f_n(x) = \frac{x - x^{n+1}}{1 - x} \text{ for all } x \neq 1 \text{ and } f_n(1) = n.$$

To decide whether the **series**  $\sum_{n=1}^{\infty} x^n$  converges uniformly on a set  $E$ , we have to study the uniform convergence of the sequence  $\left\{ \frac{x - x^{n+1}}{1 - x} \right\}_{n \in \mathbb{N}}$  on the same set  $E$ . If  $E$  is any set contained in  $(-1, 1)$  it is clear that the function  $S(x)$  must be given by  $S(x) = \frac{x}{1-x}$ . It can be shown that  $f_n$  converges to  $S$  uniformly on the set  $E$  if and only if there exists a constant  $c \in (0, 1)$  such that  $E \subset [-c, c]$ .

#### Further remarks.

1. The “computer game” way of looking at sequences of functions  $\{f_n\}_{n \in \mathbb{Z}}$  does not work directly when the functions are complex valued. But if we define  $u_n(x)$  and  $v_n(x)$  to be, respectively, the real and imaginary parts of  $f_n(x)$  for each  $x \in E$  and each  $n \in \mathbb{N}$ , then we have two real valued sequences of functions  $\{u_n\}_{n \in \mathbb{Z}}$  and  $\{v_n\}_{n \in \mathbb{Z}}$  which contain all the information about the sequence  $\{f_n\}_{n \in \mathbb{Z}}$ . In particular it is easy to show that

(i)  $f_n$  converges pointwise at some point  $x_0$  if and only if  $u_n$  and  $v_n$  both converge pointwise at the same point.

(ii)  $f_n$  converges uniformly on  $E$  (or on some subset  $F$  of  $E$ ) if and only if  $u_n$  and  $v_n$  both converge uniformly on  $E$  (or on  $F$ ).

2. In all this discussion we have considered only functions of one variable. But appropriate analogues of the same definitions and theorems all apply to functions of  $N$  variables. Then of course the set  $E$  is contained in  $\mathbb{R}^N$  instead of  $\mathbb{R}$ .

We can even consider functions and sequences of functions defined on a more general set  $E$  which might not be a subset of any  $\mathbb{R}^N$ . If we want to be able to decide whether these functions are continuous we have to be able to decide when points of  $E$  are “close” to each other. This is possible if  $E$  is a “metric space” or a “topological space”, notions which are defined in more advanced courses.