

A very interesting example of Fedor Nazarov related to the complex interpolation of compact operators.

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1. INTRODUCTION

This informal document reports on and discusses a remarkable new example, discovered recently by Fedor Nazarov, which relates to attempts to answer the following two questions about interpolation of compact operators by Calderón's complex interpolation method which have been open for some forty years.

Question 1: *Suppose that A_0 and A_1 are compatible Banach spaces, i.e., they form a Banach pair, and that so are B_0 and B_1 . Suppose that $T : A_0 + A_1 \rightarrow B_0 + B_1$ is a linear operator such that $T : A_0 \rightarrow B_0$ compactly and $T : A_1 \rightarrow B_1$ boundedly. Does it follow that $T : [A_0, A_1]_\theta \rightarrow [B_0, B_1]_\theta$ compactly for each $\theta \in (0, 1)$?*

Question 2: *Same as Question 1, but under the stronger hypothesis that $T : A_1 \rightarrow B_1$ is also compact.*

These old questions were posed and discussed yet again, by myself in Oberwolfach last August. See page 2111 of the report [CN0] from that meeting. In my talk I also formulated a new question, which is labelled as Question 4 on page 2112 of [CN0] and is also restated below in Subsection 5.1. An affirmative answer for this new question would have implied an affirmative answer to one version (Question 2) of the original Calderón question. However Fedja's very ingenious example now shows that the answer to Question 4 of [CN0] is no.

A detailed explanation of why an affirmative answer to Question 4 would have been sufficient to imply an affirmative answer to Question 2 can be found in the "pre-preprint" [CN1]. In [CN1] Question 4 and the variant of it for the special case when $\psi(\varepsilon) = \varepsilon^r$, are reformulated in somewhat more elaborate ways and referred to as Question Y1 and Question Y2. (See the first section of [CN1].) Fedja's example shows (as will be explained in detail below in Section 3) that the answers to both these questions are also no.

Question 4 of [CN0] and Questions Y1 and Y2 of [CN1] can all be considered as "quantitative" versions of the following weaker and rather more "qualitative" question which is closely related to another question which is formulated on page 362 of [CKM] (where, sorry about the confusion, it also bears the label "Question 2").

Question Q: **Suppose that, for each $\theta \in [0, 2\pi)$ we are given a subset $M(e^{i\theta})$ of the unit ball of ℓ^∞ . Define the set $M(0)$ to consist of all elements $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ of ℓ^∞ which are of the form $\{f_n(0)\}_{n \in \mathbb{N}}$ for some sequence of functions f_n which are continuous on the closed unit disk and analytic in its interior and for which $\{f_n(e^{i\theta})\}_{n \in \mathbb{N}} \in M(e^{i\theta})$ for each $\theta \in [0, 2\pi)$. If $M(e^{i\theta})$ is compact for every $\theta \in [0, 2\pi)$, does it follow that $M(0)$ is contained in a compact subset of ℓ^∞ ?**

Using arguments similar to those given in [CKM], it can be shown that an affirmative answer to Question Q would also suffice to imply an affirmative answer to Question 2 on page 2111 of [CN0]. Furthermore this would also follow from an affirmative answer to a special case of Question Q, in the case where one makes

the additional assumption that the sets $M(e^{i\theta})$ are “uniformly compact” on $[0, 2\pi)$, i.e., that for each $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that, for each $\theta \in [0, 2\pi)$, the set $M(e^{i\theta})$ is contained in the union of $N(\varepsilon)$ balls in ℓ^∞ of radius ε .

It will be convenient sometimes to use the notation $\mathcal{N}_*(\varepsilon, A)$, where $\varepsilon > 0$ and A is a subset of a metric space (usually ℓ^∞ for us), for the ε -covering number of A . I.e., $\mathcal{N}_*(\varepsilon, A)$ is the smallest integer q such that A is contained in the union of q open balls of radius ε in the metric space (i.e., ℓ^∞), and $\mathcal{N}_*(\varepsilon, A) = \infty$ if no such integer exists.

A negative answer to Question Q would, like Fedja’s recent example, still not be sufficient to give a negative answer to the original classical questions, i.e., Questions 1 and 2 of [CN0].

Fedja’s example provides us, apparently for the first time, with “partial evidence” that the answer to Question Q and to questions similar to it, might be no. Fedja shows that, for each $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ and a family $\{M(e^{i\theta})\}_{\theta \in [0, 2\pi)}$ of subsets of ℓ^∞ each of which satisfy $\mathcal{N}_*(\varepsilon, M(e^{i\theta})) \leq N$, and such that the set $M(0)$ defined by the procedure specified in Question Q, is nevertheless not contained in any compact subset of ℓ^∞ .

It should be stressed that the particular sets $M(e^{i\theta})$ constructed via Fedja’s example depend on the particular choice of ε and probably $\mathcal{N}_*(\varepsilon_0, M(e^{i\theta}))$ is not finite for sufficiently small positive $\varepsilon_0 < \varepsilon$. Nevertheless, it now begins to seem conceivable that ideas connected to Fedja’s new example might perhaps ultimately provide tools for giving a negative answer to Question Q and the related Question 2 of [CKM] p. 362. Then that might perhaps be the first step to finding a negative answer to Question 1 or Question 2 of [CN0].

So, for several reasons, it now seems rather natural (but by no means easy) to try to resolve Question Q of [CKM]. We will make some very preliminary comments about attempts to do that in Section 4.

2. THE DETAILS OF FEDJA’S EXAMPLE

The first ingredient is the following essentially classical estimate for trigonometric series.

Lemma 1. *There exists an absolute constant C_1 such that*

$$\left| \sum_{k=1}^K \frac{\sin kx}{k} \right| \leq C_1 \text{ for all } K \in \mathbb{N} \text{ and all } x \in \mathbb{R}.$$

For the reader’s convenience we provide a proof of this lemma below in an appendix (Subsection 5.2).

Let us set $\gamma(m, z) = z^m$ and $\gamma_k(m, z) = \gamma(m, z^k) = z^{km}$ for each $z \in \mathbb{C}$ and each k and m in \mathbb{N} . Then, for each fixed $z \in \mathbb{C}$ and each fixed $k \in \mathbb{N}$ we define the sequences $\gamma(z) = \{\gamma(m, z)\}_{m \in \mathbb{N}} = \{z^m\}_{m \in \mathbb{N}}$ and $\gamma_k(z) = \gamma(z^k) = \{\gamma(m, z^k)\}_{m \in \mathbb{N}} = \{z^{km}\}_{m \in \mathbb{N}}$. These are of course elements of $\mathcal{B}_{\ell^\infty}$ for each $z \in \mathbb{D}$ and each $k \in \mathbb{N}$.

Let us fix a (very small) positive number ε . After doing this we choose a number $K \in \mathbb{N}$ which is sufficiently large so that

$$(1) \quad \frac{2C_1}{\sum_{k=1}^K \frac{1}{k}} < \frac{\varepsilon}{2}.$$

(This implies that we have, approximately, that $\ln K > 4C_1/\varepsilon$, i.e., K can be approximately $e^{4C_1/\varepsilon}$.)

Clearly, for some fixed integer $\alpha \in \mathbb{N}$, depending only on ε , we can find a finite collection $\{\lambda_1, \lambda_2, \dots, \lambda_\alpha\}$ of elements in \mathbb{T} such that, for each $\omega \in \mathbb{T}$ there exists at least one $j = j(\omega) \in \{1, 2, \dots, \alpha\}$ such that

$$(2) \quad \max_{k=1,2,\dots,K} |\lambda_j^k - \omega^k| < \frac{\varepsilon}{2}.$$

For possible later use, let us indicate more specifically how this can be done. First we choose α to be the smallest integer such that $\alpha \geq 4\pi K/\varepsilon$. Then we choose $\lambda_j = e^{2\pi i j/\alpha}$ for $j = 1, 2, \dots, \alpha$. Then, given $\omega \in \mathbb{T}$, we choose $\theta \in [0, 2\pi)$ such that $\omega = e^{i\theta}$ and then $j = j(\omega) \in \{1, 2, \dots, \alpha\}$ such that $2\pi(j-1)/\alpha \leq \theta \leq 2\pi j/\alpha$.

To show that (2) holds for these choices, we first observe that $|\lambda_j - \omega| \leq |\lambda_j - \lambda_{j-1}| \leq \frac{2\pi}{\alpha} \leq \frac{\varepsilon}{2K}$. Consequently, for each $k \in \mathbb{N}$, since $\lambda_j^k - \omega^k = (\lambda_j - \omega) \sum_{j=0}^{k-1} \lambda_j^{k-1-j} \omega^j$, we have $|\lambda_j^k - \omega^k| \leq |\lambda_j - \omega| \sum_{j=0}^{k-1} |\lambda_j^{k-1-j} \omega^j| \leq \frac{\varepsilon}{2K} k$. So (2) indeed is satisfied.

From here onwards it will be convenient to use the notation $M_K = \sum_{k=1}^K \frac{1}{k}$. For $j = 1, 2, \dots, \alpha$, define the functions, $u_j : \mathbb{D} \rightarrow \mathcal{B}_{\ell^\infty}$ by $u_j(z) = \frac{1}{M_K} \sum_{k=1}^K \frac{\lambda_j^k}{k} \gamma(z^k)$. Clearly each component of each $u_j(z)$ is an analytic polynomial in z . Furthermore, for each fixed $z \in \mathbb{T}$ and each $n \in \mathbb{N}$, there exists some $j = j(z^{-n}) \in \{1, 2, \dots, \alpha\}$ such that $\max_{k=1,2,\dots,K} |\lambda_j^k - z^{-kn}| < \frac{\varepsilon}{2}$.

(Note that as z varies, each point $\lambda_{j(z^{-n})}$ is approximately equal to z^{-n} i.e., the points $\lambda_{j(z^{-n})}$ therefore also the points $u_{j(z^{-n})}(z)$ depend on z in a very "NON-ANALYTIC" way. This is apparently one of the key properties which enables this example to have its at first sight surprising behaviour.)

For each $n \in \mathbb{N}$, we define the function $g_n : \mathbb{T} \rightarrow \mathcal{B}_{\ell^\infty}$ by $g_n(z) = \frac{1}{M_K} \sum_{k=1}^K \frac{z^{-kn}}{k} \gamma(z^k)$. These functions g_n do not have analytic extensions to \mathbb{D} . But, by our definition of $j(z^{-n})$, we see that, for each fixed $z \in \mathbb{T}$ and $n \in \mathbb{N}$, we have

$$(3) \quad \|g_n(z) - u_{j(z^{-n})}(z)\|_{\ell^\infty} \leq \frac{1}{M_K} \sum_{k=1}^K \frac{|\lambda_j^k - z^{-kn}|}{k} \|\gamma(z^k)\|_{\ell^\infty} < \frac{\varepsilon}{2}.$$

Next, for each $n \in \mathbb{N}$ we define the sequence $\phi_n(z) = \{z^{|m-n|}\}_{m \in \mathbb{N}}$. Then we define the function $f_n : \mathbb{D} \rightarrow \mathcal{B}_{\ell^\infty}$ by $f_n(z) = \frac{1}{M_K} \sum_{k=1}^K \frac{1}{k} \phi_n(z^k)$. Each component of each of these functions is of course an analytic polynomial in z .

For each $z \in \mathbb{T}$ and each $n \in \mathbb{N}$ we have

$$f_n(z) - g_n(z) = \frac{1}{M_K} \sum_{k=1}^K \frac{\phi_n(z^k) - z^{-kn} \gamma(z^k)}{k}.$$

If $m \leq n$ then the m^{th} component of the sequence $f_n(z) - g_n(z)$ is $\frac{1}{M_K} \sum_{k=1}^K \frac{z^{k(n-m)} - z^{-kn} z^{km}}{k}$ which equals $\frac{1}{M_K} \sum_{k=1}^K \frac{2i \sin kx}{k}$ where $z^{n-m} = e^{ix}$. In view of Lemma 1 and (1) this

sum has absolute value not exceeding $\frac{2C_1}{M_K} < \varepsilon/2$. If $m > n$ then the m^{th} component of the sequence $f_n(z) - g_n(z)$ is simply $\frac{1}{M_K} \sum_{k=1}^K \frac{z^{k(m-n)} - z^{-kn} z^{km}}{k} = 0$. From these remarks, and from (3), we see that, for each $z \in \mathbb{T}$ and each $n \in \mathbb{N}$,

$$(4) \quad \min_{j=1,2,\dots,\alpha} \|f_n(z) - u_j(z)\|_{\ell^\infty} < \varepsilon.$$

Thus, for each fixed $z \in \mathbb{T}$, the set $\{f_n(z) : n \in \mathbb{N}\}$ is contained in the union of α balls in ℓ^∞ , each of radius not exceeding ε .

On the other hand, at $z = 0$, we need infinitely many open balls in ℓ^∞ of radius $1/2$ to contain the set $\{f_n(0) : n \in \mathbb{N}\}$. This is because, at $z = 0$, the sequence $f_n(0)$ equals e_n , i.e., it is the sequence which has n^{th} component equal to 1 and all other components equal to 0.

Remark 2. *As we have seen, the integer α depends only on ε , and we can, for example, suppose that $\alpha \leq 4\pi K/\varepsilon + 1$ where K is approximately $e^{4C_1/\varepsilon}$. For reasons which will be elaborated later, in Subsection 4.2, also for each $\varepsilon_0 > \varepsilon$ we would like to be able control the size of $\mathcal{N}_*(\varepsilon_0, \{f_n(0) : n \in \mathbb{N}\})$, i.e., the number of balls of radius ε_0 needed to cover this same set $\{f_n(0) : n \in \mathbb{N}\}$. It would be very interesting if $\mathcal{N}_*(\varepsilon_0, \{f_n(0) : n \in \mathbb{N}\})$ could be bounded by a number depending only on ε_0 , i.e., if it did not increase as we choose smaller and smaller values of ε . This does not seem possible with the current definition of the functions f_n . Maybe there is some variant of this construction which would yield functions f_n which still have most of the other properties of these functions and also have this additional property.*

3. A PROOF, USING FEDJA'S EXAMPLE, THAT THE ANSWERS TO QUESTION 4, QUESTION Y1 AND QUESTION Y2 ARE ALL NO.

Let us first recall some general facts about subspaces of ℓ^∞ in versions which are appropriate for our needs here.

Let N be any positive integer. Let S be an arbitrary subspace of ℓ_N^∞ . Then the following facts are apparently well known and also easy to show (see [CN1] for more details):

- For each $x \in \overline{\mathcal{B}_{\ell_N^\infty}}$ we have $\text{dist}_{\ell_N^\infty}(x, S) = \text{dist}_{\ell_N^\infty}(x, \overline{\mathcal{B}_S(2)})$ where $\overline{\mathcal{B}_S(2)} = \{y \in S : \|y\|_{\ell_N^\infty} \leq 2\}$.

- Suppose that $\mathcal{N}_*\left(\frac{1}{4}, \overline{\mathcal{B}_S(2)}\right)$ denotes the smallest integer m such that $\overline{\mathcal{B}_S(2)}$ is contained in the union of m open balls in ℓ_N^∞ , each of radius $\frac{1}{4}$. There exists a number $\rho(n)$ depending only on $n \in \mathbb{N}$ such that, for every subspace S of ℓ_N^∞ ,

$$(5) \quad \mathcal{N}_*\left(\frac{1}{4}, \overline{\mathcal{B}_S(2)}\right) \leq \rho(\dim S).$$

In particular, $\rho(\dim S)$ does not depend on N . For example, simple arguments ([CN1]) show that we can take $\rho(n) = (16n^2((n-1)^{(n-1)/2} + 1))^n$.

We are now ready to use the functions described in Section 2 to show that the answers to Question 4 of [CN0] and to Question Y2 of [CN1] are both NO.

(i) Suppose that the answer to Question 4 is yes. Then there exists some particular function $\psi : (0, \infty) \rightarrow (0, \infty)$ which satisfies $\lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = 0$ and there exists some integer valued function $\xi : (0, \infty) \times \mathbb{N} \rightarrow \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and every collection F_1, F_2, \dots, F_k of k functions in $H^\infty(\mathbb{D}, \mathbb{C}^N)$, there exists a subspace

S of \mathbb{C}^N with dimension not exceeding $\xi(\varepsilon, k)$ such that whenever $G \in H^\infty(\mathbb{D}, \mathbb{C}^N)$ with $\|G(z)\|_\infty \leq 1$ for all $z \in \mathbb{D}$ and $\varepsilon = \sup_{z \in \mathbb{T}} \text{dist}_{\ell_N^\infty}(G(z), M(z))$, where

$$(6) \quad M(z) = \text{span}\{F_1(z), F_2(z), \dots, F_k(z)\}$$

then it follows that $\text{dist}_{\ell_N^\infty}(G(0), S) \leq \psi(\varepsilon)$.

(ii) Suppose that the answer to Question Y2 of [CN1] is yes. Then, in the terminology of [CN1], there exists an Ysaye pair (ψ, ξ) which satisfies $\lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = 0$.

The following argument will show, simultaneously, that there cannot exist functions ψ and ξ with the properties mentioned in (i), nor with the properties mentioned in (ii).

Our first step is to choose a particular $\varepsilon > 0$ above, so that $\psi(\varepsilon) < 1/4$, where ψ is the function whose existence has been postulated in either (i) or (ii). Then we choose a positive integer α depending on that value of ε and construct the functions $u_j : \mathbb{D} \rightarrow \mathcal{B}_{\ell^\infty}$ for $j = 1, 2, \dots, \alpha$ and $f_n : \mathbb{D} \rightarrow \mathcal{B}_{\ell^\infty}$ for all $n \in \mathbb{N}$ by exactly the same procedures as in Section 2. Now let N be a positive integer satisfying

$$(7) \quad N > \rho(\xi(\varepsilon, \alpha))$$

where $\rho : \mathbb{N} \rightarrow \mathbb{N}$ is the function introduced in (5). Let $M = N + \alpha$. Let $P : \ell^\infty \rightarrow \mathbb{C}^N$ be the projection map defined by $P\{\sigma_n\}_{n \in \mathbb{N}} = (\sigma_1, \sigma_2, \dots, \sigma_N)$ for each $\{\sigma_n\}_{n \in \mathbb{N}} \in \ell^\infty$.

Now we define the functions F_1, F_2, \dots, F_M in $H^\infty(\mathbb{D}, \mathbb{C}^N)$ by $F_j = P \circ u_j$ for $j = 1, 2, \dots, \alpha$, and $F_{\alpha+j} = P \circ f_j$ for $j = 1, 2, \dots, N$. All components of all of these functions are analytic polynomials in the unit ball of $A(\mathbb{D})$. Furthermore, using (4), we have

$$(8) \quad \min \left\{ \|F_n(z) - F_j(z)\|_{\ell_N^\infty} : j = 1, 2, \dots, \alpha \right\} \leq \min \left\{ \|f_{n-\alpha}(z) - u_j(z)\|_{\ell_N^\infty} : j = 1, 2, \dots, \alpha \right\} < \varepsilon$$

for every $n \in \{\alpha + 1, \alpha + 2, \dots, \alpha + N\}$ and the left side of (8) is of course 0 when $n \in \{1, 2, \dots, \alpha\}$.

We now use these numbers and functions in the two rather similar particular cases which have to be considered.

On the one hand, if we are supposing, as in (ii) that (ψ, ξ) is an Ysaye pair, then we would have to be able to deduce that there exists a subspace S of \mathbb{C}^N of dimension $\xi(\varepsilon, \alpha)$, such that $\text{dist}_{\ell_N^\infty}(F_{\alpha+n}(0), S) = \text{dist}_{\ell_N^\infty}(Pe_n, S) < \frac{1}{4}$ for every $n = 1, 2, \dots, N$. Thus, as indicated above, we also have $\text{dist}_{\ell_N^\infty}(Pe_n, \overline{\mathcal{B}_S(2)}) < \frac{1}{4}$. We recall that $\overline{\mathcal{B}_S(2)}$ is contained in the union of $\rho(\xi(\varepsilon, \alpha))$ open balls $B_1, B_2, \dots, B_{\rho(\xi(\varepsilon, \alpha))}$ in ℓ_N^∞ each of radius $1/4$. If $2B$ denotes the ball having the same centre and twice the radius as the ball B , then the union of the open balls $2B_1, 2B_2, \dots, 2B_{\rho(\xi(\varepsilon, \alpha))}$ contains each of the N points Pe_n . But each open ball $2B_j$ since its radius is $1/2$, can contain at most one of the points Pe_n . This contradicts (7) and completes the proof that the answer to Question Y2 is no.

On the other hand, if we are supposing, as in (i), that the answer to Question 4 is yes, then we obtain a contradiction in almost the same way. We choose $k = \alpha$ and again take $F_j = P \circ u_j$ for $j = 1, 2, \dots, \alpha$ and define $M(z)$ by (6). It follows from (8) that, for each $G_n = P \circ f_n$ for $n = 1, 2, \dots, N$, we have $\text{dist}_{\ell_N^\infty}(G_n(z), M(z)) \leq \varepsilon$ for each $z \in \mathbb{T}$. Obviously we also have $\|G_n(z)\|_{\ell_N^\infty} \leq 1$ for each $z \in \mathbb{T}$. So there has to

be a subspace S of dimension $\xi(\varepsilon, \alpha)$ such that $\text{dist}_{\ell_N^\infty}(G_n(0), S) = \text{dist}_{\ell_N^\infty}(Pe_n, S) < \frac{1}{4}$ for every $n = 1, 2, \dots, N$. As already explained, this is impossible.

4. FURTHER THOUGHTS AND QUESTIONS

It might be interesting and instructive to try to prove that Question Q has an affirmative answer under the additional assumption that the sets $M(e^{i\theta})$ are “uniformly ultra-compact”, i.e., if the expression $\sup_{\theta \in [0, 2\pi)} \mathcal{N}_*(\varepsilon, M(e^{i\theta}))$ tends to ∞ in some suitably slow manner as ε tends to 0. Perhaps, analogously, the answer to Question 1 (or Question 2) might be yes if the set $T(\mathcal{B}_{A_0})$ is contained in some “ultra-compact” subset of B_0 (and also the set $T(\mathcal{B}_{A_1})$ is contained in some “ultra-compact” subset of B_1 .)

Here are some tentative remarks and thoughts about how one might perhaps extend Fedja’s example to show that the answer to Question Q is also no.

4.1. FIRST APPROACH:. To answer Question Q in the negative, it would suffice if we could produce an example of a family $\{E_z\}_{z \in \mathbb{T}}$ of compact subsets of $\mathcal{B}_{\ell^\infty}$ such that, for some fixed infinite subset $G \subset \mathbb{N}$ the above ℓ^∞ -valued functions f_n introduced in Fedja’s example satisfy $f_n(z) \in E_z$ for each $z \in \mathbb{T}$ and each $n \in G$.

It does not seem very likely, but we could do this if we could somehow produce strictly increasing sequences of integers $\{K_\alpha\}_{\alpha \in \mathbb{N}}$ and $\{n_\alpha\}_{\alpha \in \mathbb{N}}$ such that

$$v_\alpha(z) := \frac{1}{M_{K_\alpha}} \sum_{k=1}^{K_\alpha} \frac{z^{-kn_\alpha}}{k} \gamma_k(z) = \frac{1}{M_{K_\alpha}} \sum_{k=1}^{K_\alpha} \frac{1}{k} \{z^{k(m-n_\alpha)}\}_{m \in \mathbb{N}}$$

converges in ℓ^∞ as $\alpha \rightarrow \infty$, for every $z \in \mathbb{T}$. Rewriting $z = e^{i\theta}$, we see that $v_\alpha(e^{i\theta}) = \frac{1}{M_{K_\alpha}} \sum_{k=1}^{K_\alpha} \frac{1}{k} \{\cos k(m-n_\alpha)\theta\}_{m \in \mathbb{N}} + \frac{i}{M_{K_\alpha}} \sum_{k=1}^{K_\alpha} \frac{1}{k} \{\sin k(m-n_\alpha)\theta\}_{m \in \mathbb{N}}$ and the ℓ^∞ norm of the second sequence (cf. Lemma 1) is dominated by C_1/M_{K_α} and will thus tend to 0 for any choice of the sequences $\{K_\alpha\}_{\alpha \in \mathbb{N}}$ and $\{n_\alpha\}_{\alpha \in \mathbb{N}}$. Thus we have to consider the sequence $w_\alpha(\theta) = \frac{1}{M_{K_\alpha}} \sum_{k=1}^{K_\alpha} \frac{1}{k} \{\cos k(m-n_\alpha)\theta\}_{m \in \mathbb{N}}$ for each θ .

In general,

$$\sum_{k=1}^K \frac{\cos kx}{k} = \sum_{k=1}^K \left(\frac{1}{k} - \int_0^x \sin kt \, dt \right) = M_K - \int_0^x \sum_{k=1}^K \sin kt \, dt.$$

Now

$$\begin{aligned} \sum_{k=1}^K \sin kt &= \sum_{k=0}^K \sin kt = \text{Im} \sum_{k=0}^K e^{ikt} = \text{Im} \frac{e^{i(K+1)t} - 1}{e^{it} - 1} = \text{Im} \frac{e^{it/2} (e^{i(K+\frac{1}{2})t} - e^{-it/2})}{e^{it/2} (e^{it/2} - e^{-it/2})} \\ &= \text{Im} \frac{(e^{i(K+\frac{1}{2})t} - e^{-it/2})}{(e^{it/2} - e^{-it/2})} = \text{Im} \frac{(e^{i(K+\frac{1}{2})t} - e^{-it/2})}{2i \sin \frac{t}{2}} = -\frac{\cos(K + \frac{1}{2})t - \cos \frac{t}{2}}{2 \sin \frac{t}{2}} \\ &= \frac{2 \sin \frac{K}{2}t \sin(\frac{K+1}{2})t}{2 \sin \frac{t}{2}} = \frac{\sin \frac{K}{2}t \sin(\frac{K+1}{2})t}{\sin \frac{t}{2}}. \end{aligned}$$

So

$$w_\alpha(\theta) = \left\{ 1 - \frac{1}{M_{K_\alpha}} \int_0^{(m-n_\alpha)\theta} \frac{\sin \frac{K_\alpha}{2}t \sin(\frac{K_\alpha+1}{2})t}{\sin \frac{t}{2}} dt \right\}_{m \in \mathbb{N}}.$$

We have reduced the problem to finding strictly increasing sequences of integers $\{K_\alpha\}_{\alpha \in \mathbb{N}}$ and $\{n_\alpha\}_{\alpha \in \mathbb{N}}$ such that the sequence

$$\beta_\alpha(\theta) = \left\{ \frac{1}{M_{K_\alpha}} \int_0^{(m-n_\alpha)\theta} \frac{\sin \frac{K_\alpha}{2} t \sin \left(\frac{K_\alpha+1}{2} \right) t}{\sin \frac{t}{2}} dt \right\}_{m \in \mathbb{N}}$$

converges in ℓ^∞ for each $\theta \in [-\pi, \pi]$. Note that

$$\begin{aligned} \frac{1}{M_{K_\alpha}} \int_0^{(m-n_\alpha)\theta} \frac{\sin \frac{K_\alpha}{2} t \sin \left(\frac{K_\alpha+1}{2} \right) t}{\sin \frac{t}{2}} dt &= \frac{1}{M_{K_\alpha}} \int_0^{(m-n_\alpha)\theta} \frac{\sin \frac{K_\alpha}{2} t \sin \left(\frac{K_\alpha+1}{2} \right) t}{\frac{t}{2}} dt \\ &+ \frac{1}{M_{K_\alpha}} \int_0^{(m-n_\alpha)\theta} \sin \frac{K_\alpha}{2} t \sin \left(\frac{K_\alpha+1}{2} \right) t \left[\frac{1}{\sin \frac{t}{2}} - \frac{1}{\frac{t}{2}} \right] dt. \end{aligned}$$

It is far from evident that this will work.

4.2. SECOND APPROACH: Since the sequence $f_n(z)$ defined as in Section 2 depends on a particular choice of a positive number ε , let us now use the more explicit notation $f_n(z, \varepsilon)$ for it. Then let us use the notation $f_{nm}(z, \varepsilon)$ for the m^{th} element of the sequence $f_n(z, \varepsilon)$. By standard arguments using normal families, we can see that, for each fixed m and n in \mathbb{N} , the sequence of functions $\{f_{nm}(z, 1/k)\}_{k \in \mathbb{N}}$ has a subsequence which converges uniformly on each compact subset of the open unit disk. By suitable “diagonalization” we can find a “sparser”, subsequence of k 's, i.e., some increasing sequence of integers $\{p_k\}_{k \in \mathbb{N}}$ such that $\{f_{nm}(z, 1/p_k)\}_{k \in \mathbb{N}}$ converges uniformly on each compact subset of the open unit disk for each fixed m and n in \mathbb{N} . The pointwise limit of this sequence is a bounded analytic function $g_{n,m}(z)$ on the open unit disk which must necessarily have radial (or even non tangential) limits at almost every boundary point. We shall also denote these limits by $g_{n,m}(z)$ for (a.e.) $z \in \mathbb{T}$. For each $z \in \mathbb{D}$, except possibly for those points z in some null subset of \mathbb{T} , let $M(z)$ be the subset of $\mathcal{B}_{\ell^\infty}$ consisting of all sequences of the form $\{g_{nm}(z)\}_{m \in \mathbb{N}}$ as n ranges over \mathbb{N} . Clearly $g_{nm}(0) = \delta_{m,n}$ for all $m, n \in \mathbb{N}$ and so $M(0)$ is not contained in any compact subset of $\mathcal{B}_{\ell^\infty}$. If $M(z)$ happens to be compact, or contained in a compact subset of ℓ^∞ for all or almost all $z \in \mathbb{T}$, then we have something close to a negative answer for Question Q, with H^∞ functions instead of analytic functions which are continuous up to the boundary. We might expect somehow that $M(z)$ is compact for such $z \in \mathbb{T}$ since, as k tends to ∞ , the set of sequences $f_n(z, 1/p_k)$ as n ranges over \mathbb{N} , satisfies $\mathcal{N}_*(\varepsilon, \{f_n(z, 1/p_k) : n \in \mathbb{N}\}) < \infty$ for smaller and smaller values of ε , i.e., for $\varepsilon = 1/p_k$. The problem (at least one problem) is that each time when we make k larger, so that we indeed obtain a new set of sequences with a finite ε net for some smaller ε , we have no reasonable control of the size of $\mathcal{N}_*(\varepsilon, \{f_n(z, 1/p_k) : n \in \mathbb{N}\})$ for larger values of ε for this new set of sequences. (In this connection, see Remark 2.)

So the challenge here is to try to use the above process, but with the ℓ^∞ valued functions $f_n(z, \varepsilon)$ of Section 2 replaced by “nicer” variants which still satisfy $f_n(0, \varepsilon) = e_n$ and still are such that the set $\{f_n(z, \varepsilon) : n \in \mathbb{N}\}$ has a finite ε net. But now we also want to also be able to assert the existence of positive integers q_k depending only on k such that, for each k and each $k_1 \geq k$, the set $\{f_n(z, 1/p_{k_1}) : n \in \mathbb{N}\}$ satisfies

$$(9) \quad \mathcal{N}_*(1/p_k, \{f_n(z, 1/p_{k_1}) : n \in \mathbb{N}\}) \leq q_k.$$

Simple variants of the calculations and estimates of Section 2 suggest that, for each positive ε_1 and ε_2 and for $f_n(z, \varepsilon)$ as defined in Section 2, we have that $\mathcal{N}_* \left(\frac{1}{2} (\varepsilon_1 + \varepsilon_2), \{f_n(z, \varepsilon_1) : n \in \mathbb{N}\} \right)$ is bounded by an expression approximately equal to $\frac{4\pi}{\varepsilon_2} e^{4C_1/\varepsilon_1}$ for each $z \in \mathbb{T}$. But such estimates seem far too weak to give anything like (9).

5. APPENDICES

5.1. The formulation of Question 4 in [CN0]. In [CN0] this question was stated for ℓ^p for general $p \in [1, \infty]$. Here we restrict ourselves to the case $p = \infty$, which is, or would have been, the relevant case for our purposes.

Let N and k be arbitrary positive integers. Let \mathbb{D} denote the closed unit disk. Let $H^\infty(\mathbb{D}, \mathbb{C}^N)$ be the space of all \mathbb{C}^N valued functions $f(z) = (\phi_1(z), \phi_2(z), \dots, \phi_N(z))$ where each ϕ_j is in $H^\infty(\mathbb{D})$. Let $\|\cdot\|_\infty$ denote the ℓ^∞ norm on \mathbb{C}^N .

Question 4 (in the case $p = \infty$): Let f_1, f_2, \dots, f_k be k functions in $H^\infty(\mathbb{D}, \mathbb{C}^N)$. For each fixed $z \in \mathbb{D}$, let $M(z)$ be the subspace of \mathbb{C}^N defined by $M(z) = \text{span}\{f_1(z), f_2(z), \dots, f_k(z)\}$. For each positive ε , does there exist some subspace S of \mathbb{C}^N , whose dimension depends only on k and ε , with the following property: Whenever g is a function in $H^\infty(\mathbb{D}, \mathbb{C}^N)$ which satisfies $\|g(z)\|_\infty \leq 1$ for all $z \in \mathbb{D}$ and also

$$\text{dist}_{\ell^\infty}(g(z), M(z)) \leq \varepsilon \text{ for almost every } z \in \mathbb{T},$$

then $\text{dist}_{\ell^\infty}(g(0), S) \leq \psi(\varepsilon)$?

Here $\psi : (0, \infty) \rightarrow (0, \infty)$ is a fixed function of one variable (i.e., it does **not** depend on k or N) which satisfies $\lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = 0$.

A positive answer to Question 4 for $p = \infty$, would have implied a positive answer to Question 2. If, furthermore, such an answer could have been obtained with $\psi(\varepsilon) = \varepsilon^r$ for some positive constant r , then this would have implied a positive answer to Question 1.

5.2. THE PROOF OF LEMMA 1. This result is part of Lemma 8.2 of [Z] p. 57 (cf. also [Z] p. 61.) Here is a self contained proof, using various steps in [Z] pp. 57 and 61). First we recall the very classical result that the limit $\lim_{R \rightarrow \infty} \int_0^R \frac{\sin x}{x} dx$ exists and is finite. (This can of course be immediately deduced, using integration by parts, from the fact that $\frac{\cos x}{x^2}$ is absolutely integrable on $[1, \infty)$.)

We have, for each $x \in [0, \pi]$, that

$$\begin{aligned} \frac{x}{2} + \sum_{k=1}^K \frac{\sin kx}{k} &= \int_0^x \left(\frac{1}{2} + \sum_{k=1}^K \cos kt \right) dt = \int_0^x \frac{\sin(K + \frac{1}{2})t}{2 \sin \frac{t}{2}} dt \\ (10) \qquad \qquad \qquad &= \int_0^x \left(\frac{1}{2 \sin \frac{t}{2}} - \frac{1}{t} \right) \sin(K + \frac{1}{2})t dt + \int_0^x \frac{\sin(K + \frac{1}{2})t}{t} dt. \end{aligned}$$

The first integral in the preceding line has absolute value dominated by

$$\int_0^\pi \left| \frac{1}{2 \sin \frac{t}{2}} - \frac{1}{t} \right| dt = \int_0^\pi \left| \frac{t - 2 \sin \frac{t}{2}}{2t \sin \frac{t}{2}} \right| dt = c_1 < \infty$$

since the integrand tends to 0 as t tends to 0. The second integral equals $\int_0^{(K + \frac{1}{2})x} \frac{\sin u}{u} du$

and so its absolute value is bounded by the finite quantity $c_2 = \sup_{y \in [0, \infty)} \left| \int_0^y \frac{\sin u}{u} du \right|$. Combining these estimates, we see that

$$\sup_{x \in \mathbb{R}} \left| \sum_{k=1}^K \frac{\sin kx}{k} \right| = \sup_{x \in [0, \pi]} \left| \sum_{k=1}^K \frac{\sin kx}{k} \right| \leq \frac{\pi}{2} + c_1 + c_2,$$

which completes the proof. \square

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