Revised version (September 2014) of a letter sent long long ago to a number of colleagues.

Dear Friends:
We very much hope someone will solve the question discussed in the paper \[3\] of which we have sent you a preprint with this letter.
We thought we would add these pages to give an immediately accessible and self contained formulation of one of the questions for which a positive answer will suffice to give a positive answer to THE question whose answer can bring you fame(?) and fortune(??????).

Let \(\Omega\) be the annulus \(\{ z \in \mathbb{C} : 1 \leq |z| \leq e \}\). Let \(A(\Omega)\) denote the space of all continuous functions \(f : \Omega \to \mathbb{C}\) which are analytic on the interior of \(\Omega\) with norm \(\|f\|_{A(\Omega)} = \sup_{z \in \Omega} |f(z)|\).

Let \(\ell^\infty(A(\Omega))\) be the space of \(A(\Omega)\) valued sequences \(\{f_n\}_{n \in \mathbb{N}}\) for which the norm \(\sup_{n \in \mathbb{N}} \|f_n\|_{A(\Omega)}\) is finite.

**Question 2**
Suppose that for each point \(\sigma\) of the boundary of the annulus \(\Omega\) we are given an absolutely convex subset \(M_\sigma\) of the unit ball of \(\ell^\infty\) and that \(M_\sigma\) is relatively compact for each \(\sigma\) on the circle \(|\sigma| = 1\). Let \(\mathcal{M}\) be the space of all functions \(\phi \in \ell^\infty(A(\Omega))\) such that \(\phi(\sigma) \in M_\sigma\) for each \(\sigma \in \partial \Omega\). For \(0 < \theta < 1\), let \(M_{[e^\theta]}\) be the subset of \(\ell^\infty\) consisting of all elements of the form \(\phi(e^\theta)\) where \(\phi \in \mathcal{M}\). Is \(M_{[e^\theta]}\) relatively compact in \(\ell^\infty\)?

**Remark.** See Proposition 6 (page 363 or 11) of \[3\] for a proof of a special case of Question 2. In fact it would suffice to resolve another special case of Question 2 where the (relative) compactness of the sets \(M_\sigma\) is “uniform” for all \(\sigma, |\sigma| = 1\), in the sense that for each \(\epsilon > 0\), there exists an integer \(N(\epsilon)\) such that for each \(\sigma, |\sigma| = 1\), \(M_\sigma\) is contained in the union of \(N(\epsilon)\) balls of radius \(\epsilon\) in \(\ell^\infty\). It would also suffice (via reiteration) to consider the particular case where the elements of \(\mathcal{M}\) have the additional property that their restrictions to the circle \(|\sigma| = e\) are continuous \(\ell^\infty\) valued functions on that circle. Obviously it is equivalent but perhaps somehow more comfortable to work on the unit disc instead of the annulus with the point 0 instead of \(e^\theta\). Then you have to assume that \(M_\sigma\) is relatively compact for all \(\sigma\) in some subset of the boundary having positive arc length measure. But we would be delighted if you give us a proof even assuming compactness on the whole boundary. (That would suffice to give a positive answer to Question 1 of the paper under the additional assumption that \(T : A_1 \to B_1\) is also compact.)

It is important to mention that a remarkable calculation done by Fedor Nazarov produces something looking very much like a counterexample to Question 2. Its details can be obtained from the site \([1]\), or more directly from \([2]\).

GOOD LUCK!!

M.C., N.K and M.M.

**SOME FURTHER REMARKS**
(i) The answer to Question 2 is NO if in the definition of \( A(\Omega) \) we require the functions to be merely harmonic instead of analytic.

(ii) The answer to Question 2 is YES if we require each set \( M_\sigma \) to be finite dimensional for each \( \sigma \) in some subset of \( \partial \Omega \) having positive (arc length) measure. But even with this far more stringent condition on the \( M_\sigma \)'s the answer is still NO if we deal with harmonic instead of analytic functions.

Let us prove (i) and (ii).

We shall adopt the option, already indicated above, of working on the unit disc \( D \) instead of the annulus. Let \( H(D) \) be the space of all complex valued continuous functions on the closed unit disc which are harmonic in its interior, normed by the supremum norm. Let \( A(D) \) be the subspace of those functions which are also analytic. Let \( F \) denote a sequence (or vector) \( F = (f_1, f_2, \ldots) \) of functions in the unit ball of \( H(D) \) or \( A(D) \). We actually need to consider a sequence \( \{F_m\} \) of such \( F \)'s, so we shall use the notation: \( F_m = (f_{1,m}, f_{2,m}, \ldots) \). For each fixed \( \tau \) on the boundary \( \mathbb{T} \) of \( D \) consider the sequence \( \{F_m(\tau)\}_{m=1}^\infty \) of elements in \( \ell^\infty \), (i.e. set \( F_m(\tau) = (f_{1,m}(\tau), f_{2,m}(\tau), \ldots) \) for each fixed \( m \)). If we know that for each such \( \tau \in \mathbb{T} \) this sequence is relatively compact in \( \ell^\infty \), we can show that the corresponding sequence “at the origin”, whose elements are \( F_m(0) = (f_{1,m}(0), f_{2,m}(0), \ldots) \) for \( m = 1, 2, \ldots \) has a convergent subsequence in \( \ell^\infty \).

To answer YES to this for the case of analytic functions would come close to solving the original question. (It would solve it for the case where one has the additional hypothesis that \( T : A_1 \to B_1 \) is compact. To answer the general question one would have to be able to get the same conclusion when the above relative compactness property holds only for each \( \tau \) in some subset of \( \mathbb{T} \) having positive arc length measure.)

The following example shows that the answer is no for the case of harmonic functions. Define each \( f_{n,m} \) on \( \mathbb{T} \) by \( f_{n,m}(\tau) = \tau^{n-m} \) and extend the function harmonically to all of \( D \). Then of course \( f_{n,m}(0) = 0 \) for all \( m \neq n \) and \( f_{n,n}(0) = 1 \), which means that the sequence \( F_m(0) = \{f_{1,m}(0), f_{2,m}(0), \ldots\} \) has no convergent subsequence in \( \ell^\infty \). On the other hand, for each fixed \( \tau \in \mathbb{T} \), the elements \( F_m(\tau) = \tau^{-m}(\tau, \tau^2, \tau^3, \ldots) \) are all contained in a compact (even bounded one dimensional) subset of \( \ell^\infty \).

With this example we have established the claim (i) above, and also the second half of the claim (ii). The first part of (ii) will now be obtained via a simple argument using determinants: To show that sequence whose elements are \( F_m(0) = (f_{1,m}(0), f_{2,m}(0), \ldots) \) for \( m = 1, 2, \ldots \) has a convergent subsequence in \( \ell^\infty \) we shall show something much stronger, namely that it is contained in a finite dimensional subset of the unit ball of \( \ell^\infty \). Obviously it is in the unit ball, so let us suppose it is not finite dimensional. Then for each \( k \in \mathbb{N} \) there exists a set \( I_k \subset \mathbb{N} \) containing \( k \) elements such that the determinant \( \Delta(I_k,0) \neq 0 \). Here we are using the notation \( \Delta(I,z) \) for each finite subset \( I \subset \mathbb{N} \) and for each \( z \in D \) to denote the determinant of the submatrix of \( \{f_{n,m}(z)\}_{n,m \in \mathbb{N}} \) obtained by restricting \( m \) and \( n \) to \( I \).

By the hypothesis we are making here, there exists a subset of positive measure of \( \mathbb{T} \), which we shall denote by \( E \), such that the span of the elements \( F_m(\tau) = (f_{1,m}(\tau), f_{2,m}(\tau), \ldots) \) is finite dimensional for each \( \tau \in E \). Consequently, \( E = \bigcup_{k=1}^{\infty} E_k \) where \( E_k = \{\tau \in E : \Delta(I_k,\tau) = 0\} \). It follows that for at least one \( k \in \mathbb{N} \) the set \( E_k \) must have positive (arc length) measure. Thus, since \( \Delta(I_k,z) \) is a function in \( A(D) \),
we obtain that $\Delta(I_k,0) = 0$. This is a contradiction, and so we have established the first part of (ii).

*Remark.* The paper [4] contains some very interesting ideas connected with this problem.

**References**