LIFTS OF SIMPLE CURVES IN FINITE REGULAR COVERINGS OF CLOSED SURFACES

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ABSTRACT. Suppose S is a closed orientable surface and $\tilde{S}$ is a finite sheeted regular cover of S. When studying mapping class groups, the following question arose: Do the lifts of simple curves from S generate $H_1(\tilde{S},\mathbb{Z})$? A family of examples is given for which the answer is "no".

CONTENTS

1. INTRODUCTION

Let S be a genus g closed, orientable surface with base point and no boundary. Fix $p\colon \tilde{S} \to S$, a finite-sheeted regular covering of S, where $\tilde{S}$ is connected. The *simple curve homology of* p (denoted by $sc_p(H_1(\tilde{S};\mathbb{Z}))$ is the span of $[\tilde{\gamma}]$ in $H_1(\tilde{S};\mathbb{Z})$ such that $\tilde{\gamma}$ is a connected component of $p^{-1}(\gamma)$ and γ a simple closed curve in S.

Recall that the Torelli group of a surface is the subgroup of the mapping class group that consists of surface diffeomorphisms that act trivially on homology with integer coefficients. A survey of the Torelli group can be found in [5]. The following question was posed by Julien Marché on Mathoverflow, [10], and arose while studying the ergodicity of the action of the Torelli group on SU(2)-character varieties of surfaces, [3]:

Question 1 (see [10]). *(1) Does $sc_p(H_1(\tilde{S};\mathbb{Z})) = H_1(\tilde{S},\mathbb{Z})$?*
(2) If not, how can we characterize the submodule $sc_p(H_1(\tilde{S};\mathbb{Z}))$?

Since writing the first version of this paper, there has been much progress on answering Question 1 and related questions. By using quantum SO(3)-representations coming from

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TQFT, the authors of [6] show that equality in part (i) of Question 1 does not always hold. However, such covers are not explicitly constructed in [6].

Denote by $Mod(\Sigma)$ the mapping class group of a surface Σ. When Σ has p punctures, genus g and n boundary components, we will sometimes denote it by $\Sigma^p_{g,n}$.

We now explain a relation between a rational coefficient version of Question 1

$$sc_p(H_1(\tilde{S};\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q} = H_1(\tilde{S},\mathbb{Q}) \tag{1}$$

and the Ivanov Conjecture [4]. The latter states that $H_1(\Gamma,\mathbb{Q})=0$ for all finite index subgroup of $Mod(\Sigma^p_{g,n})$ with $g\geq 3$.

Fix a finite-sheeted regular cover $p\colon \tilde{\Sigma}\to\Sigma$ of Σ, and let Γ_p denote the subgroup of $Mod(\Sigma)$ generated by all $\phi\in Mod(\Sigma)$ such that ϕ lifts to $\tilde{\phi}\in Mod(\tilde{\Sigma})$. Let $\tilde{\Gamma}_p$ denote the subgroup of $Mod(\tilde{\Sigma})$ generated by $\tilde{\phi}$ as before.

Boggi-Looijenga [8] observe that when (1) holds, the $\tilde{\Gamma}$ invariant submodule of $H_1(\tilde{\Sigma};\mathbb{Q})$ is trivial. If that happens for all finite regular covers p of Σ, then Theorem C of [11] implies Ivanov's Conjecture for mapping class groups of surfaces with genus $g+1$, $n-1$ boundary components and p punctures. A further discussion to the background of this question with integer and rational coefficients can be found in Section 8 of [2].

For surfaces with nonempty boundary or at least one puncture, Farb-Hensel [2] showed that analogues of Question 1 are equivalent to statements in representation theory, providing a framework in which to answer part (ii) of Question 1 with integer or rational coefficients. For punctured surfaces, it was shown in a recent paper of Malestein-Putman [9] that Question 1 fails.

Our main theorem is an explicit family of counterexamples to Question 1. These are iterated homology surface coverings, with at least two iterations, where the last one uses an integer $m\geq 3$.

Theorem 2. *For the examples of Section 3.5, equality in part (i) of Question 1 is false.*

The intuition that lifts of simple curves should generate homology possibly stems from the fact that counterexamples should be expected to have large genus; "small" genus coverings disproportionately satisfy a plethora of conditions that guarantee this. For example, it seems to be well-known that when the deck transformation group is Abelian, $sc_p(H_1(\tilde{S};\mathbb{Z}))= H_1(\tilde{S},\mathbb{Z})$. (As was explained to the author by Marco Boggi, [1], this claim follows from arguments of Boggi-Looijenga and [7] with rational and hence integral coefficients. It was proven directly for integral coefficients in an earlier version of this paper. For fundamental groups of surfaces with nonempty boundary or at least one puncture and complex coefficients, this claim is Proposition 3.1 of [2].)

Organisation of Paper. Section 2 recalls and provides some background and useful notation. Section 3 studies a family of covering spaces in detail. These covering spaces are compositions of well known covering spaces, and the author makes no claims of originality in this section. The properties of the covering spaces that will be needed are simple and elementary, and hence are proven directly for completeness. The same is true for Subsection 3.3, in which it is explained how to obtain spanning sets for homology of covers using relations in the deck transformation group. These results and examples are used in Subsection 3.4 to highlight differences between integral and rational homology, and in Subsection 3.5 to construct examples for which $sc_p(H_1(\tilde{S};\mathbb{Z}))$ is a proper submodule of $H_1(\tilde{S};\mathbb{Z})$.

Acknowledgments. As noted above, the author became aware of this question on MathOverflow, and is grateful to Julién Marché for posting the question, and the subsequent discussion by Richard Kent and Ian Agol. Mustafa Korkmaz and Sebastian Hensel pointed out an error in the first formulation of this paper. This paper was greatly improved as a result of communication with Marco Boggi, Stavros Garoufalidis, Neil Hoffman, Thomas Koberda, Eduard Looijenga, Andrew Putman, and the detailed comments of the anonymous referee. The author is also grateful to Ferruh Özbudak for a fascinating discussion on similar methods in coding theory. This research was funded by a Tübitak Research Fellowship 2216 and thanks the University of Melbourne for its hospitality during the initial and final stages of this project.

2. Assumptions and background

Since the 2-sphere is simply connected, hence has no nontrivial covers, Question 1 holds for genus 0 surfaces. Moreover, as pointed out by Ian Agol in [10], part (i) of Question 1 holds for genus 1 surfaces. Therefore we only need consider closed surfaces S of genus at least two.

Curves and intersection numbers. By a *curve* in S we mean the free homotopy class of the image of a smooth map (which can be taken to be an immersion) of S^1 into S. In this convention, a curve is necessarily closed and connected. A curve γ is said to be *simple* if the free homotopy class contains an embedding of S^1 in S.

When it is necessary to work with based curves, the assumption will often be made that wherever necessary, a representative of the free homotopy class is conjugated by an arc, to obtain a curve passing through the base point. The counterexamples constructed in this paper are obtained by iterated Abelian covers. When analysing how a curve lifts, it will therefore only be necessary to know the homology class of the curve in the intermediate cover in question.

If c is a curve in a surface S, then $[c] \in H_1(S, \mathbb{Z})$ will denote the corresponding homology class.

d-lifts. Given a finite sheeted regular covering $p\colon \tilde{S} \to S$, the deck transformation group is denoted by D. For an element $\gamma \in \pi_1(S)$, let $d = d(\gamma)$ denote the smallest natural number for which $\gamma^d \in \pi_1(\tilde{S}) \subset \pi_1(S)$. Note that d exists and $d \leq |D|$, where $|D|$ is the number of elements of D. In that case, we will say that γ d-lifts.

Primitivity. A homology class h is *primitive* if it is nontrivial there does not exist an integer $k > 1$ and a homology class h_{prim} such that $h = kh_{prim}$.

3. Homology coverings of a surface

In this section we recall the definition of a homology covering of a surface and its basic properties. Our goal is to show that iterated homology coverings give a counterexample to part (i) of Question 1.

3.1. **Definition of a homology covering.** We begin by recalling the well-known homology covers of a closed genus g surface S. Fix a natural number m and let $p : \tilde{\Sigma} \to S$ denote the covering space of S corresponding to the epimorphism $\phi : \pi_1(S) \to H_1(S, \mathbb{Z}/m\mathbb{Z})$ is given by the composition

$$\pi_1(S) \to H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z}/m\mathbb{Z}) \tag{2}$$

of the Hurewitz homomorphism with the reduction of homology modulo m. Let $D \simeq (\mathbb{Z}/m\mathbb{Z})^{2g}$ denote the deck transformation group of ϕ. These coverings are known as the *mod-m-homology coverings* of S. Using an Euler characteristic argument, it follows that the genus of $\tilde{S}$ is $m^{2g}(g-1)+1$. Homology coverings are characteristic in the following sense: the subgroup $\pi_1(\tilde{S})$ of $\pi_1(S)$ is invariant under all surface diffeomorphisms of S.

3.2. **Homology coverings with $g = 2$.** When $g = 2$, we can give an explicit description of the covering $\tilde{S}$ as follows. Write $S = t_1 \cup t_2$ where t_i for $i = 1, 2$ are genus 1 subsurfaces of S with one boundary component. The pre-image of each t_i under ϕ consists of m^2 copies of an m^2-holed torus, each of which is an m^2-fold cover of t_i, as illustrated in Figure 1.

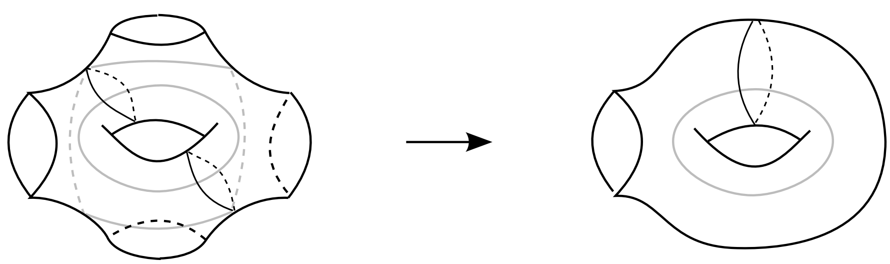

FIGURE 1. The pre-image of a 1-holed torus under the covering when $m = 2$.

The pre-images $\phi^{-1}(t_i)$ for $i = 1, 2$ are glued together as follows. Let K_{m^2,m^2} be the bipartite graph with vertices the connected components of $\phi^{-1}(t_i)$ for $i = 1, 2$. Each connected component of $\phi^{-1}(t_1)$ is glued to each connected component of $\phi^{-1}(t_2)$ along a boundary curve, and this is represented by an edge of K_{m^2,m^2}. This is illustrated in Figure 2 for

$m = 2$, where the graph K_{m^2,m^2} is shown in grey.

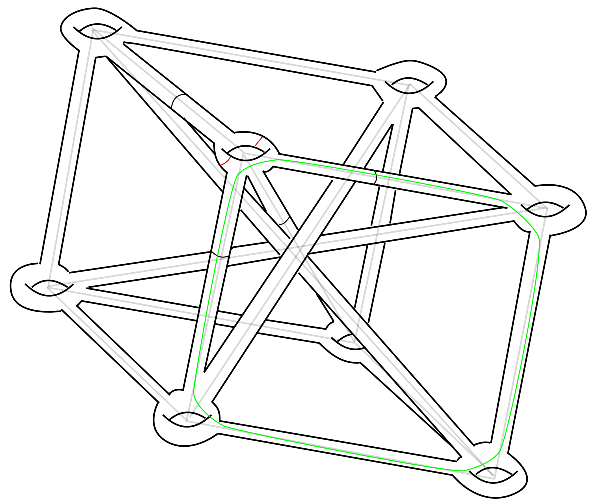

FIGURE 2. The covering space $\tilde{S}$ for $m = 2$. The red curves are some connected components of pre-images of the generator a_1 of $\pi_1(S)$. The green curve is a connected component of the lift of b_1b_2, and the black curves are connected components of the lift of $[a_1, b_1]$.

This subsection is now concluded with a useful lemma.

Lemma 3. *Let $\tilde{S} \to S$ be the homology cover of* (2) *with $g = 2$. All simple, null homologous curves in S 1-lift to nonseparating curves in $\tilde{S}$. Moreover, no curve in a primitive homology class in S 1-lifts.*

Proof. To start off with, the fact that null homologous curves 1-lift is a consequence of the fact that the cover has Abelian deck transformation group.

In Figure 2, the black curves are lifts of simple null homologous curves from S. These black curves all 1-lift to non-separating curves in $\tilde{S}$. The covering is characteristic, so this observation is true independently of the choice of basis $\{a_1, b_1, , a_2, b_2\}$ from Equation (3). It follows that all simple null homologous curves 1-lift to simple, non-separating curves in $\tilde{S}$. An analogous argument shows this is also true for $m > 3$. □

3.3. **The homology of a homology covering.** In this subsection, we will assume that S is a closed surface of genus 2. It will be useful to describe the homology covers of a surface S using a fixed presentation

$$\pi_1(S) = \langle a_1, b_1, a_1, b_2 \mid [a_1, b_1][a_2, b_2]\rangle \tag{3}$$

where a_i, b_i are curves representing a usual symplectic basis for $H_1(S; \mathbb{Z})$, satisfying $i(a_i, a_j) = i(b_i, b_j) = 0$ and $i(a_i, b_j) = \delta^i_j$.

The homomorphism (2) is given explicitly by

$$\begin{aligned} a_1 &\mapsto (1,0,0,0) \\ b_1 &\mapsto (0,1,0,0) \\ a_2 &\mapsto (0,0,1,0) \\ b_2 &\mapsto (0,0,0,1) \end{aligned} \tag{4}$$

It will now be shown how to use the relations of the deck transformation group to obtain a generating set for $H_1(\tilde{S}, \mathbb{Z})$. This subsection is only needed to show the necessity of the assumption $m \geq 3$ in the construction later on.

Suppose now that D is any group for which there is the short exact sequence

$$1 \to \pi_1(\tilde{S}) \to \pi_1(S) \xrightarrow{\phi} D \to 1 \tag{5}$$

for some covering space $\tilde{S}$. Let $\{g_1, \ldots, g_n\}$ be a set of elements of $\pi_1(S)$ whose image under ϕ is a generating set for D. Let $r = w(g_1, \ldots, g_n)$ be a word in the elements $\{g_1, \ldots, g_n\}$ that is mapped to the identity by ϕ, i.e. $w(\phi(g_1), \ldots, \phi(g_n)) = I \in D$. The word r could be either nontrivial in $\pi_1(S)$, or it could be a product of conjugates of the relation $[a_1, b_1][a_2, b_2]$ in $\pi_1(S)$.

Note that $\{\phi(a_1), \phi(a_2), \phi(b_1), \phi(b_2)\}$ is a generating set for D, where $\{a_1, a_2, b_1, b_2\}$ is the choice of generating set from Equation (3). Due to the assumption that the genus of S is two, D can always be generated by four generators.

If $r = r(g_1, \ldots, g_n)$ is a word in $g_1, \ldots, g_n$, then $\phi(r) = r(\phi(g_1), \ldots, \phi(g_n))$. A set of words $\{r_1(g_1, \ldots, g_n), r_2(g_1, \ldots, g_n), \ldots, r_k(g_1, \ldots, g_n)\}$ is a complete set of relations for D if

$$\{\phi(g_1), \ldots, \phi(g_n) \mid \phi(r_1), \ldots, \phi(r_k)\}$$

is a presentation for D.

Lemma 4. *Suppose D can be generated by no fewer than four generators. Let $\{r_1, r_2, \ldots, r_k\}$ be a set of words in $\pi_1(S)$ mapping to a complete set of relations for D. Then the set of homology classes of connected components of the pre-images of the curves representing the words $r_1, r_2, \ldots, r_k$ is a generating set for $H_1(\tilde{S}; \mathbb{Z})$.*

Proof. Consider a presentation for D given by

$$\{\phi(a_1), \phi(a_2), \phi(b_1), \phi(b_2) \mid \phi(r_1'), \phi(r_2'), \ldots, \phi(r_k')\}$$

From the exact sequence (5), we see that each of the r_i' represents an element of $\pi_1(\tilde{S})$.

When the image of $\{r_1', r_2', \ldots, r_k'\}$ under ϕ is a complete set of relations for D, it follows that any element of $\pi_1(\tilde{S})$ is a product of conjugates of elements of the set $\{r_1, r_2, \ldots, r_k\}$. Let c be a loop representing the element r_i'. The connected components of $p^{-1}(c)$ correspond to conjugates of r_i'. Therefore, the connected components of the pre-images of the closed curves represented by the words $\{r_1', r_2', \ldots, r_k'\}$ are a generating set for $H_1(\tilde{S}, \mathbb{Z})$.

We now use the assumption that D has no fewer than four generators to show that this is true for any presentation of D. Another presentation for D can be written as follows

$$\{w_1, w_2, w_3, w_4 \mid \phi(r_1), \phi(r_2), \ldots, \phi(r_m)\}$$

where w_1, w_2, w_3 and w_4 are words in $\phi(a_1)$, $\phi(a_2)$, $\phi(b_1)$ and $\phi(b_2)$, and $r_1, r_2, \ldots, r_m$ are products of conjugates of $r_1', r_2', \ldots, r_k'$. For the same reason as before, the connected components of the pre-images of the closed curves in S represented by the words $\{r_1, r_2, \ldots, r_m\}$ are a generating set for $H_1(\tilde{S}; \mathbb{Z})$. Note that this is not necessarily true for a proper subset of $\{r_1, r_2, \ldots, r_m\}$.

If one or more of w_1, w_2, w_3 or w_4 is mapped to the identity in D, there is a presentation for D with generators consisting of a proper subset of $\{w_1, w_2, w_3, w_4\}$ and relations consisting of a proper subset of $\{\phi(r_1), \phi(r_2), \ldots, \phi(r_m)\}$. However, the assumption that D is generated by no fewer than four elements rules out the possibility of a presentation for D with relations consisting of a proper subset of $\{\phi(r_1), \phi(r_2), \ldots, \phi(r_m)\}$. □

To use Lemma 4, a complete set of relations for D is needed. To start off with, there are the relations $\phi^m(a_i) = 1$ and $\phi^m(b_i) = 1$. These relations correspond to the submodule of $H_1(\tilde{S}; \mathbb{Z})$ spanned by connected components of pre-images of the generators. In Figure 2 with $m = 2$, some examples are drawn in red. Other relations are, for example, commutation relations or the relations stating that the remaining group elements have order m in the deck transformation group. When $m = 2$, the commutation relations are a consequence of the relations stating that all 16 group elements have order m. For example,

$$\begin{aligned}
\phi(a_1)\phi(b_1) &= (\phi(a_1)\phi(b_1))^{-1} && \text{since } \phi(a_1)\phi(b_1) \text{ has order 2} \\
&= \phi(b_1)^{-1}\phi(a_1)^{-1} \\
&= \phi(b_1)\phi(a_1) && \text{since } \phi(a_1) \text{ and } \phi(b_1) \text{ each have order 2}
\end{aligned}$$

It will be shown later that this is a peculiarity of $m = 2$; as shown in Lemma 5, for $m > 2$, we also need commutation relations. For $m = 2$, the relations stating that all elements of the deck tranformation group are of order two are a complete set of relations for D. By Lemma 4, this gives us a set of simple, nonseparating curves whose pre-images span $H_1(\tilde{S}; \mathbb{Z})$.

3.4. **Integral versus rational homology.** Examples for which $sc_p(H_1(\tilde{S}; \mathbb{Z}))$ can not be all of $H_1(\tilde{S}; \mathbb{Z})$ will now be constructed by showing that for $m > 2$, connected components of pre-images of separating curves are needed to span $H_1(\tilde{S}; \mathbb{Z})$. The promised examples are then obtained by taking the composition of two such covering spaces, using Lemma 3.

Lemma 5. *In the homology covering space $p : \tilde{S} \to S$ of (2) with $m \geq 3$ and $g \geq 2$, connected components of pre-images of simple, nonseparating curves do not span $H_1(\tilde{S}; \mathbb{Z})$.*

Proof. The lemma will be proven for $m = 3$ and it is claimed that analogous arguments work for $m > 3$.

We will show by contradiction that $[p^{-1}([a_1, b_1])] \in H_1(\tilde{S}, \mathbb{Z})$ is not in the span of homology classes of pre-images of simple, nonseparating curves of S.

Suppose a connected component of $p^{-1}([a_1, b_1])$ is in the span of connected components of pre-images of simple, nonseparating curves. In the group $\pi_1(S)$ there is therefore the relation

$$[a_1, b_1]^{-1}\gamma_1^3\gamma_2^3\gamma_3^3 \dots \gamma_k^3\kappa = 1 \in \pi_1(\tilde{S}) \tag{6}$$

where κ is in the subgroup $[\pi_1(\tilde{S}), \pi_1(\tilde{S})]$ of $\pi_1(S)$ and γ_i are elements of $\pi_1(S)$ representing simple closed curves in S.

Let N denote the subgroup of $\pi_1(S)$ normally generated by words of Equation (7).

$$[w, a_i^{\pm 3}],\ [w, b_i^{\pm 3}] \text{ for } i \in 1, \dots, g,\ w \in \pi_1(S) \qquad \text{and} \qquad [w, [\pi_1(S), \pi_1(S)]],\ w \in \pi_1(S)\,. \tag{7}$$

Claim 6. $\gamma_1^3\gamma_2^3\gamma_3^3 \dots \gamma_k^3 \in N$.

Proof. (of the Claim) The product $\gamma_1^3\gamma_2^3\gamma_3^3 \dots \gamma_k^3$ is null homologous in S, so for every generator a_i (respectively b_i), $i \in \{1, 2\}$, the sum of the powers in the product $\gamma_1^3\gamma_2^3\gamma_3^3 \dots \gamma_k^3$ must be zero. This implies that the elements of the set $\{\gamma_h\}$ can not all be elements of the generating set $\{a_1, a_2, b_1, b_2\}$. Assume otherwise: then for every a_i^3 $i \in \{1, 2\}$, (respectively b_i^3) there must be an a_i^{-3} (respectively b_i^3), in which case $\gamma_1^3\gamma_2^3\gamma_3^3 \dots \gamma_k^3$ would be in the commutator subgroup of $\pi_1(\tilde{S})$. Since Lemma 3 states that a connected component of $p^{-1}([a_1, b_1])$ is nonseparating, this would contradict Equation (6).

Suppose

$$\gamma_h = x_1x_2 \dots x_n, \text{ where each } x_j \in \{a_1, a_2, b_1, b_2\} \text{ for } j \in \{1, 2, \dots, n\}.$$

Then

$$\begin{aligned}
\gamma_h^3 &= x_1x_2 \dots x_nx_1x_2 \dots x_nx_1x_2 \dots x_n \\
&= x_1x_2 \dots x_nx_1^2x_2 \dots x_n[(x_2 \dots x_n)^{-1}, x_1^{-1}]x_2 \dots x_n \text{ using } zy = yz[z^{-1}, y^{-1}] \\
&= x_1^3x_2 \dots x_n[(x_2 \dots x_n)^{-1}, x_1^{-2}]x_2 \dots x_n[(x_2 \dots x_n)^{-1}, x_1^{-1}]x_2 \dots x_n \\
&= x_1^3x_2 \dots x_n[(x_2 \dots x_n)^{-1}, x_1^{-2}](x_2 \dots x_n)^2[(x_2 \dots x_n)^{-1}, x_1^{-1}] \\
&\quad [[(x_2 \dots x_n)^{-1}, x_1^{-1}]^{-1}, x_2 \dots x_n] \\
&= x_1^3(x_2 \dots x_n)^3[(x_2 \dots x_n)^{-1}, x_1^{-2}][[(x_2 \dots x_n)^{-1}, x_1^{-2}]^{-1}, (x_2 \dots x_n)^{-2}] \\
&\quad [(x_2 \dots x_n)^{-1}, x_1^{-1}][[(x_2 \dots x_n)^{-1}, x_1^{-1}]^{-1}, (x_2 \dots x_n)^{-1}] \\
&= x_1^3(x_2 \dots x_n)^3[(x_2 \dots x_n)^{-1}, x_1^{-2}]^{(x_2 \dots x_n)^{-2}}[(x_2 \dots x_n)^{-1}, x_1^{-1}]^{(x_2 \dots x_n)^{-1}}
\end{aligned} \tag{8}$$

where for $x, y \in \pi_1(S)$, we define $x^y := yxy^{-1}$.

The product

$$A := [(x_2 \dots x_n)^{-1}, x_1^{-2}]^{(x_2 \dots x_n)^{-2}}[(x_2 \dots x_n)^{-1}, x_1^{-1}]^{(x_2 \dots x_n)^{-1}}$$

from the last line of Equation (8) is an element of $\pi_1(\tilde{S})$, because it can be written as $(x_2\dots x_n)^{-3}x_1^{-3}\gamma_h^3$. It will be shown that A is in N. It follows from the commutator identities $[x,yz]=[x,y][x,z]^y$ and $[zx,y]=[x,y]^z[z,y]$ that elements of $[\pi_1(\tilde{S}),\pi_1(\tilde{S})]$ are normally generated by words from Equation (7), hence $[\pi_1(\tilde{S}),\pi_1(\tilde{S}]\in N$. To show that A is in N, it therefore suffices to show that A is homologous in $\tilde{S}$ to an element of N.

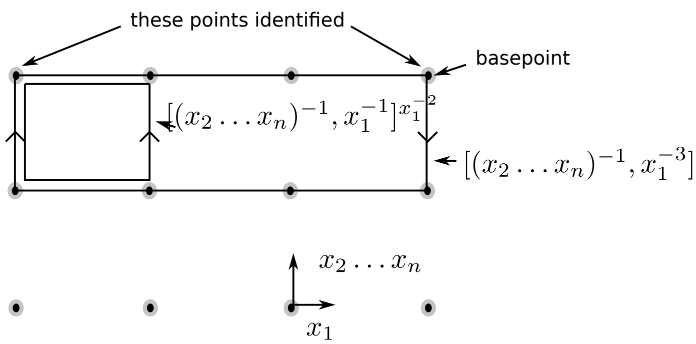

FIGURE 3. The dots represent elements of the fiber of the covering space $p:\tilde{S}\to S$. Translations in the vertical direction represent elements of the deck transformation corresponding to the lift of $x_2\dots x_n$, and translations in the horizontal direction represent elements of the deck transformation group corresponding to x_1.

Assuming a choice of base point in $\tilde{S}$, as shown in Figure 3, $[(x_2\dots x_n)^{-1},x_1^{-2}]$ is homologous to

$$[(x_2\dots x_n)^{-1},x_1^{-3}]-[(x_2\dots x_n)^{-1},x_1^{-1}]^{x_1^{-2}}$$

It follows that A is homologous to

$$[[(x_2\dots x_n)^{-1},x_1^{-1}],x_1^{-2}(x_2\dots x_n)^{-1}]^{(x_2\dots x_n)^{-1}}+[(x_2\dots x_n)^{-1},x_1^{-3}]$$

as required.

The argument from Equation (8) is then repeated on the shorter word $(x_2\dots x_n)^3\in\pi_1(\tilde{S})$, to show it is a product of the cube of a generator, the cube of a shorter word, and an element of N. Rearranging the order of elements of $\pi_1(S)$ that 1-lift to $\tilde{S}$, such as commutators and cubes, only introduces terms in $[\pi_1(\tilde{S}),\pi_1(\tilde{S})]$, which we saw is contained in N. It follows by induction that γ_h^3 is a product of cubes of generators, and an element of N.

Write $\gamma_1^3\gamma_2^3\gamma_3^3\dots\gamma_k^3$ as a product of cubes of generators, and elements of N. Rearranging the orders of cubes of generators and elements of N only introduces more elements of N. It follows that $\gamma_1^3\gamma_2^3\gamma_3^3\dots\gamma_k^3$ is a product of cubes of generators, and an element of N. Also, the products of the cubes of generators must be in $\pi_1(\tilde{S})$, by construction. In fact, the product of cubes must be in $[\pi_1(\tilde{S}),\pi_1(\tilde{S})]$ because Equation (6) implies the sum of the powers of any generator in the product must be zero, otherwise $\gamma_1^3\gamma_2^3\gamma_3^3\dots\gamma_k^3$ could not be null homologous in S. The claim follows. ∎

A contradiction will now be obtained by showing that $[a_1, b_1]$ cannot be contained in N.

Let ψ be the homomorphism taking an element of $\pi_1(S)$ to its coset in $\pi_1(S)/N$, and ψ_1 be the homomorphism taking an element of $\pi_1(S)$ to its coset in $\pi_1(S)/[\pi_1(S), [\pi_1(S), \pi_1(S)]]$. We now compute the image of $[\pi_1(S), \pi_1(S)]$ under ψ.

It follows from the commutator identities $[x, yz] = [x, y][x, z]^y$ and $[zx, y] = [x, y]^z[z, y]$ that $[\pi_1(S), \pi_1(S)]$ is generated by conjugates of commutators of generators. Since $[\pi_1(S), \pi_1(S)]$ maps to a subgroup in the center of the image of ψ_1, it follows that $\psi_1([\pi_1(S), \pi_1(S)])$ is generated by the image of commutators of generators of $\pi_1(S)$. The group $\psi_1([\pi_1(S), \pi_1(S)])$ is therefore a finitely generated, Abelian group.

Again using the commutator identities, it follows that $\psi_1([w, a_i^3]) = \psi_1([w, a_i]^3)$. Similarly for $\psi_1([w, b_i^3])$. If the word w is not a generator, it follows from the commutator identities and the fact that the image of the commutator subgroup under ψ_1 is in the center, that $\psi_1([w, a_i]^3)$ can be written as a product of cubes of commutators of generators. Since the word w can be taken to be any of the four generators, it follows that the image of $[\pi_1(S), \pi_1(S)]$ under ψ is a finitely generated Abelian group, each element of which has order three. In particular, $\psi([a_1, b_1])$ is not the identity. This proves the promised contradiction from which the lemma follows. □

Remark 7. *The argument in Lemma 5 does not work when $m = 2$. This is because in this case we do not get any commutators of commutators in the expression for A, hence there is no contradiction to the existence of Equation (6).*

It follows from arguments of Boggi-Looijenga, [1] and [7], that when D is Abelian, $H_1(\tilde{S}; \mathbb{Q})$ is generated by homology classes of lifts of simple, nonseparating curves. In Figure 1, for example, it is not hard to see that pairs of connected components of pre-images of a simple null homologous curve n are in the span of connected components of pre-images of generators. When $m > 3$, $m[\tilde{n}]$ is in the integral span of homology classes of connected components of lifts of simple, nonseparating curves. Hence $[\tilde{n}]$ is in the rational span, but not, as shown in Lemma 5, in the integral span.

3.5. **Proof of Theorem 2.** The promised families of examples for which lifts of simple curves do not span the integral homology of the covering space will now be constructed. The examples are iterated homology coverings of S, with at least two iterations, where the last homology covering uses an integer $m \geq 3$.

Let $\tilde{S} \to S$ be the covering with $m \geq 2$ just studied. Repeat the same construction, only with larger genus, and $m > 2$, on $\tilde{S}$ to obtain a cover $\tilde{\tilde{S}} \to S$ factoring through $\tilde{S}$. That the result is a regular cover follows from the fact that it is a composition of two characteristic covers.

It is possible to see almost immediately that $sc_p(H_1(\tilde{\tilde{S}}, \mathbb{Z}))$ can not be all of $H_1(\tilde{\tilde{S}}, \mathbb{Z})$. In Lemma 5 we saw that 1-lifts of simple null homologous curves from $\tilde{S}$ were needed to generate $H_1(\tilde{\tilde{S}};\mathbb{Z})$. However, by Lemma 3, no simple null homologous curves in $\tilde{S}$ project onto simple curves in S.

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