Theorem 1. *Suppose that $f:(\alpha,\beta)\to\mathbb{R}$ is a continuous function, whose derivatives of all orders $0,1,2,...,\ell$ are all continuous on (α,β). Let a be a point in (α,β). Then the function $R(x)$ which is defined by*

$$f(x)=\sum_{n=0}^{\ell}\frac{f^{(n)}(a)}{n!}(x-a)^n+R(x)$$

satisfies

$$\lim_{x\to a}\frac{R(x)}{(x-a)^\ell}=0. \tag{1}$$

Proof. Without loss of generality, by translating the functions, we can suppose that $a=0$.

Consider the function $g(x)=f(x)-\sum_{r=0}^{\ell}\frac{f^{(r)}(0)}{r!}x^r$. It is clear that all of the derivatives of g of order $0,1,...,\ell$ are continuous, and they all vanish at $x=0$. It follows that $g^{(r-1)}(x)=\int_0^x g^{(r)}(t)dt$ for all $x\in\mathbb{R}$ and all $r=1,2,...\ell$.

So, for each $x\in\mathbb{R}$ we have

$$\begin{aligned} g(x) &= \int_0^x g'(t_1)dt_1=\int_0^x\left(\int_0^{t_1} g''(t_2)dt_2\right)dt_1\\ &= \int_0^x\left(\int_0^{t_1}\left(\int_0^{t_2} g^{(3)}(t_3)dt_3\right)dt_2\right)dt_1\\ &= \ldots\ldots\\ &= \int_0^x\left(\int_0^{t_1}\left(\int_0^{t_2}\ldots\ldots\left(\int_0^{t_{\ell-1}} g^{(\ell)}(t_\ell)dt_\ell\right)\ldots\ldots\right)dt_2\right)dt_1. \end{aligned} \tag{2}$$

We observe that, at all the stages of calculation of this repeated integral, the variables $t_1,t_2,....,t_{\ell-1}$ all have the same sign as x and all lie between 0 and x.

If $x\geq 0$ then it follows from (2) that

$$|g(x)|\leq\int_0^x\left(\int_0^x\left(\int_0^x\ldots\ldots\left(\int_0^x|g^{(\ell)}(t_\ell)|dt_\ell\right)\ldots\ldots\right)dt_2\right)dt_1=x^{\ell-1}\int_0^x|g^{(\ell)}(t_\ell)|dt_\ell.$$

Similarly, if $x\leq 0$, then

$$|g(x)|=\int_{-x}^0\left(\int_{-x}^0\left(\int_{-x}^0\ldots\ldots\left(\int_{-x}^0|g^{(\ell)}(t_\ell)|dt_\ell\right)\ldots\ldots\right)dt_2\right)dt_1=|x|^{\ell-1}\int_{-x}^0|g^{(\ell)}(t_\ell)|dt_\ell.$$

Thus in general we have

$$|g(x)|\leq|x|^{\ell-1}\int_{-|x|}^{|x|}|g^{(\ell)}(t_\ell)|dt_\ell. \tag{3}$$

Since $g^{(\ell)}$ is a continuous function we have $\lim_{y\searrow 0}\frac{1}{2y}\int_{-y}^{y}|g^{(\ell)}(t_\ell)|dt_\ell=|g^{(\ell)}(0)|=0$. This, together with (3) shows that the two sided limit $\lim_{x\to 0}x^{-\ell}g(x)$ exists and equals 0. Via an obvious translation, this is exactly the same as (1). □