

ESTIMATES FOR COVERING NUMBERS IN SCHAUDER'S THEOREM ABOUT ADJOINTS OF COMPACT OPERATORS

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ABSTRACT. Let $T : X \rightarrow Y$ be a bounded linear map between Banach spaces X and Y . Let $T^* : Y^* \rightarrow X^*$ be its adjoint. Let \mathcal{B}_X and \mathcal{B}_{Y^*} be the closed unit balls of X and Y^* respectively. We obtain apparently new estimates for the covering numbers of the set $T^*(\mathcal{B}_{Y^*})$. These are expressed in terms of the covering numbers of $T(\mathcal{B}_X)$, or, more generally, in terms of the covering numbers of a “significant” subset of $T(\mathcal{B}_X)$. The latter more general estimates are best possible. These estimates follow from our new quantitative version of an abstract compactness result which generalizes classical theorems of Arzelà–Ascoli and of Schauder. Analogous estimates also hold for the covering numbers of $T(\mathcal{B}_X)$, in terms of the covering numbers of $T^*(\mathcal{B}_{Y^*})$ or in terms of a suitable “significant” subset of $T^*(\mathcal{B}_{Y^*})$.

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1. INTRODUCTION

The main motivation for the main result of this note is to give quantitative versions of the celebrated Schauder theorem about adjoints of compact operators. In fact our versions also apply to operators which are not compact.

When we first obtained these results it seemed hard to imagine that they are not already known. But so far we have not found any references to similar results in the literature. We invite the reader to inform us of any such references. In future versions of this note we hope to include a more extensive survey of related previous results.

Of course there are remarkable recent results by Shiri Artstein–Avidan, Vitali Milman, Stanislaw Szarek and Nicole Tomczak–Jaegermann [1, 2] which give estimates for covering numbers which look much much stronger than the ones we shall give here. But they deal with a slightly different kind of question. Still more recently, Emanuel Milman [9] has obtained results which in many cases also give substantially better estimates than those given below in our Corollary 8. However his estimates contain factors and exponents which depend on the dimension of the underlying space, and ours do not, at least not in any explicit way. Moreover,

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we obtain a quantitative version of a natural variant of Schauder's Theorem, (see Corollary 9) where our estimates are best possible.

Our point of departure is a theorem which is general enough to contain the classical theorems of Arzelà-Ascoli and of Schauder as special cases. But it, in turn, can be considered as a special case of considerably more abstract and general results presented by Robert G. Bartle in [3], and which have their roots in earlier work of Kakutani, R. S. Phillips and Šmulian.

The said theorem appears in [7] as a prelude to other results dealing with finitely additive means and semigroups of operators. It also appears in [5], where it is used as a tool to study complex interpolation of compact operators. Each of us obtained the theorem independently, and before we became aware of the earlier and more general results of [3].

We will state the theorem in the same formulation as is used in [5]. Perhaps we first need to recall that a *semimetric space* (E, d) (also often referred to as a *pseudometric space*) is a set E equipped with a function $d : E \times E \rightarrow [0, \infty)$ which is a *semimetric*, this meaning that it satisfies all the usual conditions for a metric, except that the condition $d(x, y) = 0$ does not imply that $x = y$. The definition of a totally bounded semimetric space (which will not be explicitly needed anyway in the sequel) is exactly analogous to that of a totally bounded metric space.

Theorem 1. (Cf. [5, 7]) *Let A and B be two sets and let $h : A \times B \rightarrow \mathbb{C}$ be a function with the properties that*

$$(1) \quad \sup_{a \in A} |h(a, b)| < \infty \text{ for each fixed } b \in B, \text{ and}$$

$$(2) \quad \sup_{b \in B} |h(a, b)| < \infty \text{ for each fixed } a \in A.$$

Define $d_A(a_1, a_2) := \sup_{b \in B} |h(a_1, b) - h(a_2, b)|$ for each pair of elements a_1 and a_2 in A .

Define $d_B(b_1, b_2) = \sup_{a \in A} |h(a, b_1) - h(a, b_2)|$ for each pair of elements b_1 and b_2 in B .

Then

$$(3) \quad (A, d_A) \text{ and } (B, d_B) \text{ are semimetric spaces}$$

and

$$(4) \quad (A, d_A) \text{ is totally bounded if and only if } (B, d_B) \text{ is totally bounded.}$$

The claim (3) in the previous theorem is of course a trivial consequence of the definitions of d_A and d_B and we shall use it in the sequel here without further comment.

There are proofs of Theorem 1 in both [5] and [7]. The proof in [7] (on p. 2) is shorter and simpler and has the further advantage that it can be adapted to give a quantitative version of the theorem, and therefore also of Schauder's Theorem. That is what we do in this note.

Before we can state our new results we need to fix and discuss some more (essentially standard) notation and terminology:

For each Banach space X we denote the closed unit ball of X by \mathcal{B}_X .

We use the standard notation of the "ceiling function", i.e., for each $t \in \mathbb{R}$ we let $\lceil t \rceil$ denote the smallest integer which dominates t .

By a *cube* in \mathbb{R}^n we will always mean a cube with sides parallel to the axes.

Let (E, d) be a semimetric space. We wish to define balls in (E, d) , the diameters of subsets of E , and two kinds of covering numbers for E . Our definitions will be mostly obvious variants of familiar ones for metric spaces. Covering numbers in metric spaces are often defined in terms of coverings by open balls of fixed given radius, (whose centres may or may not be required to be in some particular subset being considered). Here we will find it convenient to instead use coverings by “closed balls” or arbitrary sets of fixed given diameter. We have permitted ourselves these slight “perturbations” of the usual definitions in order to enable the convenient formulation of examples showing that at least some of our results are best possible. Of course our results can easily be translated into results corresponding to covering numbers defined via coverings using open balls.

For each subset G of E we define the *diameter* of G to (of course!) be the quantity

$$\text{diam}(G) = \sup_{x, y \in G} d(x, y).$$

For each $x \in E$ and each $r > 0$ we refer to the sets $\{y \in E : d(x, y) < r\}$ and $\{y \in E : d(x, y) \leq r\}$ respectively as the *open* and *closed balls of radius r centred at x* .

For each semimetric space (E, d) and each $\epsilon > 0$, the *intrinsic covering number* $N_E(\epsilon)$ is the least positive integer n for which there exists a finite subset $F \subset E$ of cardinality n such that

$$(5) \quad \min_{y \in F} d(x, y) \leq \epsilon \text{ for each } x \in E,$$

i.e., $N_E(\epsilon)$ is the smallest n such that E is contained in some union of n closed balls of radius ϵ . If no such $n \in \mathbb{N}$ exists, then $N_E(\epsilon) = \infty$.

We use the word “intrinsic” in this definition to reflect the fact that if E happens to be contained in a larger semimetric space and d is the restriction to $E \times E$ of the semimetric for that larger space, then we are requiring the centres of all the closed balls to be in E . This requirement will be relevant in the settings of Corollaries 8 and 9.

For each semimetric space (E, d) and each $\epsilon > 0$ we define the *diameter covering number* $N_E^\Delta(\epsilon)$ to be the smallest positive integer n for which there exist n subsets E_1, E_2, \dots, E_n of E , each having diameter not exceeding 2ϵ and for which $E \subset \bigcup_{j=1}^n E_j$. If no such positive integer n exists, then $N_E^\Delta(\epsilon) = \infty$.

The triangle inequality for the semimetric d immediately gives us some simple connections between intrinsic covering numbers and diameter covering numbers, namely

$$(6) \quad N_E(2\epsilon) \leq N_E^\Delta(\epsilon) \leq N_E(\epsilon) \text{ for all } \epsilon > 0.$$

Both of the inequalities in (6) can be strict. An example in Subsection 3.2 shows that we can even have $N_E(2\epsilon) = 1$ when $N_E^\Delta(\epsilon) = \infty$. The naive hope that each arbitrary subset of E having diameter not exceeding 2ϵ might be contained in a ball of radius ϵ is immediately shattered by the following example: In \mathbb{R}^2 with euclidean norm take E to be the interior of an equilateral triangle of side length 2ϵ . (In that case we have $N_E^\Delta(\epsilon) = 1$ and $N_E(\epsilon) = 2$.) However we can make the following simple observation, which will turn out to be relevant later in the setting where we will show that one of our results is best possible.

Fact 2. Let m be a positive integer, let $d(x, y) = \|x - y\|_{\ell_m^\infty}$ for each $x, y \in \mathbb{R}^m$ and let E be a subset of \mathbb{R}^m . Let G be a subset of E with $\text{diam}(G) \leq 2\epsilon$. Then there exists a point $x \in \mathbb{R}^m$ such that $G \subset \{y \in E : d(x, y) \leq \epsilon\}$.

Proof. For each $k = 1, 2, \dots, m$ and each $x \in \mathbb{R}^m$, let $\pi_k(x)$ denote the k^{th} coordinate of x . We let $\alpha_k = \inf \{\pi_k(x) : x \in G\}$ and $\beta_k = \sup \{\pi_k(x) : x \in G\}$. The finiteness of $\text{diam}(G)$ ensures that both α_k and β_k are finite. Then the bound on $\text{diam}(G)$ ensures that $0 \leq \beta_k - \alpha_k \leq 2\epsilon$. The point $x = \left(\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_2 + \beta_2}{2}, \dots, \frac{\alpha_m + \beta_m}{2}\right)$ clearly has the required property. \square

Remark 3. Note that the point x in Fact 2 does not have to belong to G or even to E . If G happens to be convex and closed then we can have $x \in G$ for $m \leq 2$. But this need not happen for $m \geq 3$, as one can see by considering the set $G \subset \mathbb{R}^3$ which is the convex hull of $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$.

We are now ready to state our main result, and then its corollaries. Their proofs will be given in Section 2.

Theorem 4. Let A and B be two sets and let $h : A \times B \rightarrow \mathbb{C}$ be a function with the properties stated in Theorem 1. Let d_A and d_B be the semimetrics defined on A and B respectively, as in Theorem 1.

Suppose that the intrinsic covering number $N_A(\epsilon)$ is finite for some positive number ϵ . Then

- (i) The quantity $C := \sup_{a \in A, b \in B} |h(a, b)|$ is also finite.
- (ii) The diameter covering number $N_B^\Delta(\rho)$ is finite for each $\rho > \epsilon$.
- (iii) Furthermore,

$$(7) \quad N_B^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2}C}{\delta} \right\rceil \right)^{2N_A(\epsilon)} \quad \text{for each } \delta > 0,$$

and, if h is real valued, the following stronger estimate also holds.

$$(8) \quad N_B^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{C}{\delta} \right\rceil \right)^{N_A(\epsilon)} \quad \text{for each } \delta > 0.$$

(iv) By symmetry, the roles of A and B can be interchanged and so exactly analogous estimates hold for $N_A^\Delta(\epsilon + \delta)$ in terms of $N_B(\epsilon)$.

Remark 5. Note that in this theorem we do not make any ‘‘compactness’’ or ‘‘total boundedness’’ assumptions about (A, d_A) or (B, d_B) .

Remark 6. In general, in Theorem 4, we cannot expect to have any estimate for the supremum C just in terms of $N_A(\epsilon)$. However we do have an estimate for C in the following particular case: Assume that A is an absolutely convex subset of a linear space V and that the semimetric d_A defined as above also satisfies

$$(9) \quad d_A(a, a') = p(a - a')$$

for some seminorm p on V and all $a, a' \in A$. Assume furthermore that the function $h : A \times B \rightarrow \mathbb{C}$ satisfies

$$(10) \quad h(0, b) = 0 \quad \text{for all } b \in B.$$

In this case, from straightforward “geometric” considerations, one can expect the supremum C to satisfy

$$(11) \quad C \leq \epsilon N_A(\epsilon),$$

and we will prove in an appendix (Section 4) that (11) indeed holds. Consequently, in this case, Theorem 4 gives us that

$$(12) \quad N_B^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2}\epsilon N_A(\epsilon)}{\delta} \right\rceil \right)^{2N_A(\epsilon)},$$

and, if we also have that h is real valued, we can sharpen this to

$$(13) \quad N_B^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{\epsilon N_A(\epsilon)}{\delta} \right\rceil \right)^{N_A(\epsilon)}.$$

In our first application (Corollary 8) of Theorem 4 to the context of Schauder's Theorem, the set A , the function h and the semimetric d_A satisfy the conditions of this particular case. But of course there we have a simpler and, in general, better way to estimate C . We apparently cannot invoke an analogue of (11) in Corollary 9.

Remark 7. One of the examples to be presented in Section 3 will show that the estimate (8) cannot be sharpened. On the other hand, it will be clear from the proof below that the estimates (7) and (12) can be somewhat improved, either by making some small efforts in planar geometry or by consulting some appropriate references, in order to sharpen the claim (ii) in Fact 12. Another example in Section 3 will show that the requirement that $\rho > \epsilon$ in part (ii) of Theorem 4 cannot be weakened to $\rho \geq \epsilon$.

Here is the obvious simplest way that we can apply Theorem 4.

Corollary 8. *Let X and Y be Banach spaces and let \mathcal{B}_X and \mathcal{B}_{Y^*} be the closed unit balls of X and Y^* respectively.*

Let $T : X \rightarrow Y$ be a bounded linear operator with adjoint $T^ : Y^* \rightarrow X^*$.*

For each $\epsilon > 0$ let $N_T(\epsilon)$ denote the least number of closed balls in Y of radius ϵ with centres in $T(\mathcal{B}_X)$ which are required to cover the set $T(\mathcal{B}_X)$, and let $N_T^\Delta(\epsilon)$ denote the least number of subsets of Y each with Y -norm diameter not exceeding 2ϵ which are required to cover $T(\mathcal{B}_X)$.

Analogously, let $N_{T^}(\epsilon)$ denote the least number of closed balls in X^* of radius ϵ with centres in $T^*(\mathcal{B}_{Y^*})$ which are required to cover the set $T^*(\mathcal{B}_{Y^*})$ and let $N_{T^*}^\Delta(\epsilon)$ denote the least number of subsets of X^* each with X^* -norm diameter not exceeding 2ϵ which are required to cover $T^*(\mathcal{B}_{Y^*})$.*

Suppose that $N_T(\epsilon)$ is finite for some particular $\epsilon > 0$. Then $N_{T^}^\Delta(\rho)$ is finite for all $\rho > \epsilon$ and the estimate*

$$(14) \quad N_{T^*}^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{\sqrt{2} \|T\|_{X \rightarrow Y}}{\delta} \right\rceil \right)^{2N_T(\epsilon)}$$

holds for all $\delta > 0$. If X and Y are real Banach spaces, then this estimate can be sharpened to

$$(15) \quad N_{T^*}^\Delta(\epsilon + \delta) \leq \left(\left\lceil \frac{\|T\|_{X \rightarrow Y}}{\delta} \right\rceil \right)^{N_T(\epsilon)}.$$

Furthermore, if $N_{T^*}(\epsilon)$ is finite for some $\epsilon > 0$, then $N_T^\Delta(\rho)$ is finite for all $\rho > \epsilon$ and the quantity $N_T^\Delta(\epsilon + \delta)$ can be estimated in terms of $N_{T^*}(\epsilon)$ via formulae exactly analogous to (14) and (15), where T and T^* are interchanged.

Apparently other results will give much better estimates than (14) and (15). But here is a slightly more subtle variant of Corollary 8 for which, in some cases, our estimates are best possible. With the perspective of Theorem 4 we can see that it may be just as appropriate and just as easy to work with the covering numbers of certain “significant” subsets of $T(\mathcal{B}_X)$ and of $T^*(\mathcal{B}_{Y^*})$, instead of working with the covering numbers of these sets themselves. We will obtain new versions of the estimates (14) and (15) for $N_T^\Delta(\epsilon + \delta)$, which are stronger in the sense that the number $N_T(\epsilon)$ is replaced by a smaller, in some cases very much smaller number, which is the covering number of a suitable subset K of $T(\mathcal{B}_X)$. Similarly the estimates for $N_T^\Delta(\epsilon + \delta)$, which were stated implicitly in Corollary 8, can be replaced by stronger results where $N_{T^*}(\epsilon)$ is replaced by the covering number of a suitable subset K^* of $T^*(\mathcal{B}_{Y^*})$.

Corollary 9. *Let $X, Y, T, N_T^\Delta(\epsilon)$, and $N_{T^*}^\Delta(\epsilon)$ all be as specified in the statement of Corollary 8. Let K be a “norming” subset of $T(\mathcal{B}_X)$, i.e., a subset with the property that*

$$(16) \quad \sup \{ |\langle u, y \rangle| : u \in K \} = \sup \{ |\langle u, y \rangle| : u \in T(\mathcal{B}_X) \} \text{ for each } y \in Y^* .$$

Analogously, let K^* be a subset of $T^*(\mathcal{B}_{Y^*})$ with the property that

$$(17) \quad \sup \{ |\langle x, v \rangle| : v \in K^* \} = \sup \{ |\langle x, v \rangle| : v \in T^*(\mathcal{B}_{Y^*}) \} \text{ for each } x \in X .$$

For each $\epsilon > 0$ let $N[K, \epsilon]$ be the least number of closed balls in Y of radius ϵ with centres in K which are required to cover the set K . Analogously, let $N[K^*, \epsilon]$ denote the least number of closed balls in X^* of radius ϵ with centres in K^* which are required to cover the set K^* .

Suppose that $N[K, \epsilon]$ is finite for some particular $\epsilon > 0$. Then $N_{T^*}^\Delta(\rho)$ is finite for all $\rho > \epsilon$ and the estimate

$$(18) \quad N_{T^*}^\Delta(\epsilon + \delta) \leq \left(\left[\frac{\sqrt{2} \|T\|_{X \rightarrow Y}}{\delta} \right] \right)^{2N[K, \epsilon]} .$$

holds for all $\delta > 0$. If X and Y are real Banach spaces then this estimate can be sharpened to

$$(19) \quad N_{T^*}^\Delta(\epsilon + \delta) \leq \left(\left[\frac{\|T\|_{X \rightarrow Y}}{\delta} \right] \right)^{N[K, \epsilon]} .$$

Analogously, if $N[K^*, \epsilon]$ is finite for some particular $\epsilon > 0$, then $N_T^\Delta(\rho)$ is finite for all $\rho > \epsilon$ and the estimate

$$(20) \quad N_T^\Delta(\epsilon + \delta) \leq \left(\left[\frac{\sqrt{2} \|T\|_{X \rightarrow Y}}{\delta} \right] \right)^{2N[K^*, \epsilon]}$$

holds for all $\delta > 0$, and in the case where X and Y are real Banach spaces, it can be sharpened to

$$(21) \quad N_T^\Delta(\epsilon + \delta) \leq \left(\left[\frac{\|T\|_{X \rightarrow Y}}{\delta} \right] \right)^{N[K^*, \epsilon]} .$$

Remark 10. Obviously Corollary 8 is nothing more than a special case of Corollary 9 since of course the sets $K = T(\mathcal{B}_X)$ and $K^* = T^*(\mathcal{B}_{Y^*})$ satisfy (16) and (17). But it seems better and clearer to have begun this discussion by stating that special case separately.

What reasonable kinds of sets might play the roles of K and of K^* in Corollary 9? Obviously we can take $K = \{Tx : \|x\|_X = 1\}$ and $K^* = \{T^*y : \|y\|_{Y^*} = 1\}$. But for such choices we cannot expect $N[K, \epsilon]$ and $N[K^*, \epsilon]$ to be substantially smaller than, respectively, $N_T(\epsilon)$ and $N_{T^*}(\epsilon)$. In finite dimensional spaces, if \mathcal{B}_X is the convex hull (or absolutely convex hull) of some finite set F , then K can be chosen to be $T(F)$. (We are aware of at least two papers, namely [4] and [6], which could be applied to give connections between the covering numbers, or entropy numbers, of $T(F)$ and of $T(\mathcal{B}_X)$ in such cases.) If T has some special additional properties, then it might be possible to choose an even smaller set than $T(F)$ in the role of K .

Here is what is probably the most natural example of a choice of K which satisfies (16) and for which $N[K, \epsilon]$ is very significantly smaller than $N_T(\epsilon)$. Fix a positive integer n and let X and Y both be \mathbb{R}^n equipped with the ℓ^1 norm, and let T be the identity operator on \mathbb{R}^n . Let K be the subset of \mathcal{B}_X which consists of the n points e_j for $j = 1, 2, \dots, n$, where $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, 0, \dots, 0, 1)$. Of course $N_T(\epsilon)$ is arbitrarily large for small values of ϵ . But $N[K, \epsilon] = n$ for all ϵ in the range $0 < \epsilon < 1$. Of course in (16) we take $\langle \cdot, \cdot \rangle$ to be the usual inner product on \mathbb{R}^n , and so X^* and Y^* are both \mathbb{R}^n equipped with the ℓ^∞ norm. Clearly (16) holds here since, for each $y \in \mathbb{R}^n$, both sides of (16) equal $\|y\|_{\ell^\infty}$.

Let us now use this example to show that in general the estimate (19) cannot be improved. In our context here (cf. Fact 2), for each $\rho > 0$, we see that $N_{T^*}^\Delta(\rho)$ is the minimal number of closed cubes in \mathbb{R}^n of side length 2ρ required to cover all of the closed cube $Q = [-1, 1]^n$. The interval $[-1, 1]$ is contained in the union of $\left\lceil \frac{2}{2\rho} \right\rceil$ non overlapping closed intervals of length 2ρ , and so it is clear that $N_{T^*}^\Delta(\rho) \leq \left(\left\lceil \frac{1}{\rho} \right\rceil \right)^n$. By trivial considerations of volume, we must also have $(2\rho)^n N_{T^*}^\Delta(\rho) \geq 2^n$. It follows that, for each positive integer m , we have $N_{T^*}^\Delta(\rho) = m^n$ for all numbers ρ which satisfy $m^n - 1 < \rho^{-n} \leq m^n$. In particular this will hold whenever we choose $\rho = (m^n - \theta)^{-1/n}$ for some number $\theta \in (0, 1)$. For our purposes we will choose a particular value of $\theta \in (0, 1)$ which is sufficiently small to ensure that

$$(22) \quad m - 1 < (m^n - \theta)^{1/n} < m.$$

For each number ϵ which is in the range $0 < \epsilon < \min \left\{ 1, (m^n - \theta)^{-1/n} \right\}$ we have $N[K, \epsilon] = n$ and we also have that the number $\delta = \delta(\epsilon)$ which satisfies $\epsilon + \delta = (m^n - \theta)^{-1/n}$ is positive. For each such number ϵ and for each corresponding $\delta = \delta(\epsilon)$, the left side of (19) equals m^n and the right side equals $\left(\left\lceil \frac{\|T\|_{X \rightarrow Y}}{\delta} \right\rceil \right)^n = \left(\left\lceil \frac{1}{(m^n - \theta)^{-1/n} - \epsilon} \right\rceil \right)^n = \left(\left\lceil \frac{(m^n - \theta)^{1/n}}{1 - \epsilon(m^n - \theta)^{1/n}} \right\rceil \right)^n$. Since we have chosen θ to satisfy (22), we can now choose ϵ sufficiently small, so that the number $\frac{(m^n - \theta)^{1/n}}{1 - 2\epsilon(m^n - \theta)^{1/n}}$ also lies in the open interval $(m - 1, m)$. This makes the right side of (19) also equal to m^n . So indeed the estimate (19) is best possible for certain values of the numbers ϵ and δ , and in fact for infinitely many such values, which can be taken arbitrarily small.

Perhaps an analogous example and similar arguments might show that a variant of (18), in which the quantity $\left[\frac{\sqrt{2}\|T\|_{X \rightarrow Y}}{\delta}\right]^2$ is replaced by the intrinsic covering number $N_{\mathbb{D}}\left(\frac{\delta}{\|T\|_{X \rightarrow Y}}\right)$, is also best possible. Here \mathbb{D} denotes the unit disk in \mathbb{C} with the usual metric. But we leave further thoughts about this to a subsequent version of this paper.

Remark 11. If we change this example so that $X = Y$ is \mathbb{R}^n equipped with the ℓ^∞ norm, then we can of course take K to consist of the 2^n vertices of the cube of side length 2 centred at the origin, i.e., the extreme points of \mathcal{B}_X . Or K may consist merely of half of these points. Perhaps we will look at this example in more detail in a subsequent version of this paper.

2. PROOFS

2.1. The proof of Corollary 9 (and therefore also of Corollary 8) .

It is rather obvious what needs to be done, even more so in the case where $K = T(\mathcal{B}_X)$ and $K^* = T^*(\mathcal{B}_{Y^*})$. But let us write out the proof explicitly, at least in this preliminary version of our paper:

Let $A = \{x \in \mathcal{B}_X : Tx \in K\}$. We clearly have $T(A) = K$. We also let $B = \mathcal{B}_{Y^*}$. Define $h : A \times B \rightarrow \mathbb{C}$ by $h(a, b) = \langle Ta, b \rangle$. Then we have

$$(23) \quad \sup_{a \in A, b \in B} |h(a, b)| \leq \|T\|_{X \rightarrow Y} = \|T^*\|_{Y^* \rightarrow X^*} ,$$

and also, by the Hahn–Banach Theorem,

$$(24) \quad d_A(a_1, a_2) = \|Ta_1 - Ta_2\|_Y \text{ for all } a_1, a_2 \in A .$$

Let ϵ be a positive number for which $N[K, \epsilon]$ is finite and equals n . Then there exists a set of n points $\{y_1, y_2, \dots, y_n\}$ in K such that $\min\{\|y - y_j\|_Y : 1 \leq j \leq n\} \leq \epsilon$ for each $y \in K$. Since $K = T(A)$, we have a set of n points $\{x_1, x_2, \dots, x_n\}$ in A such that $Tx_j = y_j$ for each $j = 1, 2, \dots, n$.

For each $a \in A$ we have $Ta \in K$ and therefore

$$\min\{d_A(a, x_j) : 1 \leq j \leq n\} = \min\{\|Ta - Tx_j\|_Y : 1 \leq j \leq n\} \leq \epsilon .$$

This shows that

$$(25) \quad N_A(\epsilon) \leq N[K, \epsilon] \text{ for all } \epsilon > 0 .$$

In order to obtain a formula for d_B , we use the fact that $h(a, b)$ is also equal to $\langle a, T^*b \rangle$. For each pair of elements $b_1, b_2 \in B$ we again use the fact that $T(A) = K$ and we apply (16) to obtain that

$$(26) \quad \begin{aligned} d_B(b_1, b_2) &= \sup_{a \in A} |\langle Ta, b_1 - b_2 \rangle| = \sup_{u \in K} |\langle u, b_1 - b_2 \rangle| = \sup_{u \in T(\mathcal{B}_X)} |\langle u, b_1 - b_2 \rangle| \\ &= \sup_{x \in \mathcal{B}_X} |\langle Tx, b_1 - b_2 \rangle| = \sup_{x \in \mathcal{B}_X} |\langle x, T^*(b_1 - b_2) \rangle| = \|T^*b_1 - T^*b_2\|_{X^*} . \end{aligned}$$

Next we will show that

$$(27) \quad N_{T^*}^{\Delta}(\rho) \leq N_B^{\Delta}(\rho) \text{ for all } \rho > 0 .$$

Let ρ be an arbitrary positive number. If $m = N_B^{\Delta}(\rho)$ then there exists a collection W_1, W_2, \dots, W_m of m subsets of $B = \mathcal{B}_{Y^*}$ such that $\sup_{b, b' \in W_j} d_B(b, b') \leq 2\rho$ for

each $j = 1, 2, \dots, m$ and $B \subset \bigcup_{j=1}^m W_j$. Let $V_j = T^*(W_j)$ for each j . Each set V_j is contained in $T^*(\mathcal{B}_{Y^*})$ and, in view of (26),

$$(28) \quad \sup_{v, v' \in V_j} \|v - v'\|_{X^*} = \sup_{b, b' \in W_j} \|T^*b - T^*b'\|_{X^*} = \sup_{b, b' \in W_j} d_B(b, b') \leq 2\rho.$$

We also have $T^*(\mathcal{B}_{Y^*}) \subset \bigcup_{j=1}^m V_j$ which, together with (28), gives us (27).

Now to obtain (18) and (19) we simply have to substitute (27), (23) and (25) in (7) and (8) respectively.

The rest of the proof, i.e., of the estimates (20) and of (21), is a variant of the arguments to obtain (18) and (19) that we have just presented. For this last part of the proof we will choose $A = \{y \in \mathcal{B}_{Y^*} : T^*y \in K^*\}$ and $B = \mathcal{B}_X$ and $h(a, b) = \langle Tb, a \rangle = \langle b, T^*a \rangle$, and then use reasoning which is almost exactly analogous to that used above. beginning with the observation that $T^*(A) = K^*$. We will only need to make two small and obvious changes:

- The formula for d_A , i.e., $d_A(a_1, a_2) = \|T^*a_1 - T^*a_2\|_{X^*}$, whose analogue in (24) followed from the Hahn–Banach Theorem, here is simply a consequence of the definition of the norm of a linear functional.
- The last step in (26) above simply used the definition of the norm of a linear functional. The analogous step here, which shows that $d_B(b_1, b_2) = \|Tb_1 - Tb_2\|_Y$, will use the Hahn–Banach theorem.

We leave the details to the reader. \square

2.2. How to cover big intervals by small intervals, and big disks by small disks.

This very simple geometrical observation will be one of the components in our proof of Theorem 4.

Fact 12. *Let δ and C be numbers satisfying $0 < \delta \leq C$.*

(i) *The interval $[-C, C]$ can be contained in the union of $\lceil \frac{C}{\delta} \rceil$ closed intervals I_q , $q = 1, 2, \dots, \lceil \frac{C}{\delta} \rceil$, each of length 2δ .*

(ii) *The closed disk $\{z \in \mathbb{C} : |z| \leq C\}$ can be contained in the union of $\lceil \frac{\sqrt{2}C}{\delta} \rceil^2$ closed disks D_q , $q = 1, 2, \dots, \lceil \frac{\sqrt{2}C}{\delta} \rceil^2$ each of radius δ .*

Proof. The claim (i) is trivial and needs no further comment.

The number of disks mentioned in the claim (ii) can certainly be reduced. However, in this preliminary version of the paper, we will not seek the optimal value of this quantity, but simply content ourselves with some quite crude, simple minded estimates. First we remark that each closed disc of radius δ contains a closed square of side length $\sqrt{2}\delta$. If the positive integer k satisfies $k \geq 2C/\sqrt{2}\delta$ then we will be able to cover a closed square of side length $2C$ with k^2 closed squares of side length $\sqrt{2}\delta$. So we will certainly be able to cover a closed disc of radius C with k^2 closed discs of radius δ , where $k^2 = \lceil \frac{\sqrt{2}C}{\delta} \rceil^2$. \square

2.3. The proof of Theorem 4.

Throughout this proof the number $\epsilon > 0$, for which we suppose that $N_A(\epsilon)$ is finite, will remain fixed, and n will always equal $N_A(\epsilon)$. We also fix a finite n element subset $F = \{a_1, a_2, \dots, a_n\}$ of A which has the property

$$(29) \quad \min \{d_A(a, a_j) : j \in \{1, 2, \dots, n\}\} \leq \epsilon.$$

At least one such subset necessarily exists.

C will always denote the supremum $C := \sup_{a \in A, b \in B} |h(a, b)|$.

Our first task is to show that the supremum C is finite: Let a and a' be two arbitrary elements of A . Let $C_F = \max_{j, k \in \{1, 2, \dots, n\}} d_A(a_j, a_k)$. Then, for each j and k in $\{1, 2, \dots, n\}$ we have

$$d_A(a, a') \leq d_A(a, a_j) + d_A(a_j, a_k) + d(a_k, a').$$

By making appropriate choices of j and k we obtain that

$$(30) \quad d_A(a, a') \leq 2\epsilon + C_F.$$

Now, for arbitrary $a \in A$ and $b \in B$, we have

$$\begin{aligned} |h(a, b)| &\leq |h(a, b) - h(a_1, b)| + |h(a_1, b)| \\ &\leq d_A(a, a_1) + \sup_{b' \in B} |h(a_1, b')|. \end{aligned}$$

In view of (30) and (2) this last expression is finite and does not depend on a or b . So our assumption that $N_A(\epsilon)$ is finite indeed ensures the finiteness also of C .

Let δ be an arbitrary positive number. It remains to show that $N_B^\Delta(\epsilon + \delta)$ is finite and satisfies the required estimates.

It will be convenient to simultaneously consider both the general case and the special case where h is real valued. To facilitate this we will let \mathbb{K} denote the real field \mathbb{R} if h is real valued, and otherwise we will have $\mathbb{K} = \mathbb{C}$.

Let us now consider the set $G = \{g_b : b \in B\}$ of functions $g_b : F \rightarrow \mathbb{K}$ defined by the formula $g_b(f) = h(f, b)$. G is of course contained in the set $Q \subset \ell^\infty(F)$ consisting of those functions $g : F \rightarrow \mathbb{K}$ for which $|g(f)| \leq C$ for each f .

If $\mathbb{K} = \mathbb{C}$ then we let $M = \left\lceil \frac{\sqrt{2}C}{\delta} \right\rceil^2$ and let D_1, D_2, \dots, D_M be the M closed discs of radius δ in the complex plane provided by Fact 12, whose union contains the closed disk $\{z \in \mathbb{C} : |z| \leq C\}$. For each n -tuple $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, where each μ_j is an integer in the range $1 \leq \mu_j \leq M$, let $Q(\mu_1, \mu_2, \dots, \mu_n) = Q(\vec{\mu})$ be the set of all functions $g : F \rightarrow \mathbb{C}$ such that $g(a_j) \in D_{\mu_j}$ for $j = 1, 2, \dots, n$. Furthermore, let $B(\mu_1, \mu_2, \dots, \mu_n) = B(\vec{\mu})$ be the set of all $b \in B$ for which the function g_b defined as above is in $Q(\mu_1, \mu_2, \dots, \mu_n)$.

The inclusion $\{z \in \mathbb{C} : |z| \leq C\} \subset \bigcup_{q=1}^M D_q$ implies that $G \subset \bigcup_{\vec{\mu}} Q(\vec{\mu})$ which in turn gives us that $B \subset \bigcup_{\vec{\mu}} B(\vec{\mu})$, i.e., that B is contained in the union of the M^n sets $B(\vec{\mu})$.

Analogously, if $\mathbb{K} = \mathbb{R}$ then we choose a different value for M , namely $M = \left\lceil \frac{C}{\delta} \right\rceil$, and we let I_1, I_2, \dots, I_M be the M closed intervals of length 2δ provided by Fact 12, whose union contains the closed interval $[-C, C]$. For each n -tuple $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, where each μ_j is an integer in the range $1 \leq \mu_j \leq M$, let $Q(\mu_1, \mu_2, \dots, \mu_n) = Q(\vec{\mu})$ be the set of all functions $g : F \rightarrow \mathbb{R}$ such that $g(a_j) \in I_{\mu_j}$ for $j = 1, 2, \dots, n$. Furthermore, let $B(\mu_1, \mu_2, \dots, \mu_n) = B(\vec{\mu})$ be the set of all $b \in B$ for which the function g_b defined as above is in $Q(\mu_1, \mu_2, \dots, \mu_n)$. As before, this time using the inclusion $[-C, C] \subset \bigcup_{q=1}^M I_q$, we show that B is contained in the union of the M^n sets $B(\vec{\mu})$, this time of course for this different choice of M .

Now we shall estimate the diameter of each set $B(\vec{\mu})$. We will use the same argument for both of the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$. Fix some $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$ and let b and b' be two arbitrary elements of $B(\vec{\mu})$. Fix some arbitrary positive

number ρ and let a be an element of A (possibly depending on ρ) for which

$$d_B(b, b') \leq |h(a, b) - h(a, b')| + \rho.$$

Next, using (29), we pick some $a_j \in F$ (possibly depending on a and therefore on ρ) for which $d_A(a, a_j) \leq \epsilon$. Then we have

$$\begin{aligned} & |h(a, b) - h(a, b')| \\ & \leq |h(a, b) - h(a_j, b)| + |h(a_j, b) - h(a_j, b')| + |h(a_j, b') - h(a, b')| \\ & \leq \epsilon + |g_b(a_j) - g_{b'}(a_j)| + \epsilon. \end{aligned}$$

By definition, when $\mathbb{K} = \mathbb{C}$ the two numbers $g_b(a_j)$ and $g_{b'}(a_j)$ are either both in the same disk D_{μ_j} of radius δ , and when $\mathbb{K} = \mathbb{R}$ they are both in the same interval I_{μ_j} of length 2δ . So, for both $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$, we have

$$|g_b(a_j) - g_{b'}(a_j)| \leq 2\delta$$

and the previous three displayed estimates combine to give us that $d_B(b, b') \leq 2\epsilon + 2\delta + \rho$. Since b and b' are arbitrary and ρ can be chosen arbitrarily small, we obtain that the diameter of $B(\bar{\mu})$ does not exceed $2\epsilon + 2\delta$ and therefore $N_B^\Delta(\epsilon + \delta) \leq M^n$, which is exactly (7) when $\mathbb{K} = \mathbb{C}$ or exactly (8) when $\mathbb{K} = \mathbb{R}$. \square

3. TWO MORE EXAMPLES

The two examples which we present here apply to the rather more general context of Theorem 4, rather than to Corollary 9.

3.1. The estimate (8) in Theorem 4 is best possible.

Our example in this subsection is a simpler version of the example discussed in Section 1. It shows (as also follows indirectly from the example of Section 1) that (8) cannot be improved:

Let m and n be positive integers. Let A be the set of n canonical vectors \vec{e}_j , $j = 1, 2, \dots, n$ in \mathbb{R}^n , i.e., $\vec{e}_1 = (1, 0, 0, \dots, 0)$, $\vec{e}_2 = (0, 1, 0, \dots, 0)$, \dots , $\vec{e}_n = (0, 0, 0, \dots, 0, 0, 1)$. Divide the cube $[-1, 1]^n$ in \mathbb{R}^n into m^n non overlapping closed cubes, each of side length $2/m$. Let B be the set of m^n points $\vec{b} = (\beta_1, \beta_2, \dots, \beta_n)$ which are the centres of these cubes. Let $h(\vec{a}, \vec{b})$ be the usual inner product of vectors in \mathbb{R}^n . Thus $d_B(\vec{b}, \vec{b}') = \left\| \vec{b} - \vec{b}' \right\|_{\ell_n^\infty}$ for each \vec{b} and \vec{b}' in B , and $d_A(\vec{e}_j, \vec{e}_k) = 2 - 2/m$ whenever $j \neq k$. This means that $N_A(\rho) = n$ for all $\rho \in (0, 1 - 1/m)$. We also have $\sup_{\vec{a} \in A, \vec{b} \in B} |h(\vec{a}, \vec{b})| = 1 - 1/m$. Since any set in \mathbb{R}^n whose ℓ_n^∞ diameter is less than $2/m$ can contain at most one of the points of B , we see that $N_B^\Delta(\rho) = m^n$ for each $\rho \in (0, 1/m)$.

Suppose that $m \geq 2$, and choose positive numbers $\epsilon = \frac{1}{3m^2}$ and $\delta = \frac{1}{m} - \frac{1}{2m^2}$. Note that $\epsilon + \delta < 1/m$ and $\epsilon < 1 - 1/m$. So, in this case, $N_B^\Delta(\epsilon + \delta) = m^n$ and $N_A(\epsilon) = n$. Furthermore the number

$$\frac{C}{\delta} = \frac{1 - \frac{1}{m}}{\frac{1}{m} - \frac{1}{2m^2}} = \frac{m-1}{1 - \frac{1}{2m}}$$

lies in the interval $(m-1, m]$. Consequently, $(\lceil \frac{C}{\delta} \rceil)^{N_A(\epsilon)} = m^n = N_B^\Delta(\epsilon + \delta)$ and in this case equality holds in (8).

3.2. The statement (ii) in Theorem 4 is best possible.

Our example in this subsection addresses the statement (ii) in Theorem 4. In order to show that (ii) cannot be sharpened in any way, we will give an example of sets A and B and a function $h : A \times B \rightarrow \mathbb{R}$ satisfying the hypotheses of Theorem 4, such that, when $\epsilon = 1/2$, we have $N_A(\epsilon) < \infty$ and therefore $N_B^\Delta(\rho) < \infty$ for all $\rho > \epsilon$, but $N_B^\Delta(\epsilon) = \infty$ and therefore of course $N_B^\Delta(\rho) = \infty$ for all $\rho \in (0, \epsilon]$.

In view of (6), statement (ii) of Theorem 4 implies the following similar statement, expressed solely in terms of intrinsic covering numbers:

$$(31) \quad \text{If } N_A(\epsilon) < \infty \text{ then } N_B(\rho) < \infty \text{ for all } \rho > 2\epsilon.$$

The same example which we are about to present will also show that the number 2ϵ in (31) cannot be replaced by an smaller number. I.e., we can have $N_A(\epsilon) < \infty$ but $N_B(\rho) = \infty$ for all $\rho \in (0, 2\epsilon)$. Here again we will be choosing $\epsilon = 1/2$. Since in our example $N_B(1) = 1$, this does not settle the question of whether or not we can always replace $\rho > 2\epsilon$ by $\rho \geq 2\epsilon$ in (31).

Here are the details of the example:

We take A to be the set of all real valued sequences $a = \{\alpha_n\}_{n \in \mathbb{N}}$ which take values in $[0, 1]$. We take B to be the set of all real valued sequences $b = \{\beta_n\}_{n \in \mathbb{N}}$ which are finitely supported and satisfy $\sum_{n=1}^{\infty} |\beta_n| \leq 1$. For each $a = \{\alpha_n\}_{n \in \mathbb{N}}$ in A and each $b = \{\beta_n\}_{n \in \mathbb{N}}$ in B , let $h(a, b) = \sum_{n=1}^{\infty} \alpha_n \beta_n$. Then of course $d_A(a - a') = \|a - a'\|_{\ell^\infty}$ for all $a, a' \in A$, and, for each $b = \{\beta_n\}_{n \in \mathbb{N}}$ and $b' = \{\beta'_n\}_{n \in \mathbb{N}}$, we have $d_B(b, b') = \max \left\{ \sum_{n=1}^{\infty} (\beta_n - \beta'_n)_+, \sum_{n=1}^{\infty} (\beta'_n - \beta_n)_+ \right\}$.

Let x be the sequence $x = \{\xi_n\}_{n \in \mathbb{N}}$ where $\xi_n = 1/2$ for all n . Then $d_A(a, x) \leq 1/2$ for all $a \in A$. This means that $N_A(1/2) = 1 < \infty$. Since $d_B(b, 0) \leq 1$ for all $b \in B$, we see that $N_B(1) = 1$ and also (cf. 6) that $N_B^\Delta(1) = 1$.

Now let us show that $N_B(\rho) = \infty$ whenever $0 < \rho < 1$: Suppose on the contrary that $N_B(\rho) = m < \infty$ for some $\rho \in (0, 1)$. Then there exist m sequences in B , which we will denote by $b_j = \{\beta_{j,n}\}_{n \in \mathbb{N}}$ for $j = 1, 2, \dots, m$, such that

$$(32) \quad \min \{d_B(b, b_j) : 1 \leq j \leq m\} \leq \rho \text{ for all } b \in B.$$

Since each of the sequences b_j is finitely supported, there exists an integer q such that $\beta_{j,q} = 0$ for $j = 1, 2, \dots, m$. Now let $b = \{\beta_n\}_{n \in \mathbb{N}}$ be the sequence for which $\beta_q = 1$ and $\beta_n = 0$ for all $n \neq q$. Then $d_B(b, b_j) \geq 1$ for all j , which contradicts (32). Thus indeed, for each $\rho \in (0, 1)$, we have $N_B(\rho) = \infty$.

For each $m \in \mathbb{N}$ let b_m be the sequence $\{\beta_{m,n}\}_{n \in \mathbb{N}}$ which has only two non zero elements, namely $\beta_{m,2m-1} = \frac{1}{2} + \frac{1}{2m}$ and $\beta_{m,2m} = -\frac{1}{2} + \frac{1}{2m}$. Then $\sum_{n=1}^{\infty} |\beta_{m,n}| = 1$ so that $b_m \in B$. For each $m, k \in \mathbb{N}$ with $m \neq k$ we have

$$\begin{aligned} d_B(b_m, b_k) &= \max \left\{ \left(\frac{1}{2} + \frac{1}{2m} \right) + \left(\frac{1}{2} - \frac{1}{2k} \right), \left(\frac{1}{2} + \frac{1}{2k} \right) + \left(\frac{1}{2} - \frac{1}{2m} \right) \right\} \\ &= 1 + \frac{1}{2} \left| \frac{1}{m} - \frac{1}{k} \right| > 1. \end{aligned}$$

This shows that each subset of B whose diameter does not exceed 1 can contain at most one of the elements b_m . Therefore $N_B^\Delta(1/2) = \infty$.

We have thus shown that the covering numbers N_A , N_B^Δ and N_B have all the properties which were promised at the beginning of this subsection.

4. APPENDIX: FURTHER DETAILS ABOUT THE CASE WHERE A IS ABSOLUTELY CONVEX

Here is a proof of the estimate (11) which was mentioned in Remark 6:

Let ϵ be the positive number appearing in (11), let $n = N_A(\epsilon)$, and let $F = \{a_1, a_2, \dots, a_n\}$ be the finite subset of A depending on ϵ which is introduced at the beginning of the proof of Theorem 4. If $\Gamma_j = \{a \in A : d_A(a, a_j) \leq \epsilon\}$ for each j , then (29) can be restated as $A \subset \bigcup_{j=1}^n \Gamma_j$.

Since $h(0, b) = 0$ for all $b \in B$ and since d_A is related to the seminorm p by (9), we see that

$$(33) \quad C := \sup_{a \in A, b \in B} |h(a, b)| = \sup_{a \in A, b \in B} |h(a, b) - h(0, b)| = \sup_{a \in A} d_A(a, 0) = \sup_{a \in A} p(a).$$

Let a_* be an arbitrary fixed element of A . The set $L = \{(1 - 2t)a_* : t \in [0, 1]\}$, i.e., the line segment in V from a_* to $-a_*$, is contained in A and therefore also in $\bigcup_{j=1}^n \Gamma_j$. So the interval $[0, 1]$ satisfies

$$(34) \quad [0, 1] \subset \bigcup_{j=1}^n I_j,$$

where the sets I_j are defined by

$$I_j = \{t \in [0, 1] : (1 - 2t)a_* \in \Gamma_j\} = \{t \in [0, 1] : p(a_* - 2ta_* - a_j) \leq \epsilon\}.$$

For each j the non negative function $t \mapsto p(a_* - 2ta_* - a_j)$ is continuous, and in fact it is also convex. So each I_j is a closed (possibly empty) subinterval of $[0, 1]$. It follows from (34) that at least one of these intervals, say I_m , must have length no less than $1/n$. Let s be a point in I_m such that $s + 1/n$ is also in I_m . Then

$$\begin{aligned} \frac{2}{n}p(a_*) &= p\left(\frac{2a_*}{n}\right) = p\left((a_* - 2sa_* - a_m) - \left(a_* - 2\left(s + \frac{1}{n}\right)a_* - a_m\right)\right) \\ &\leq p(a_* - 2sa_* - a_m) + p\left(a_* - 2\left(s + \frac{1}{n}\right)a_* - a_m\right) \leq \epsilon + \epsilon. \end{aligned}$$

This shows that $p(a_*) \leq \epsilon n$. So now we simply take the supremum in this last inequality as a_* ranges over A and use (33) to obtain (11).

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