Global Surjection and Global Inverse Mapping Theorems in Banach Spaces

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It is well known that a nonlinear operator that is one-to-one locally (i.e., in a neighborhood of any point at which it is defined) may fail to be one-to-one on the entire domain, even if the latter is very regular (say, a convex body; see reference 1, p. 25). Global inverse mapping theorems, therefore, are of considerable interest to us. We refer to reference 1 for a detailed discussion that is mainly concerned with the finite dimensional situation and its applications. It is probably in reference 2 that a general theorem for C^1-maps in Banach spaces was first proved. The assumption of the theorem is a combination of the standard criterion for local univalence and a condition of a global nature. The second principal result of this paper has a similar structure, but it deals with arbitrary continuous mappings and uses a slightly weakened form of the global condition (in this part, however, without changing essentially the techniques developed by Plastock[2]). This theorem is accompanied by a number of local univalence criteria for nondifferentiable maps.

The first result that we prove here is a global surjection theorem that offers a lower estimate for the image of a map with a closed graph (which is a guaranteed radius of a ball contained within the image). As with the inverse map theorem of which we spoke of above, this one uses a combination of a local sufficient condition (this time for surjection) and a global condition similar to that in the other theorem. We refer to references 3 and 4 for specific local surjection criteria for nondifferentiable maps.

In what follows, X and Y are Banach spaces, and F is a map from the whole of X into Y.

GLOBAL SURJECTION THEOREM

We set[4]

$$\mathrm{Sur}\,(F, x)(t) = \sup\{r \geq 0: B[F(x), r] \subset F[B(x, t)]\}, \qquad t > 0$$

(the modulus of surjection of F at x), where $B(w, a)$ is the closed ball of radius a around w. Thus, for any $t > 0$, the value of the modulus of surjection of F at x is the maximal radius of a ball around y contained in the F-image of the ball of radius t around x.

We further introduce the constant of surjection of F at x:

$$\mathrm{sur}\,(F, x) = \liminf_{t \to 0} t^{-1}\,\mathrm{Sur}\,(F, x)(t).$$

Obviously, $\mathrm{sur}\,(F, x) > 0$ is a sufficient condition for F to be surjective at x, that is, for $\mathrm{Sur}\,(F, x)(t)$ to be positive for small t.

Theorem 1: Suppose the graph of F is closed and there is a positive lower semicontinuous (l.s.c.) function $m(t)$ on $[0, \infty]$ such that

$$\text{sur}\,(F, x) \geq m(\|x\|), \forall x. \tag{1}$$

Then,

$$\text{Sur}\,(F, 0)(t) \geq \int_0^r m(t)\,dt, \forall r > 0.$$

Proof: For simplicity, we suppose that $F(0) = 0$. Fix a $y \in Y$ and set

$$k_y(r) = \sup\{\lambda \geq 0: \mu y \in F(rB_X), \forall \mu \in [0, \lambda]\}.$$

It suffices to show that

$$k_y(r) \geq \int_0^r m(t)\,dt \tag{2}$$

if $\|y\| = 1$. For $r = 0$, this is obvious, so we suppose that $r > 0$.

As a function of r, $k_y(r)$ is nondecreasing. Therefore, to prove equation 2, it suffices to show that

$$d^-k_y(t) = \liminf_{s \to +0} s^{-1}[k_y(t + s) - k_y(t)] \geq m(t) \tag{3}$$

for all $t > 0$.

Consider a sequence $\{x_n\}$ such that $\|x_n\| \leq r$ and $F(x_n) = \mu_n y$, where $\mu_n \to k_y(r)$. We claim that $\|x_n\| \to r$.

Indeed, assuming the contrary, we find a subsequence of $\{x_n\}$ with norms separated from r. With no loss of generality, we assume that the subsequence coincides with the entire sequence. Thus, there is $\tau > 0$ such that $\|x_n\| \leq r - \tau$ for all n. Set $m = \inf\{m(t): 0 \leq t \leq r\}$. Then $m > 0$ because $m(t)$ is l.s.c. and positive. Fix a positive $\gamma < \tau$. Then $\|x\| \leq r$ if $\|x - x_n\| \leq \gamma$ for some n. By this assumption, sur $(F, x) \geq m$ for any such x. It follows from proposition 2 of reference 4 that Sur $(F, x_n)(\gamma) \geq m\gamma$. Therefore, for any positive $\delta < \gamma$, we have

$$\mu_n y + (m\delta)B_Y \subset F(x_n + \delta B_X) \subset F(rB_X).$$

In particular, $\lambda y \in F(rB_X)$ if $\lambda < \mu_n + m\gamma$, which contradicts the definition of $k_y(r)$ because $\mu_n + m\gamma > k_y(r)$ if n is sufficiently large. The contradiction proves the claim.

Take an arbitrary $\delta > 0$ and set $m_\delta(r) = \inf\{m(t): |t - \tau| < \delta\}$. Clearly, $m_\delta(r) \to m(r)$ as $\delta \to 0$. Take a positive $s < \delta$ and set $s_n = -r + \|x_n\| + s$. Then,

$$r - \delta \leq \|x_n\| - s_n, \quad \|x_n\| + s_n \leq r + s \leq r + \delta.$$

By proposition 2 of reference 4,

$$F(x_n) + \gamma B_Y \subset F(x_n + s_n B_X) \subset F[(r + s)B_X]$$

if $s_n \leq m_\delta(r)$ [because sur $(F, x) \geq m_\delta(r)$ if $\|x - x_n\| \leq s_n$]. It follows that $\lambda y \in F[(r + s)B_X]$ if $\lambda < \mu_n + m_\delta(r)$, so $k_y(r + s) \geq \mu_n + s_n m_\delta(r)$. Consequently, $k_y(r + s) \geq k_y(r) + s m_\delta(r)$, which implies $d^-k_y(r) \geq m_\delta(r)$ for any $\delta > 0$; thus, equation 3 (QED).

By replacing equation 1 by one or another local surjection criterion, we obtain various specific surjection theorems as corollaries. Let $\mathcal{A}$ be a homogeneous set-valued mapping from X into Y. We set

$$C(\mathcal{A}) = \sup_{\|y\|=1} \inf\{\|x\|: y \in \mathcal{A}(x)\},$$

$$C^*(\mathcal{A}) = \inf\{\|y\|: y \in \mathcal{A}(x), \|x\| = 1\}.$$

Corollary 1.1: Suppose F has a closed graph and is everywhere Gateaux differentiable. If there is a positive l.s.c. function $m(t)$ on $[0, \infty)$ such that

$$C[F'(x)]m(\|x\|) \leq 1, \forall x,$$

then the conclusion of the theorem holds.

Proof: Because $m(\ )$ is l.s.c., it follows from corollary 1.6 of reference 4 that sur $(F, x) \geq m(\|x\|)$ for all x. A more general result can be obtained if we apply theorem 2 of reference 4.

Corollary 1.2: Given a subdifferential ∂ on the class of l.s.c. functions, if $F: X \to Y$ has a closed graph and there is a positive l.s.c. function $m(t)$ on $[0, \infty)$ such that

$$C^*(D^*[F(x)]) \geq m(\|x\|), \forall x,$$

then the conclusion of the theorem holds. Here, $D^*F(x)$ is the coderivative of F associated with ∂,

$$D^*F(x)(y^*) = \{x^*: (x^*, y^*) \in \partial\chi_{\text{Graph}F}[x, F(x)]\},$$

and χ_S is the indicator function of S (which is the function equal to 0 on S and equal to ∞ outside of S). We refer to reference 4 for those (few) properties of subdifferentials that are needed for applications like corollary 1.2. In fact, every subdifferential now used has these properties. It has to be mentioned, however, that the smaller the subdifferential, the better is the result. Therefore, the G-subdifferential described in reference 4 and the Dini subdifferential are preferable.

We mention one more corollary. Corollary 1.3: Suppose F is locally Lipschitz and Y^* has an equivalent uniformly convex norm. Then, the conclusion of the theorem holds provided that

$$x^* \in \partial_A(y^* \circ F)(x) \to \|x^*\| \geq m(\|x\|), \forall x,$$

where $m(\ )$ is as mentioned above and ∂_A is the A-subdifferential (see references 4 and 5, and references cited therein).

GLOBAL INVERSE MAPPING THEOREM

Theorem 2: Suppose that $F: X \to Y$ is a continuous mapping that is locally one-to-one (i.e., every x has a neighborhood in which F is one-to-one) and there exists a positive l.s.c. function $m(t)$ on $[0, \infty)$ such that

$$\int_0^\infty m(t)\,dt = \infty \quad \text{and} \quad \text{sur}\,(F, x) \geq m(\|x\|), \quad \forall x.$$

Then F is a homeomorphism onto Y, the inverse mapping F^{-1} is locally Lipschitz, and, for every y, the Lipschitz constant of F^{-1} at y is not greater than $m(\|F^{-1}(y)\|)^{-1}$.

Proof: According to Plastock,[2] F is a homeomorphism onto Y if and only if F is a local homeomorphism and the following property holds:

Property I. If $p(t)$: $[0, \alpha) \to X$ is a continuous curve such that

$$F[p(t)] = ty_1 + (1 - t)y_2,$$

then there is a sequence $t_n \to \alpha$ such that $\lim p(t_n)$ exists.

We begin by showing that for every x, the local inverse of F (we also denote it F^{-1} for simplicity) is a Lipschitz mapping with a Lipschitz constant at $y = F(x)$ not exceeding $m(\|x\|)^{-1}$.

Indeed, fix an $\epsilon \in [0, m(\|x\|)]$. Then sur $(F, u) \geq m(\|x\|) - \epsilon = m_\epsilon$ for all u of a neighborhood of x. Using proposition 1.2 of reference 4, we can find a $t_0 > 0$ such that F is one-to-one on $B(x, 2t_0)$ and Sur $(F, u)(t) \geq m_\epsilon t$ if $\|u - x\| \leq t_0$, $0 < t \leq t_0$.

Take a u such that $\|u - x\| \leq t_0$, set $v = F(u)$, and let z be such that $\|z - v\| = m_\epsilon t$, $t \leq t_0$. Then, there is a unique w such that $z = F(w)$ and $\|w - u\| \leq t = \|z - v\|/m$. It follows that

$$\|F^{-1}(z) - F^{-1}(y)\| \leq m_\epsilon^{-1}\|y - v\|$$

if z and v are sufficiently close to $y = F(x)$.

Thus, F^{-1} is Lipschitz and $F(\quad)$ is a local homeomorphism. We also have $m_\epsilon^{-1} \to m(\|x\|)^{-1}$ when $\epsilon \to 0$.

Suppose now that $p(\quad)$ is given as in Property I. Fix a $t \in [0, \alpha)$. It follows from the just proven property that for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\begin{aligned} \|p(\tau) - p(\tau')\| &\leq (m[\|p(t)\|] - \epsilon)^{-1}\|F[p(\tau)] - F[p(\tau')]\| \\ &= (m[\|p(t)\|] - \epsilon)^{-1}|\tau - \tau'| \cdot \|y_2 - y_1\| \end{aligned} \tag{4}$$

if $|\tau - t| < \delta$, $|\tau' - t| < \delta$, and $\tau, \tau' < \alpha$. This implies that the function $t \to p(t)$ is locally Lipschitz on $[0, \alpha)$ and that

$$\frac{d}{dt}\|p(t)\| \leq \frac{\|y_2 - y_1\|}{m[\|p(t)\|]}$$

almost everywhere on $[0, \alpha)$. In particular, for every positive $\bar{t} < \alpha$, we have

$$\int_{\|p(0)\|}^{\|p(\bar{t})\|} m(r)\,dr = \int_0^{\bar{t}} m[\|p(t)\|]\,d\|p(t)\| \leq \bar{t}\|y_2 - y_1\| \leq \alpha\|y_2 - y_1\|.$$

By this assumption, the integral of $m(\quad)$ between 0 and ∞ equals ∞. It follows that $\|p(t)\| \leq k < \infty$ for all $t \in [0, \alpha)$. In view of equation 4, this implies that $p(\quad)$ itself is locally Lipschitz on $[0, \alpha)$ with a Lipschitz constant nowhere greater than $c = \|y_2 - y_1\|/m$, where m is the lower bound of the values of $m(\quad)$ corresponding to $r \leq \max\{\|y_2\|, \|\alpha y_1 + (1 - \alpha)y_2\|\}$. The latter, in turn, implies that $p(\quad)$ is Lipschitz on the whole of $[0, \alpha)$ with a Lipschitz constant not greater than c. Indeed, let $\bar{t}$ be the upper bound of those $t \in [0, \alpha)$ for which $\|p(\tau) - p(\tau')\| \leq c|\tau - \tau'|$ if $0 \leq \tau, \tau' \leq t$. We claim that $\bar{t} = \alpha$.

Next, if we assume the contrary, then, as follows from equation 4, a $\delta > 0$ exists such that $\bar{t} + \delta < \alpha$ and $\|p(\tau) - p(\tau')\| \leq c|\tau - \tau'|$ if $|\tau - \bar{t}| < \delta$, $|\tau' - \bar{t}| < \delta$. Suppose now that $\tau', \tau'' \leq \bar{t} + \delta$ and $\tau' < \tau''$. If $\tau'' < \bar{t}$, then

$$\|p(\tau') - p(\tau'')\| \leq c|\tau' - \tau''| \tag{5}$$

by definition of $\bar{t}$; if $\tau' > \bar{t} - \delta$, then equation 5 is valid due to the choice of δ. Finally, for $\tau'' \geq \bar{t}$, $\tau' \leq \bar{t} - \delta$, we choose a $\tau(\bar{t} - \delta, \bar{t})$ and have

$$\begin{aligned} \|p(\tau') - p(\tau'')\| &\leq \|p(\tau') - p(\tau)\| + \|p(\tau) - p(\tau'')\| \\ &\leq c(\tau - \tau') + c(\tau'' - \tau) = c|\tau'' - \tau'|. \end{aligned}$$

Thus, $p(\quad)$ is a Lipschitz curve and, therefore, $\lim_{t\to\alpha} p(t)$ exists. This completes the proof of the theorem.

Remark: We observe that the Lipschitz constant of the inverse map is completely defined by surjection properties of F, while the property of being one-to-one enters the statements in a purely qualitative way.

CRITERIA FOR LOCAL UNIVALENCE

Suppose that F: $X \to Y$ is defined in a neighborhood of x_0. Recall that a homogeneous set-valued mapping $\mathcal{A}$ from X into Y [i.e., such that $\mathcal{A}(\lambda x) = \lambda\mathcal{A}(x)$ for $\lambda > 0$] is a strict prederivative of F at x_0 if

$$F(x + h) \subset F(x) + \mathcal{A}(h) + r(x; h)\|h\|B, \tag{6}$$

where $r(x; h) \to 0$ as $x \to x_0$ and $h \to 0$. We refer to reference 6 for more details.

Proposition 1: Suppose F has a strict prederivative $\mathcal{A}$ at x_0 such that $\rho[0, \mathcal{A}(h)] \geq c\|h\|$ $(c > 0)$. Then, F is one-to-one in a neighborhood of x_0. [Here, $\rho(x, S)$ is the distance from x to S.]

Proof: Take an $\epsilon > 0$ such that $r(x; h) < c$ for $\|x - x_0\| < \epsilon$, $\|h\| < 2\epsilon$. If u and x belong to the ϵ-ball around x_0 and $h = u - x$, then $0 \notin \mathcal{A}(h) + r(x; h)\|h\|B$ (otherwise, $\mathcal{A}(h)$ would contain a vector with a norm less than $c\|h\|$) and consequently $F(u) = F(x + h) \neq F(x)$.

A linear operator A is a strict prederivative if and only if it is a strict derivative [i.e., if $\|h\|^{-1}\|F(x + h) - F(x) - Ah\| \to 0$ as $h \to 0$, $x \to x_0$]. The condition $\rho[0, \mathcal{A}(h)] \geq c\|h\|$ assumes the form $\|Ah\| \geq c\|h\|$, and it means that A is a linear homeomorphism onto a closed subspace of Y and the norm of the inverse operator (from the subspace into X) is not greater than c^{-1}.

Thus (in view of the Banach open mapping theorem), proposition 1 contains the fact that F is one-to-one near x_0 provided that F is strictly differentiable at x_0, ker $F'(x_0) = \{0\}$, and Im $F'(x_0)$ is a closed subspace of Y.

In general, the condition $\rho[0, \mathcal{A}(h)] \geq c\|h\|$ is equivalent to $C^*(\mathcal{A}) = \inf\{\|y\|: y \in \mathcal{A}(h), \|h\| = 1\} \geq c$. If $\mathcal{A}$ is convex-valued, then

$$C^*(\mathcal{A}) = -\sup_{\|h\|=1}\inf_{\|y^*\|\leq 1} s(y^*, h),$$

where $s(y^*, h) = \sup\{\langle y^*, y\rangle : y \in \mathcal{A}(h)\}$ is the support function of $\mathcal{A}$ (see reference 6). The following proposition is obvious.

Proposition 2: If $\mathcal{A}_1$, $\mathcal{A}_2$ are strict prederivatives of F at x, then so is $\mathcal{A}(x) = \mathcal{A}_1(x) \cap \mathcal{A}_2(x)$. If $\mathcal{A}_1(x) \subset \mathcal{A}_2(x)$ for all x, then $C^*(\mathcal{A}_1) \geq C^*(\mathcal{A}_2)$.

Strict prederivatives can often be represented in the form

$$\mathcal{A}(h) = \mathrm{cl}\,\{Ah : A \in \mathfrak{A}\}, \tag{7}$$

where $\mathfrak{A}$ is a collection of linear continuous operators and "cl" denotes the closure operation. This is particularly true for $D^*F(x)$ if Y is reflexive and F is Lipschitz near the point in question.

Proposition 3: Let $\mathcal{A}$ be a homogeneous set-valued mapping defined by equation 7. Then $C^*(\mathcal{A}) > 0$ if and only if every $A \in \mathfrak{A}$ is a homeomorphism onto a closed subspace of Y (depending on A) and the norms of the inverse operators are bounded by the same constant for all $A \in \mathfrak{A}$. In this case,

$$C^*(\mathcal{A}) = \inf\{\|A^{-1}\|^{-1} : A \in \mathcal{A}\}.$$

Proof: We have

$$\begin{aligned} C^*(\mathcal{A}) &= \inf\{\|y\| : y \in \mathcal{A}(h),\ \|h\| = 1\} \\ &= \inf\{\|Ah\| : A \in \mathfrak{A},\ \|h\| = 1\} \\ &= \inf_{A \in \mathfrak{A}} \inf\{\|Ah\| : \|h\| = 1\} = \inf_{A \in \mathfrak{A}} C^*(A). \end{aligned}$$

Thus, $C^*(\mathcal{A}) \geq k > 0$ if and only if $C^*(A) \geq k$ for all $A \in \mathfrak{A}$. The latter means that every A is a homeomorphism onto a closed subspace of Y and $\|A^{-1}\| \leq 1/k$.

For locally Lipschitz mappings, an important class of strict prederivatives can be defined in terms of the generalized gradients of Clarke functions $y^* \circ F$, $y^* \in Y^*$. Namely, for any $\epsilon > 0$, the function

$$s_\epsilon(y^*, h) = \sup_{\|x - x_0\| < \epsilon} d^0(y^* \circ F)(x; h)$$

[where $d^0\varphi(x; h)$ stands for Clarke's directional derivative of φ at x] is the support function of a strict prederivative of F at x_0 that will be denoted $D^0_\epsilon F(x_0)$:

$$D^0_\epsilon F(x_0)(h) = \{y : \langle x^*, h\rangle \leq s_\epsilon(y^*, h),\ \forall y^*\}.$$

This prederivative is by definition convex-valued and, moreover, it has an additional property of being a fan [a convex-valued homogeneous mapping such that $\mathcal{A}(x + u)$ belongs to the closure of $\mathcal{A}(x) + \mathcal{A}(u)$ for every x, u.[6]]

If X and Y are both finite dimensional and F is Lipschitz near x_0, then

$$s(y^*, h) = \sup\{\langle x^*, h\rangle : x^* \in \partial_c(y^* \circ F)(x_0)\}$$

is the support function of the smallest strict prederivative of F at x_0 (where ∂_c denotes Clarke's generalized gradient).

By $C^*(F^*, x)$, we denote the upper bound of "dual Banach constants" $C^*(\mathcal{A})$ over the collection of all strict prederivatives $\mathcal{A}$ of F at x. Then, proposition 1 can be

reformulated in the following way: if $C^*(F, x) > 0$, then F is one-to-one in a neighborhood of x.

If F is strictly (or continuously) differentiable at x, then $C^*(F, x) = C^*[F'(x)]$ easily follows from proposition 2. It is difficult, however, to find a recipe to calculate $C^*(F, x)$ in a more general situation. In certain important cases, a smaller quantity is calculable: namely, the upper bound of $C^*(\mathcal{A})$ over all convex-valued strict prederivatives. We shall denote it by $c^*(F, x)$.

Proposition 4: Suppose F is Lipschitz near x_0. Then,

$$\begin{aligned} c^*(F, x_0) &= \lim_{\epsilon \to 0} C^*[D^0_\epsilon F(x_0)] \\ &= -\lim_{\epsilon \to 0} \sup_{\|h\| = 1} \inf_{\|y^*\| = 1} \sup_{\|x - x_0\| < \epsilon} d^0(y^* \circ F)(x; h). \end{aligned}$$

In addition, if $\dim X = \dim Y < \infty$, then

$$c^*(F, x_0) = C^*[D^0F(x_0)] = -\sup_{\|h\| = 1} \inf_{\|y^*\| \leq 1} d^0(y^* \circ F)(x_0; h),$$

and in cases where this quantity is positive,

$$c^*(F, x_0) = \inf\{\|A^{-1}\|^{-1} : A \in \partial_c F(x_0)\},$$

where $\partial_c F(x)$ is Clarke's generalized gradient of F at x.

Proof: The first equality of $c^*(F, x_0) = \lim C^*[D^0_\epsilon F(x_0)]$ follows from proposition 9.8 of reference 6; the rest follow from the discussion preceding the statement.

Proposition 5: Suppose F is Gateaux differentiable everywhere in a neighborhood of x_0. Then,

$$c^*(F, x_0) = -\lim_{\epsilon \to 0} \sup_{\|h\| = 1} \inf_{\|y^*\| \leq 1} \sup_{\|x - x_0\| < \epsilon} \langle y^*, F'(x)h\rangle. \tag{8}$$

Remark: In the case where F is continuously Gateaux differentiable, proposition 5 is an immediate consequence of proposition 4 because, in this case,

$$d^0(y^* \circ F)(x; h) = \langle y^*, F'(x)h\rangle.$$

Proof: Set

$$s_\epsilon(y^*, h) = \sup\{\langle y^*, F'(x)h\rangle : \|x - x_0\| < \epsilon\}.$$

This function is sublinear and l.s.c. in each argument; hence, the set

$$\mathcal{A}_\epsilon(h) = \{y : \langle y^*, y\rangle \leq s_\epsilon(y^*, h),\ \forall y^*\}$$

is (obviously convex-closed and) nonempty for any h. It follows from the well-known equality

$$\langle y^*, F(u) - F(x)\rangle = \int_0^1 \langle y^*, F'[x + t(u - x)](u - x)\rangle\, dt$$

that $F(u) - F(x) \in \mathcal{A}_\epsilon(u - x)$ if both u and x belong to the ϵ-ball around x_0; thus, $\mathcal{A}_\epsilon$ is a strict prederivative of F at x_0.

Suppose now that $\mathcal{A}$ is an arbitrary strict prederivative of F at x_0; that is, equation 6 holds. Fix a $\delta > 0$ and choose $\epsilon > 0$ so small that $r(x; h) < \delta$ if $\|x - x_0\| < \epsilon$, $\|h\| < \epsilon$.

For such x and h,

$$\langle y^*, F'(x)h\rangle = \lim_{t\to 0} t^{-1}\langle y^*, F(x+th) - F(x)\rangle \leq s(y^*, h) + \delta\|h\|$$

[where $s(y^*, h)$ is the support function of $\mathcal{A}$]; hence,

$$s_\varepsilon(y^*, h) \leq s(y^*, h) + \delta\|h\|.$$

It follows that $C^*(\mathcal{A}_\varepsilon) \geq C^*(\mathcal{A}) - \delta$ and—as $\mathcal{A}$ is an arbitrary convex-valued strict prederivative and δ is an arbitrary positive number—that

$$\lim_{\varepsilon\to 0} C^*(\mathcal{A}_\varepsilon) = c^*(F, x_0).$$

It remains for us to recall that $\mathcal{A}_\varepsilon$ is convex-valued and to apply the formula for C^* in the case of the convex-valued mapping that was mentioned before the statement of proposition 2.

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