EXAMPLES OF COVERING PROPERTIES OF BOUNDARY POINTS OF SPACE-TIMES

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Abstract. The problem of classifying boundary points of space-time, for example singularities, regular points and points at infinity, is an unexpectedly subtle one. Due to the fact that whether or not two boundary points are identified or even "nearby" is dependant on the way the space-time is embedded, difficulties occur when singularities are thought of as an inherently local aspect of a space-time, as an analogy with electromagnetism would imply. The completion of a manifold with respect to a pseudo-Riemannian metric can be defined intrinsically, [SS94]. This is done via an equivalence relation, formalising which boundary sets cover other sets. This paper works through the possibilities, providing examples to show that all covering relations not immediately ruled out by the definitions are possible.

Contents

1. Introduction

Defining the completion of a manifold with respect to a pseudo-Riemannian metric is much more complicated than the familiar completion of a metric space. The standard mathematical tool for defining boundaries of topological spaces is called a net, and a definition based on nets, taken from [SS94], is given in Section 2.

The singularity theorems of Hawking, Penrose et. all show that in general relativity, one is forced to work with manifolds with nonempty boundary. Understanding the geometry and topology of a space time will therefore reduce to a large extent to the problem of understanding the boundary. There are many different tools for illuminating different aspects of this boundary using the additional structure present on the topological spaces, for example, using causal structures, [FHS11], incomplete geodesics, [Ger68], or Sasaki metrics [Sch71].

Even if true, it is not satisfactory to have to rely on cosmic censorship to be able to determine whether a space-time is singular or not. Anyone who has studied the Schwarzschild solution will be familiar with the concept of a removable singularity. Informally, this is often described as a singularity that can be removed by a "change of coordinates". More precisely, these "coordinate changes" are re-embeddings of the pseudo-Riemannian manifold into different Riemannian manifolds with nonequivalent metrics. In order to distinguish between boundary "points" that are intrinsically singular, such as curvature or cone singularities, and those that are not, it is necessary to understand the different ways the pseudo-Riemanninan manifold can be enveloped by a larger Riemannian manifold of the same dimension. Re-envelopments of the pseudo-Riemannian manifold give rise to the notion of what boundary sets cover which other boundary sets. This gives an equivalence relationship used to define boundary points.

The purpose of this paper is to show by example how the different types of boundary points may cover each other. This is done to a large extent using cut and paste techniques from low dimensional topology.

Section 2 introduces the different types of boundary points, following the classification from [SS94]. Section 3 then goes through all the different types of boundary points, and provides examples to show that all covering relations not explicitly ruled out by the definitions are possible.

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2. Background and Definitions

The a-boundary provides an implicit means for describing and classifying boundary points, and will be outlined briefly here. For more details, see for example [Ash02], [FS94], [WAS15] & [SS94].

Definition 1 (Enveloped manifold)**.** *An enveloped manifold is a triple* $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ *where* $\mathcal{M}$ *and* $\widehat{\mathcal{M}}$ *are differentiable manifolds of the same dimension and* ϕ *is a* C^∞ *embedding* $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$*. The enveloped manifold is also called an envelopment, where* $\widehat{\mathcal{M}}$ *is the enveloping manifold.*

Definition 2 (Extension)**.** *An extension of a pseudo-Riemannian manifold (*$\mathcal{M}$*,*g*) is an envelopment of it by a pseudo-Riemannian manifold (*$\widehat{\mathcal{M}}$*,* $\hat{g}$*) such that* $\hat{g}|_{\phi(\mathcal{M})} = g$*.*

Informally speaking, a singularity is an approachable boundary point of $\mathcal{M}$, at which the manifold structure breaks down. In the a-boundary scheme, approachability is defined with respect to a family of curves satisfying the following property. The set of curves in question is chosen to suit the aim of the investigation.

Definition 3 (bounded parameter property (b.p.p.))**.** *A family* $\mathcal{C}$ *of parametrized curves in* $\mathcal{M}$ *satisfies the b.p.p. if:*

(1) for any point $p \in \mathcal{M}$ *there is at least one curve of the family passing through* p
(2) if $\gamma(t)$ *is a curve of the family then so is any connected subset of it*
(3) if γ *and* γ' *are in* $\mathcal{C}$ *and* γ' *is obtained from* γ *by a change of parameter then either the parameter is bounded or unbounded on both curves.*

Curves satisfying the b.p.p. are a generalization of geodesics with affine parameter.

In order to define the a-boundary, it is necessary to make precise what is meant by two boundary sets representing the same abstract set in different envelopments. This is done by an equivalence relation.

Definition 4 (Covering relation)**.** *If B is a boundary set of $\phi(\mathcal{M})$ and B' is a boundary set of $\phi'(\mathcal{M})$ then B covers B' (denoted $B \rhd B'$) if for every open neighbourhood $\mathcal{U}$ of B in $\widehat{\mathcal{M}}$ there exists an open neighbourhood $\mathcal{U}'$ of B' in $\widehat{\mathcal{M}}'$ such that*

$$\phi \circ \phi'^{-1}(\mathcal{U}' \cap \phi'(\mathcal{M})) \subset \mathcal{U}. \tag{1}$$

Two boundary sets B and B' are equivalent if they cover each other. This defines an equivalence relation. An equivalence class is denoted by a representative element with a square bracket around it, for example $[B]$. In the a-boundary formulation, a boundary set is an equivalence class, and an a-boundary point is an equivalence class with a point on the boundary of some envelopment as a representative element.

Definition 5 (Abstract boundary $B(\mathcal{M})$)**.** $B(\mathcal{M}) := \{[p] \mid p \in \partial_\phi(\mathcal{M})$ *for some envelopment* $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)\}$

Points of the a-boundary are then divided into various categories.

Definition 6 (Regular point)**.** *A boundary point p of an envelopment $(\mathcal{M}, g, \widehat{\mathcal{M}}, \phi)$ is regular if there exists a manifold $(\overline{\mathcal{M}}, \overline{g})$ such that $\phi(\mathcal{M}) \cup \{p\} \subseteq \overline{\mathcal{M}} \subseteq \widehat{\mathcal{M}}$ and $(\mathcal{M}, g, \overline{\mathcal{M}}, \overline{g}, \phi)$ is an extension of $(\mathcal{M}, g)$.*

Regularity does not "pass to the a-boundary" (i.e. it is not invariant under the equivalence relation used to define the a-boundary). A regular a-boundary point is defined as follows:

Definition 7 (Regular a-boundary point)**.** *A regular a-boundary point is an equivalence class with a regular point as a representative element.*

A maximally extended space-time is essentially one whose a-boundary does not contain any regular a-boundary points.

Definition 8 (Maximally extended)**.** *A C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$ is termed C^l maximally extended $(1 \leqslant l \leqslant k)$ if there does not exist a C^l extension $(\mathcal{M}, g, \widehat{\mathcal{M}}, \widehat{g}, \phi)$ of $(\mathcal{M}, g)$ such that $\phi(\mathcal{M})$ is a proper open submanifold of $\widehat{\mathcal{M}}$*

The Misner example [HE73], is a simplification of an example given in [Mis67]. One description of it is the cylinder $\mathbb{R} \times S^1$, with the metric

$$ds^2 = -2dtd\theta + td\theta^2$$

The Misner example has a closed null curve at $t = 0$, and this closed curve is approached by null geodesics that are incomplete with respect to their affine parameters. However, the submanifold $t = 0$ would appear to be perfectly regular, interior points of the manifold. In the a-boundary, irregularity replaces the concept of curve incompleteness often used in the literature as an indicator of the existence of singularities. This has the advantage of being able to distinguish between examples such as the Misner example, in which the existence of incomplete geodesics are not really indicative of singular behaviour.

Definition 9 (Limit point of a curve). *We say that* $p \in \phi(\mathcal{M}) \cup \partial_\phi \mathcal{M}$ *is a limit point of a curve* $\gamma : [a, b) \to \phi(\mathcal{M})$ *if there exists an increasing infinite sequence of real numbers* $t_i \to b$ *such that* $\gamma(t_i) \to p$*.*

Definition 10 (Endpoint of a curve). *We say that* p *is an endpoint of the curve* γ *if* $\gamma(t) \to p$ *as* $t \to b$*.*

Definition 11 (Approachable boundary point). *A parametrised curve* $\gamma : I \to \mathcal{M}$ *approaches the boundary set* B *if the curve* $\phi \circ \gamma$ *has a limit point lying in* B*. A point* $p \in \partial_\phi \mathcal{M}$ *is approachable if it is approached by a curve from the family* $\mathcal{C}$*.*

Irregular boundary points consist of singularities, points at infinity and irregular unapproachable boundary points.

Definition 12 (Point at infinity). *A boundary point* p *of the envelopment* $(\mathcal{M}, g, \widehat{\mathcal{M}}, \mathcal{C}, \phi)$ *is a point at infinity if*

(1) p *is not a regular boundary point*
(2) p *is approachable by an element of* $\mathcal{C}$*, and*
(3) no curve of $\mathcal{C}$ *approaches* p *with bounded parameter.*

Definition 13 (Removable point at infinity). *A boundary point* p *at infinity is termed a removable point at infinity if there is a boundary set,* $B \subset \partial_\phi \mathcal{M}$ *composed purely of regular boundary points such that* $B \triangleright p$

Definition 14 (Essential point at infinity). *A point* p *at infinity is an essential point at infinity if it is not removable.*

Definition 15 (Mixed point at infinity). *An essential point* p *at infinity is a mixed point at infinity if it covers a regular boundary point.*

Definition 16 (Pure point at infinity). *An essential point at infinity is a pure point at infinity if it does not cover any regular boundary points.*

Definition 17 (Singular boundary points). *A boundary point* p *of an envelopment* $(\mathcal{M}, g, \widehat{\mathcal{M}}, \mathcal{C}, \phi)$ *is called singular or a singularity if*

(1) p *is not a regular boundary point,*
(2) p *is approachable by a curve* γ*, where* γ *is an element of* $\mathcal{C}$ *and has finite parameter.*

Definition 18 (Removable singularity). *A singular boundary point* p *will be called removable if it can be covered by a non-singular boundary set* B *of another embedding.*

Definition 19 (Essential singularity). *A singular boundary point* p *is called essential if it is not removable.*

Definition 20 (Directional and pure singularities). *An essential singularity* p *is called a directional singularity if it covers a boundary point of another embedding which is either regular or a point at infinity. Otherwise* p *is called a pure singularity.*

The definitions of directional and pure singularities pass to the a-boundary, as shown in [SS94].

The a-boundaryclassification scheme is summarised in Figure 1.

A concept that has turned out to be useful is that of a connected neighbourhood region, [Ash02].

Definition 21 (Connected Neighbourhood Region (CNR))**.** *Suppose $p \in \partial_\phi \mathcal{M}$ and $\mathcal{N}$ is a neighbourhood of p in $\widehat{\mathcal{M}}$. Then a connected component of $\mathcal{N} \cap \phi(\mathcal{M})$ is called a connected neighbourhood region of p.*

Definition 22 (The finite connected neighbourhood region property (FCNR property))**.** *We say that p has n connected neighbourhood regions if for any open neighbourhood $\mathcal{N}(p)$ there exists a sub-neighbourhood $\mathcal{U}(p) \subset \mathcal{N}(p)$ for which $\mathcal{U}(p) \cap \phi(\mathcal{M})$ is composed of exactly n connected components, and n is the smallest natural number for which this is true. The boundary point p satisfies the finite connected neighbourhood region property if it has only finitely many connected neighbourhood regions.*

It is a consequence of Theorem 4.3 of [FS94] that the FCNR property passes to the a-boundary.

3. Examples

Figure 3 is a table displaying which types of a-boundary points can cover other types. When it is not possible for a particular type of boundary point to cover another, this is an immediate consequence of the definitions. In all other cases, the entries in the table will now be confirmed by providing examples.

In this section, it will be assumed that $\mathcal{C}$ is the set of geodesics with affine parameter.

Separating out a point. When a boundary point p has a finite number k of connected neighbourhood regions, many of the examples presented in this section will be constructed by splitting boundary points up into points with fewer connected neighbourhood regions, and recombining these in different ways. The procedure outlined below for splitting up a boundary point with k connected neighbourhood regions into k boundary points, each with one connected neighbourhood region, will be referred to as "separating out a point".

Suppose $\phi(\mathcal{M})$ is an n dimensional manifold and $\mathcal{N}_1$ is a connected neighbourhood region of the point $p \in \partial_\phi \mathcal{M}$ with $k > 1$ connected neighbourhood regions, $\mathcal{N}_1, \dots, \mathcal{N}_k$. Let $S \subseteq \widehat{\mathcal{M}} \backslash \phi(\mathcal{M})$ be a closed codimension one surface with p in its interior. The surface S will also be referred to as a slit, and may have nonempty boundary in $\widehat{\mathcal{M}}$. Suppose also that there is some neighbourhood $\mathcal{N}$ of p in which S is separating, and in this neighbourhood $\mathcal{N}_1$ is on one side of S and the other connected neighbourhood regions are on the other. More precisely, S is chosen so that for every open neighbourhood $\mathcal{U}$ of p in $\widehat{\mathcal{M}}$, there exists an open neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of p in $\widehat{\mathcal{M}}$, such that $\mathcal{V} \setminus S$ has two connected components in

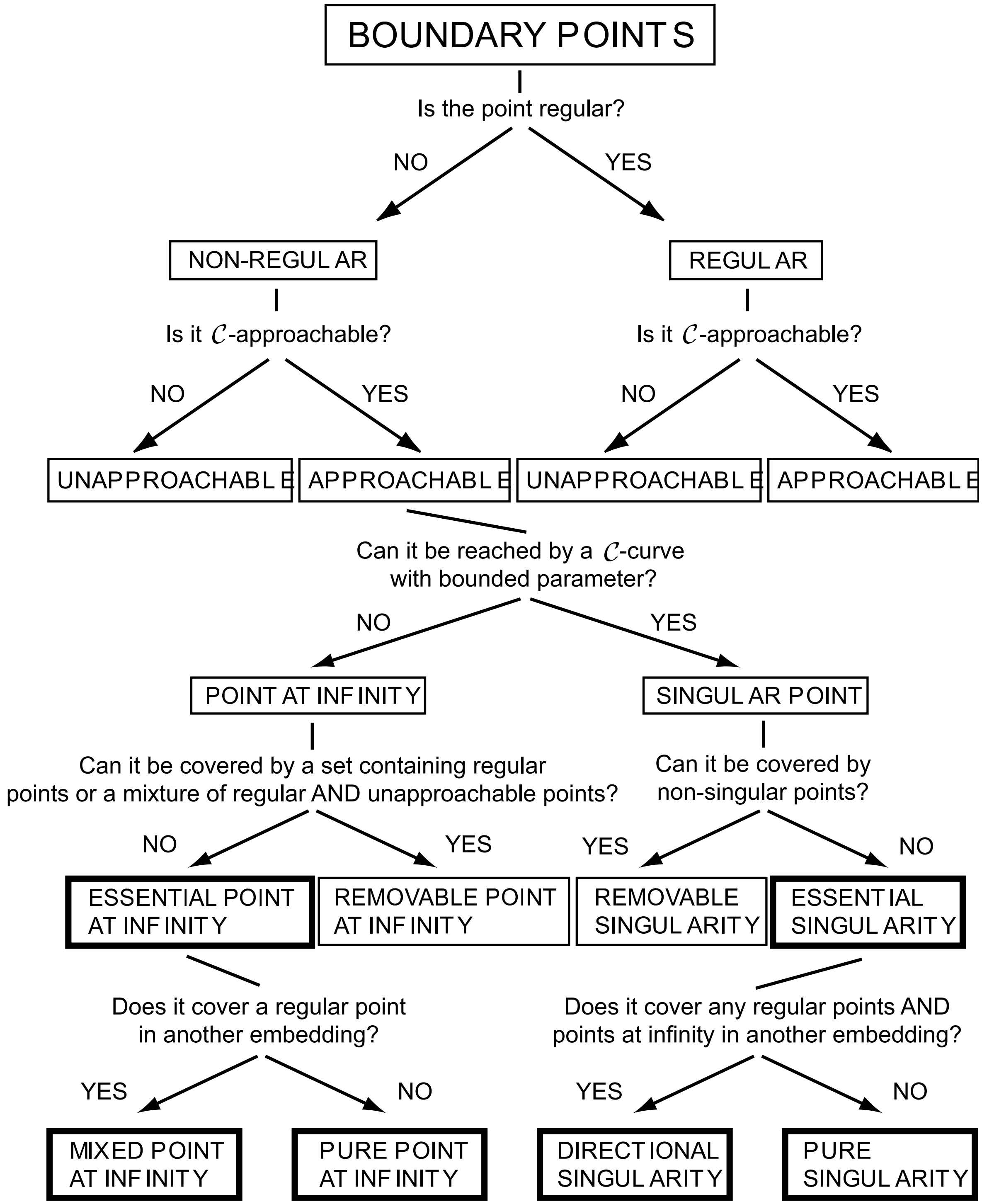

FIGURE 1. A figure taken from [SS94] summarising the a-boundary point classification scheme. Categories that pass to the a-boundary are in bold.

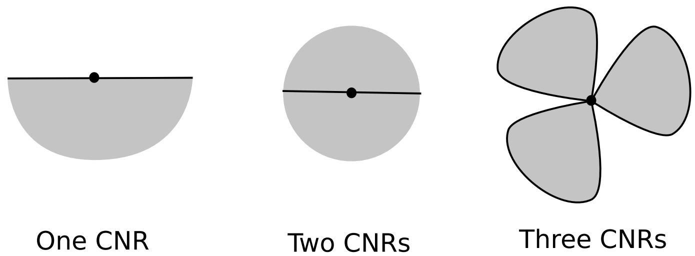

FIGURE 2. In this diagram, $\phi(\mathcal{M})$ is represented by the shaded region. This figure gives examples of different numbers of connected neighbourhood regions of the boundary point indicated by the black dot.

▷	reg.	non-reg unapp.	rem. ∞	mixed ∞	pure ∞	rem. sing.	dir. sing.	pure sing.
reg.	x	x	x			x		
non-reg unapp.	x	x						
rem. ∞	x	x	x					
mixed ∞	x	x	x	x	x			
pure ∞		x	x		x			
rem. sing.	x	x	x	x	x	x		
dir. sing.	x	x	x	x	x	x	x	x
pure sing.		x				x		x

FIGURE 3. A table summarising the types of boundary points that may and may not cover each other. An x implies the entry in the column can cover the corresponding entry in the row.

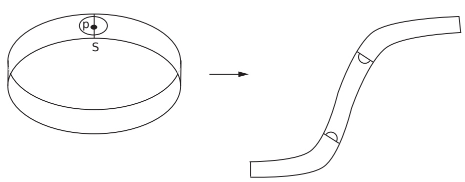

FIGURE 4. The two neighbourhood regions at p are separated out.

$\widehat{\mathcal{M}}$, $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{N}_1$ has nonempty intersection with precisely one of the components, say $\mathcal{V}_1$, and $\{\mathcal{N}_2, \ldots, \mathcal{N}_k\}$ have non-empty intersection only with $\mathcal{V}_2$. Since p is a boundary point with more that one connected neighbourhood region and $\widehat{\mathcal{M}}$ is a manifold, a set S with these properties can always be found.

Remove S from $\widehat{\mathcal{M}}$ and identify the lower edge of the slit with the upper edge of the slit in a second copy of $\widehat{\mathcal{M}}\backslash S$, as shown in Figure 4. Identify the upper edge of the slit with the

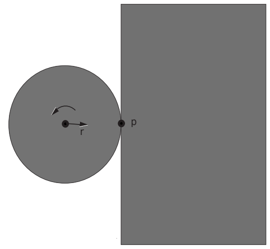

FIGURE 5. In this figure, the manifold is $\phi(\mathcal{M})$ is the unshaded region. The radial coordinates have their origin at the dot in the center of the circle.

lower edge of the slit in a third copy of $\widehat{\mathcal{M}}\backslash S$. In this envelopment, p is equivalent to two boundary points, one with only one connected neighbourhood region, and the other with $k-1$ connected neighbourhood regions. If $k>2$ this process is repeated on the boundary point with more than one connected neighbourhood region, until a single enveloping manifold is obtained, in which p is equivalent to k boundary points each with only one connected neighbourhood region.

If S can be chosen in such a way that it does not have any boundary points in common with $\phi(\mathcal{M})$, as will always be the case in this paper, then separating out a point does not destroy the regularity properties of any a-boundary points, including p. Separating out a point also does not change the original embedding ϕ, it merely enlarges the enveloping manifold, by taking the union of $\widehat{\mathcal{M}}\smallsetminus S$ with two copies $\widehat{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_2$ of $\widehat{\mathcal{M}}\smallsetminus S$. The union is joined along the copies of S as explained above.

Example 23 (A non-regular unapproachable boundary point that covers a regular point)**.** *Put coordinates (r,θ) on* $\mathbb{R}^2$ *and let* $\mathcal{M}$ *be the manifold satisfying* $r>1$, $r\cos\theta<1$, $0<\theta<2\pi$ *with metric given by*

$$ds^2 = \frac{-1}{\theta}dr^2 + r^2 d\theta^2.$$

Let p be the boundary point with coordinates $r=1$ *and* $\theta=0$, *as shown in Figure 5. Then p is non-regular. The curvature scalar is given by*

$$\frac{1}{r} - \frac{3}{2r^2\theta^2} + \frac{2\theta}{r^2}$$

Curves approaching $p=(1,2\pi)$ *with increasing* θ *have a finite limit of the curvature scalar, while curves approaching* $p=(1,0)$ *with decreasing* θ *have unbounded curvature scalar. The removal of the circle from the space-time makes p unapproachable. Separating out the two connected neighbourhood regions reveals a regular point* $(1,2\pi)$ *and a non-regular* $(1,0)$ *point covered by p.*

Example 24 (A removable point at infinity that covers a regular point)**.** *Put coordinates* (x,y) *on* $\mathbb{R}^2$ *and let* $\mathcal{M}$ *be the manifold satisfying* $y<-1$, $\mid x \mid< e^y$, *with metric*

$$ds^2 = -dy^2 + dx^2$$

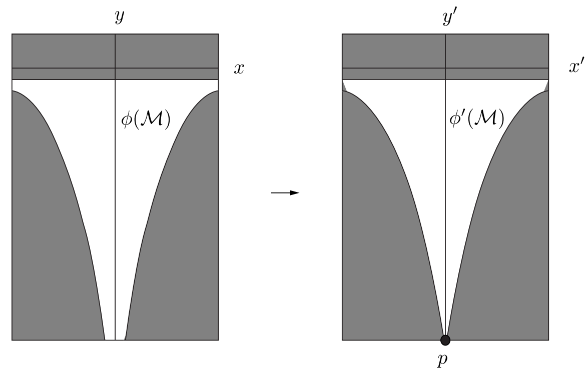

FIGURE 6. The point p is a removable point at infinity.

The re-envelopment ϕ' of $\mathcal{M}$ in $\mathbb{R}^2$ given by

$$x \to x' = x,\ y \to y' = \arctan(y).$$

has precisely one boundary point which is a point at infinity, namely the point p, with coordinates $(x', y') = (0, -\pi/2)$. To see that p is a removable point at infinity, consider a second re-envelopment ϕ'' of $\mathcal{M}$ in $\mathbb{R}^2$ given by

$$x \to x'' = x + 4e^{-1} - 4e^{y},\ y \to y'' = y \text{ mod } 1.$$

The action of this re-envelopment is depicted in Figure 7. The image $\phi''(\mathcal{M})$ is contained in the compact subset

$$\{(x'', y'')| - e^{-1} \leqslant x'' \leqslant 4e^{-1}, 0 \leqslant y'' \leqslant 1\}$$

of $\widehat{\mathcal{M}}''$. The point p is equivalent to the set of regular points

$$\{(x'', y'')|x'' = 4e^{-1}, 0 \leqslant y'' \leqslant 1\}.$$

Therefore p is removable and covers a regular boundary point.

Example 25 (A regular point that covers a removable point at infinity)**.** *Example 33 of* [SS94] *contains a regular boundary point that is only approachable by geodesics with infinite affine parameter. In this example, $\widehat{\mathcal{M}}$ is the unit torus with metric $ds^2 = dx^2 + dy^2$. On the central line $L = \{(x, 1/2)|0 \leqslant x < 1\}$ choose the points*

$$p_{\pm i} = \left(\frac{1}{2}\left(1 \pm \frac{1}{2^i}\right), \frac{1}{2}\right), \qquad i = 1, 2, 3, \ldots$$

For each $i = \pm 1, \pm 2, \ldots$ let L_i be the closed line segment of length $1/2$ and slope $\sqrt{2}$ centered on the point p_i and let L_0 be a similar line segment with center p. Here $\phi(\mathcal{M})$ is the open submanifold of $\widehat{\mathcal{M}}$ consisting of the complement in $\widehat{\mathcal{M}}$ of this infinite set of closed line

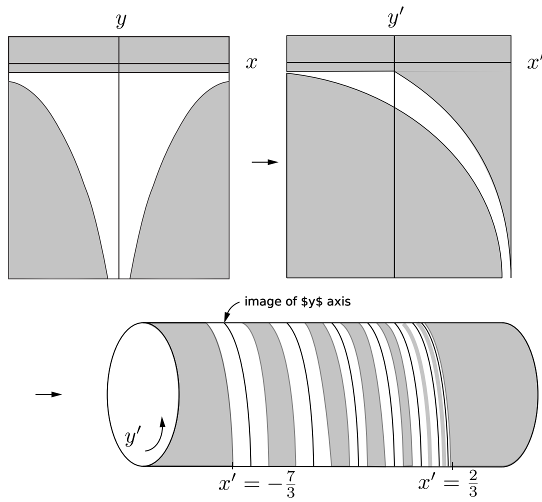

FIGURE 7. The construction from Example 24.

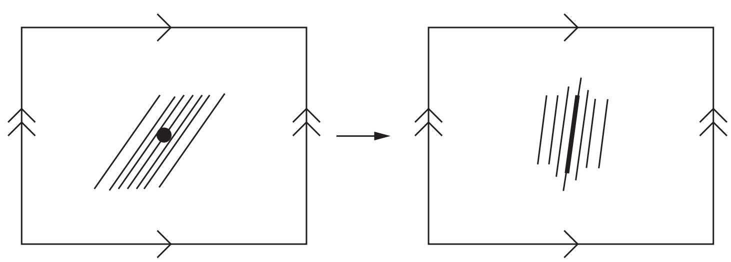

FIGURE 8. The Figure from Example 25.

segments, as shown in Figure 8. The point p is clearly a regular boundary point and is approached only by infinite geodesics with $\frac{dy}{dx} = \sqrt{2}$. *The re-envelopment*

$$x \to x' = x,\ y \to y' = \arctan\left(\frac{y}{x^2}\right)$$

blows up the point p *into a compact interval* I *of irregular boundary points. By theorem 19 of* [SS94], $[p] = [I]$, *therefore there is an irregular point covered by* p *that is only approachable by infinite geodesics, i.e.* p *covers a point at infinity.*

Example 26 (A removable singularity covering a regular point, (Example 39 of [SS94])). *Consider the manifold* $\mathbb{R}\setminus(0,0)$ *with the metric*

$$ds^2 = dr^2 + (r+1)^2 d\theta^2$$

The boundary point $p = (0,0)$ *is approachable but not regular. One way of seeing that it is not regular is to observe that for circles centered on* p, *the ratio of circumference to radius does not approach* 2π *but approaches infinity as the radius approaches zero. The coordinates* (r,θ) *therefore do not provide a coordinate patch that can be extended to* $(0,0)$, *hence* p *is not regular.*

Take the re-envelopment determined by

$$\phi'(x,y) = \frac{r+1}{r}(r\cos\theta, r\sin\theta)$$

Now $\phi'(\mathcal{M})$ *is the region* $r > 1$ *of* $\mathbb{R}^2$*, with the flat metric*

$$ds^2 = dr^2 + r^2 d\theta^2$$

The boundary can now be seen to consist of a set of regular boundary points, equivalent to p*. The point* p *is therefore a removable singularity that covers regular points.*

Example 27 (A directional singularity covering a regular point)**.** *Example 45 of* [SS94] *studies such an example in detail, however the metric is slightly complicated. An easier way to construct an example of a directional singularity covering a regular point is to take the previous example, and use a partition of unity to multiply the flat metric on* $\phi'(\mathcal{M})$ *by a conformal factor. This is done in such a way that one and only one of the boundary points of* $\phi'(\mathcal{M})$ *becomes a curvature singularity, while the others remain regular. Then take the re-embedding* $\phi \circ \phi'^{-1}$ *to obtain a directional singularity covering a regular point.*

Example 28 (A regular boundary point covering an irregular unapproachable point)**.** *Let* $\mathcal{M}$ *be the manifold* $\mathbb{R}^2 \setminus (0,0)$*. Blowing up the point* $p = (0,0)$*, i.e. taking the re-envelopment*

$$(x', y') = \begin{cases} (\arctan\left(\frac{x}{y}\right), \sqrt{x^2+y^2}) & \text{for } 0 \le x, \text{ and } 0 \le y \\ (\arctan\left(\frac{x}{y}\right) + \pi, \sqrt{x^2+y^2}) & \text{for } x < 0, \\ (\arctan\left(\frac{x}{y}\right) + \pi, \sqrt{x^2+y^2}) & \text{otherwise} \end{cases}$$

gives a re-embedding $\phi'(\mathcal{M}) \simeq S^1 \times \mathbb{R}^+$*. In this re-envelopment,* $[p]$ *has representative the ring* $y' = 0$*. Each of the boundary points of* $\phi'(\mathcal{M})$ *is approached by just one geodesic contained in* $\phi'(\mathcal{M})$*. The boundary points of* $\phi'(\mathcal{M})$ *can therefore not be regular. This is because for regular points, the inverse function theorem implies the exponential map is a local diffeomorphism. In the same way, blowing up any of the boundary points of* $\phi'(\mathcal{M})$ *give irregular unapproachable points. These points are covered by the regular point* p*.*

Example 29 (A removable point at infinity covering an irregular unapproachable point)**.** *Consider the removable point at infinity from Example 24. The width* $w(y')$ *of the manifold around the* y' *axis is given by*

$$w(y') = \frac{2}{y'^4}.$$

The envelopment $x' \to x'' = x' e^{\frac{1}{w(y')}}$, $y' \to y'' = y'$ *reveals unapproachable irregular boundary points covered by* p*, see Figure 9.*

Example 30 (A mixed point at infinity covering a removable point at infinity and an irregular unapproachable point)**.** *Start with the manifold containing a point at infinity given in example 24. Multiplying the metric by* $\frac{1}{(x'^2+(y+\pi/2)'^2)^2}$ *makes the point* $(0, -\pi/2)$ *a pure point at infinity. Attach the manifold from example 24, as shown in Figure 10.*

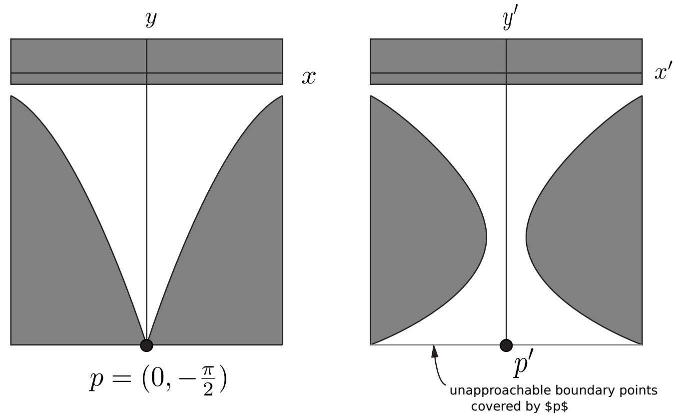

FIGURE 9. The Figure from example 29.

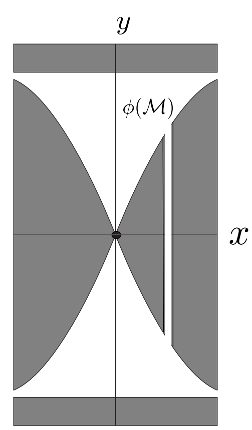

FIGURE 10. The Figure from example 30.

Let g_1 be the metric on the upper half of the diagram, and let g_2 be the metric on the lower part of the diagram. In order to make this manifold connected, attach a "bridge" from the top part to the lower part, as shown in the diagram. Let l be a parameter along the connecting piece, such that $l = 0$ on the boundary of the upper part of the example, and increases smoothly to $l = 1$ on the boundary of the other part. Then $\mathcal{M}$ is the manifold consisting of the two components + "bridge", where the metric g is given by

$$g|_{region1} = g_1,\ g|_{region2} = g_2 \text{ and } g|_{bridge} = (1-l)g_1' + lg_2'$$

The two metrics are simultaneously diagonalizable, so it is easy to verify that the metric obtained in this way is nonsingular everywhere on $\mathcal{M}$.

The origin is a mixed point at infinity, since it is not removable, but covers the removable point at infinity from example 24 that covers regular points. As shown in the previous example, this removable point at infinity covers irregular unapproachable points.

Example 31 (A pure point at infinity covering an irregular unapproachable point)**.** *This example is the same as example 29 only the removable point at infinity is made into a pure point at infinity by multiplying the metric by the conformal factor* $\frac{1}{(x'^2+(y+\pi/2)'^2)^2}$.

Example 32 (A removable singularity covering an irregular unapproachable point)**.** *Start with the region* $y > 0$ *of two dimensional space with metric* $ds^2 = -dy^2 + dx^2$*. The re-envelopment*

$$x \to x' = \frac{x}{y^2},\ y \to y' = y$$

makes the origin, p*, in these new coordinates a removable singularity, approached only by the geodesic* $x' = 0$*. Repeating this process (only with* x' *instead of* x *and* y' *instead of* y*) reveals irregular unapproachable points covered by* p*.*

Example 33 (A directional singularity covering an irregular unapproachable point and a removable singularity)**.** *Example 45 of* [SS94] *is a directional singularity. The details of this example are not required here. To show that it covers an irregular unapproachable point, choose a regular point* p *covered by the singularity. Put normal coordinates* (x, y) *around* p*. The re-envelopment*

$$x \to x' = \frac{x}{y^2},\ y \to y' = y$$

fixes the geodesic locally given by x=0, and sends all the other geodesics approaching p *off to infinity. Any point other than the origin on the* x' *axis is an irregular unapproachable point covered by the directional singularity. The origin of the* x' *axis is a removable singularity covered by a directional singularity.*

Example 25 of [SS94] is a pure singularity (To be more specific it is a cone singularity, as defined in [ES79]) which covers irregular unapproachable points.

Example 34 (A pure singularity covering a removable singularity, and a pure point at infinity covering a removable point at infinity)**.** *Start with the two dimensional manifold with metric*

$$ds^2 = \frac{1}{\sqrt{x^2+y^2}}(-dy^2 + dx^2), \qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4}.$$

Then take two dimensional manifold

$$\{(x, y) | y > 0\},\ \text{with metric } ds^2 = -dy^2 + dx^2$$

and re-envelop it as follows:

$$x \to x' = \arctan\left(\frac{x}{y}\right),\ y \to y' = y$$

The point $(x', y') = (0, 0)$ *is a removable singularity. The manifold* $\mathcal{M}$ *formed by identifying the origins of these two manifolds and inserting a connecting piece has a pure singularity at the origin which covers a removable singularity.*

An example of a pure point at infinity covering a removable point at infinity is formed from the previous example by sending the singularity off to infinity.

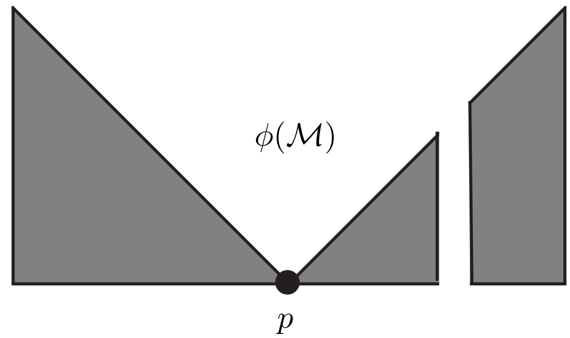

FIGURE 11. A pure singularity covering a removable singularity.

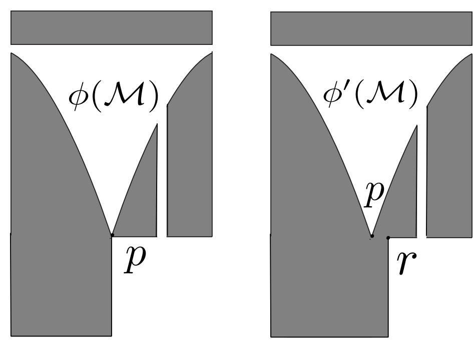

FIGURE 12. A removable singularity covering a removable point at infinity.

Example 35 (A removable singularity covering a removable point at infinity)**.** *$\mathcal{M}$ consists of the manifold in example 24 containing a removable point at infinity, connected to a quadrant of flat space, with a connecting piece, as shown in Figure 12. The point p is singular because it is irregular and approachable by finite geodesics. By theorem 19 of* [SS94] *it is equivalent to the a-boundary set consisting of $[q]$ - a removable point at infinity, and $[r]$ - a regular point. Therefore p is a removable singularity which covers a removable point at infinity.*

Example 36 (A directional singularity covering a removable point at infinity)**.** *This example is the same as the previous example, except that the metric on the quadrant is replaced by*

$$ds^2 = \frac{1}{\sqrt{x^2 + y^2}}(-dy^2 + dx^2).$$

This turns p into a directional singularity but does not influence the classification of the point q in the re-envelopment.

Example 37 (A removable singularity covering a mixed point at infinity and a pure point at infinity)**.** *Consider the manifold*

$$\{(x, y)|y < 0\} \text{ with metric } ds^2 = dx^2 + dy^2.$$

Remove a semicircle from this space as shown in Figure 13. This is done in such a way that the boundary point $(0, 0)$ has two connected neighbourhood regions, one associated with a regular unapproachable point and the other associated with a regular point approached by

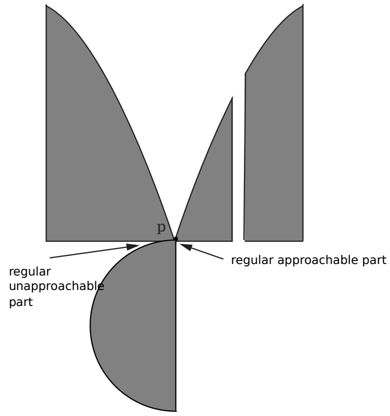

FIGURE 13. A removable singularity covering a mixed point at infinity and a pure point at infinity.

finite geodesics. Take the top half of example 30 and identify the pure point at infinity with the point $(0,0)$ *of the manifold just constructed, and call this boundary point* p*. Make this manifold connected as in the earlier examples. The point* p *is irregular, approachable by geodesics with finite affine parameter, and is equivalent to a set consisting of regular points and points at infinity. Therefore* p *is a removable singularity. Separating off the regular approachable part of* p *reveals a mixed point at infinity covered by* p*. This mixed point at infinity covers the pure point at infinity from Example 30.*

Example 38 (A directional singularity covering a mixed point at infinity and a pure point at infinity)**.** *This is the same as the previous example except with the following alteration. Break the point* p *up into its connected neighbourhood regions. Let* q *be the regular approachable point covered by* p*. Using a partition of unity, alter the metric around the point* q *by a conformal factor so that* q *becomes a curvature singularity. Putting the three connected neighbourhood regions back together again results in a directional singularity that covers a mixed point at infinity (i.e. the boundary point with the neighbourhood region around which the metric has been altered removed) and a pure point at infinity.*

The Curzon solution [SS86a] & [SS86b] contains an example of a directional singularity that covers a pure singularity.

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