1 Differential invariants

Definition 1.1. Given a point transformation \( \Psi \) acting on \( \mathbb{R}^2 \), a differential invariant of order \( n \) is an invariant function of the prolonged action \( P_r^{(n)} \Psi \).

Examples (referred to group actions on the previous page):

(A): \( \omega_0 = \sqrt{x^2 + y^2}, \omega_1 = \frac{xy}{x + yy_1} \).

(B): \( \omega_0 = \frac{y}{x}, \omega_1 = y_1 \).

(C): \( \omega_0 = y, \omega_3 = 2(y_1)^3y_3 - 3(y_1)^4(y_2)^2 \) is a complete set of third order invariants.

Are there higher order invariants?

Theorem 1. For a group action \( \Psi : G \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), if \( \alpha, \beta \) are invariants of order \( n \) then \( D_x \alpha / D_x \beta \) is an invariant of order \( n + 1 \). Here \( D_x \) is a complete derivative:

\[ D_x = \partial_x + y_1 \partial_y + \ldots + y_{n+1} \partial_{y_n} \] .

Exercises:

1. Prove that, for \( \phi = \phi(x, y, \ldots, y_n) \) and \( X = \xi \partial_x + \eta \partial_y \) on \( \mathbb{R}^2 \):

\[ P_r^{(n+1)}X(D_x \phi) = D_x \left( P_r^{(n)}X(\phi) \right) - D_x \phi \cdot D_x \xi . \]

2. Use this to prove Theorem 1.

Example:

(A) For the prolonged action of \( SO(2) \) we know that \( r = \sqrt{x^2 + y^2} \) is an invariant and \( q = \frac{y_1}{yy_1 + x} \) is a second order invariant. Then

\[ D_x(q)/D_x(r) = \frac{\sqrt{x^2 + y^2}}{(x + yy_1)^2} \left[(x^2 + y^2)y_2 - (1 + y_1)(xy_1 - y)\right] \]

is a third order invariant. We can replace it by

\[ \kappa = \frac{D_x(q)/D_x(r)}{(1 + q^2)^{3/2}} + \frac{q}{r(1 + q^2)^{1/2}} = \frac{y_2}{(1 + (y_1)^2)^{3/2}} \]

which is an expression for the curvature of the graph of the function \((x, y(x))\).

(B) For the prolonged action of the scaling group we know the invariants \( w_0 = y/x, w_1 = y_1 \).

Then a second order invariant is:

\[ D_x(w_1)/D_x(w_0) = \frac{x^2y_2}{xy_1 - y} \]
Corollary 1.1. If $w_0$ is a (zero order) invariant, $w_1$ first order invariant of an action derived by a single symmetry (vectorfield), than all differential invariants of order $n$ can be obtain, recursively, by

$$w_n = \frac{D_x w_{n-1}}{D_x w_0} = \frac{D_x^{n-1} w_1}{D_x^{n-1} w_0}.$$  

In particular, any $n$ order invariant is of the form

$$G \left( w_0, w_1, \frac{D_x w_1}{D_x w_0}, \frac{D_x^2 w_1}{D_x^2 w_0}, \ldots, \frac{D_x^{n-1} w_1}{D_x^{n-1} w_0} \right)$$

1.1 Infinitesimal formulation of invariance for ODE

Theorem 2. An ODE $H = 0$ of order $n$ is invariant under the action of the flow $\psi(t, x, y)$ generated by $X$ if and only if

$$P_{r(n)} X(H) = 0 \mod H = 0$$

provided $H^2_x + H^2_y + \ldots H^2_{y_n} \neq 0 \mod H = 0$.

Counter-Example: $H = (y_2 + y)^2$ verifies $P_r X H = 0 \mod H$ for any $X$!

In particular, the explicit ODE of order $n$

$$y_n = w(x, y, \ldots y_{n-1})$$

is $X$ invariant iff

$$P_{r(n-1)} X(w) = \eta^{(n)} (x, y, \ldots y_{n-1}, w(x, y, \ldots y_{n-1}))$$

(1.1)

(take $H = w(x, y, \ldots y_{n-1}) - y_n$).

Examples:

- A linear ODE:

  $$y_n = \sum_{i=0}^{n-1} w^{(i)}(x) y_i, \quad y_0 := y$$

  and the transformation $(\tilde{x}, \tilde{y}) = (x, e^iy)$. Its prolongation

  $$X = y \partial_y + y_1 \partial_{y_1} + y_2 \partial_{y_2} + \ldots$$

  so $\eta^{(n)} = y_n$ and $P_{r(n-1)} X(w) = w$.

- Under $(\tilde{x}, \tilde{y}) = (x, y + t)$. Then $X = \partial_y$ so $w_y = 0$ or $w = w(x, y_1, \ldots y_{n-1})$. In that case the order of the equation can be reduced by using the variable $z = y_1$.

- $(\tilde{x}, \tilde{y}) = (x + t, y)$ then $X = \partial_x$ and $w = w(y, y_1, \ldots y_{n-1})$. The order can be reduced again by inverting the dependent and independent variables $x \rightarrow y, y \rightarrow x$. 

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1.2 First order ordinary differential equations

Recall that, for \( X = \xi \partial_x + \eta \partial_y, \)

\[
\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y_1 - \xi_y(y_1)^2
\]

so, from (1.1) we get for a first order ODE \( y' = F(x, y) \)

\[
\eta_x + (\eta_y - \xi_x)F - \xi_yF^2 = \xi F_x + \eta F_y
\] (1.2)

This is a PDE for \( \xi, \eta. \) Note that any vectorfield where

\[
\eta/\xi = F
\] (1.3)

is a solution (not a big deal!)

**Proposition 1.1.** If \( Pdx + Qdy = 0 \) is an ODE and \( \xi \partial_x + \eta \partial_y \) is a generator of its symmetry, then

\[
R(x, y) := (\xi P + \eta Q)^{-1}
\]

is an integration factor, that is, \( RPdx + RQdy \) is an exact differential.

**Remark 1.1.** In case (1.3) \( R \) is not defined!

In some cases we may find other, more helpful solutions. For example

\[
y' = F(y/x)
\]

which is invariant under the scaling group action \( (x, y) \to \epsilon^\ell (x, y). \) Here \( X = x \partial_x + y \partial_y \) is a solution of (1.2). As we know, \( z_1 := y/x \) is an invariant, i.e \( X(z_1) = 0. \) So, if we take any other function \( z_2 \) of \( x \) or \( y \) as a second coordinate, then \( X \) is transformed into

\[
X(z_1)\partial_{z_1} + X(z_2)\partial_{z_2} = X(z_2)\partial_{z_2}
\]

and the ODE is transformed into

\[
dz_2/dz_1 = \tilde{F}(z_1) .
\]

For example, if \( z_2 = \ln x \) then \( y = z_1e^{z_2}, \) \( x = e^{z_2} \) so

\[
F(z_1) = \frac{dy}{dx} = \frac{d(z_1e^{z_2})}{de^{z_2}} = \frac{1 + z_1dz_2/dz_1}{dz_2/dz_1}
\]

and

\[
\frac{dz_2}{dz_1} = \frac{1}{F(z_1) - 1} .
\]

Another example:

\[
y' = \frac{y + xH(\sqrt{x^2 + y^2})}{x - yH(\sqrt{x^2 + y^2})}
\] (1.4)
is invariant under the action of $X = -y \partial_x + x \partial_y$. Indeed, it is the most general equation of this form: Any such equation must be of the form $H(w_0) = w_1$ where $w_0, w_1$ are the two first invariant of order $\leq 1$ by Theorem 1 (see (A)). Again, we take the zero-order invariant $r = \sqrt{x^2 + y^2}$ as the first coordinate, and (most conveniently) $\theta$ to be the second one. So $X = \partial_\theta$ in the new coordinate and the equation must be reduced into $d\theta/dr = F(r)$. We calculate, using this and (1.4),

$$ \frac{dy}{dx} = \frac{d(r \sin \theta)/dr}{d(r \cos \theta)/dr} = \frac{\sin \theta + r' \cos \theta}{\cos \theta - r' \sin \theta} $$

which yields

$$ \theta' = \frac{H(r)}{r} $$

### 1.3 Differential invariants revisited

Consider the first order linear PDE on $\mathbb{R}^{n+1}$:

$$ \sum a_i \partial x^i f = 0 $$

We can associate with this a vector field $A = \sum a_i \partial x^i$. We already know that it can be transformed to a new set of coordinates $s, \phi_i, i = 1, \ldots n$ where

$$ A \phi_i = 0, \quad A s = 1 $$

so $A$ is transformed into $\partial_s$ in this system.

Now, we consider an $n$ order ODE:

$$ y_n = w(x, y, y', y'', \ldots, y^{(n-1)}) $$

and the corresponding vector field on $\mathbb{R}^{n+1}$

$$ A = \partial_x + y_1 \partial_y + y_2 \partial_{y_1} + \ldots + w \partial_{y_{n-1}} $$

in the coordinates $x, y_0, y_1, \ldots y_{n-1}$. Here $w = w(x, y, y_1, \ldots y_{n-1})$.

There is a deep relation between the ODE and $A$ so defined. A solution of $A \phi = 0$ is an invariant of motion to the ODE, that is

$$ \frac{d}{dx} \phi(x, y, y', \ldots, y_{n-1}) = A \phi = 0 $$

If we found such a non-constant invariant $\phi$ then $\phi_{y_{n-1}} \neq 0$ (why?) and we solve for the implicit function $y_{n-1} = \hat{w}(x, y, y_1, \ldots, y_{n-2})$ and reduce the order of the ODE:

$$ y^{(n-1)} = \hat{w} \left( x, y, y', \ldots, y_{n-2} \right) $$

**Lemma 1.1.** If $A \neq 0$ is a v-f in $\mathbb{R}^{n+1}$ then there are (locally) $n$ functionally independent invariant functions $\phi_1, \ldots \phi_n$ verifying $A \phi_i = 0$. Moreover, any invariant function $f$ is given by $f = F(\phi_1, \ldots \phi_n)$ for some smooth $F : \mathbb{R}^n \to \mathbb{R}$.

Moreover, if $B$ is another vector field with the same invariants as $A$, then $B = \lambda(x_1, \ldots x_{n+1})A$. 4
Proof. Since we can find new variables $s_0, \ldots, s_n$ in which $A = \partial_{s_0}$, then all invariants are functions of $s_1, \ldots, s_n$. Also, any other vector field with the same invariants is of the form $\lambda \partial_{s_0}$ for some $\lambda(s_0, \ldots, s_n)$.

Moreover, if we find a complete set of $n$ functionally independent invariants $\phi_1, \ldots, \phi_n$ then we may eliminate $y = y(x, \phi_1^0, \ldots, \phi_n^0)$ form the system

$$\phi_1 = \phi_1^0 \ldots \phi_n = \phi_n^0$$

and get $n$ parameter family of solutions!

Example: consider $y'' = -y$ then $A = \partial_x + y_1 \partial_{y_1} - y_0 \partial_{y_1}$ and the invariants are

$$\phi_1 = y_0^2 + y_1^2 , \ \phi_2 = x - \arctan(y_0/y_1)$$

We can eliminate the solution $y_0 = y_1 \tan(x - \phi_1^0) \to (y_0)^2 = (y_1)^2 \tan^2(x - \phi_1^0) \to (y_0)^2 + (y_1)^2 = (y_1)^2 \cos^{-2}(x - \phi_1^0) \to (y_1)^2 = \phi_1^0 \cos^2(x - \phi_1^0) \to y_1 := y = (\phi_1^0/2)^{1/2} \sin(x - \phi_1^0)$

2 Symmetry of ODE: second formulation

Let $A = \partial_x + y_1 \partial_{y_1} + \ldots + w \partial_{y_{n-1}}$ and $X = \xi \partial_x + \eta \partial_y + \eta_1 \partial_{y_1} + \ldots + \eta_{n-1} \partial_{y_{n-1}}$ be a prolonged symmetry. Let $\phi_1, \ldots, \phi_n$ be a set of functionally independent invariants. Since $X(\phi)$ is also an invariant if $\phi$ is (prove!) and since the set of invariants are complete by Lemma 1.1, then

$$AX(\phi_i) = 0 \to X(\phi_i) = \Omega_i(\phi_1, \ldots, \phi_n)$$

It follows

$$[X, A] \phi_i = X(A(\phi_i)) - A(X(\phi_i)) = 0 , i = 1, \ldots, n .$$

So, by Lemma 1.1 again:

$$[X, A] = \lambda(x, y, y_1, \ldots, y_{n-1})A$$

(2.1)

Writing explicitly:

$$[X, A] = -(A\xi) \partial_x + [X(y_1) - A(\eta)] \partial_y + \ldots + [X(w) - A(\eta_{n-1})] \partial_{y_{n-1}}$$

so the first component (coefficient of $\partial_x$) yields

$$\lambda = -A\xi = -\xi_x - y_1 \xi_y := -\frac{d\xi}{dx}$$

Now, from (2.1)

$$\left( \eta' \frac{d\eta}{dx} \right) \partial_y + \left( \eta'' - \frac{d\eta_1}{dx} \right) \partial_{y_1} + \ldots + \left( X(w) - \frac{d\eta_{n-1}}{dx} \right) \partial_{y_{n-1}} = -\frac{d\xi}{dx} (y_1 \partial_y + y_2 \partial_{y_1} + \ldots + w \partial_{y_{n-1}})$$

(2.2)

Recall from the definition of the prolongation

$$\eta_k = \frac{d\eta_{k-1}}{dx} - y_k \frac{d\xi}{dx}$$

we obtain that (2.2) is equivalent to

$$X(w) = \eta_n \mod y_n = w$$

which is precisely $X(H) = 0$ for $H = y_n - w(x, y, y_1, \ldots, y_{n-1})$. 5