Another exercise for the course on Wavelets and Interpolation spaces.

11/6/07.

(This continues the document of 30/5/07.)

This exercise relates to various ways of proving or trying to prove Proposition 2.8 on pp. 22–23 of [W] and maybe to some difficulties with some of those ways. It should be mainly a revision of things that you should know from previous courses.

Suppose that $w : [0, 2\pi] \to [0, \infty)$ is a function in $L^1([0, 2\pi])$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^2([0, 2\pi])$ which converges in L^2 norm to a function $u \in L^2([0, 2\pi])$. We want to decide whether or not this implies that

$$\lim_{n\to\infty} \int_0^{2\pi} u_n(x)w(x)dx = \int_0^{2\pi} u(x)w(x)dx. \tag{1}$$

Prove or disprove each of the following statements.

(a) The formula (1) is true in all cases.

(b) The formula (1) is true in all cases when we are also given that $u \in L^\infty([0, 2\pi])$.

(c) The formula (1) is true in all cases when we are also given that $u = \chi_E$ where E is an arbitrary measurable subset of $[0, 2\pi]$.

(d) The formula (1) is true in all cases when we are also given that $u = \chi_{[a,b]}$ for some interval $[a, b] \subset [0, 2\pi]$.

(e) The formula (1) is true in all cases when we are also given that the sequence u_n converges to u uniformly.

(f) If each u_n is the trigonometric polynomial given by $u_n(x) = \sum_{k=-n}^{n} \widehat{u}_k e^{ikx}$, where $\widehat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and $u = \chi_{[a,b]}$ for constants a and b such that $0 < a < b < 2\pi$, then the series u_n converges to u uniformly.

(g) The formula (1) is true in all cases when we are also given that the function w is in $L^\infty([0, 2\pi])$.

If you have difficulty with any exercise that I have asked you to solve, come and talk to me. Maybe I will have difficulties too and we can resolve them together.

Reference

[W] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Mathematical Society, Student Texts 37.
Cambridge University Press, 1997.