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Discrete Sturm comparison theorems on finite and infinite intervals

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The Sturm comparison theorem for second-order Sturm–Liouville difference equations on infinite intervals is established and discussed.

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1. Introduction and main results

In [2], the Sturm comparison theorem for second-order differential equations of the form

$$u'' + p(t)u = 0 \quad \text{for } t \in (a, b)$$

was established for the two singular situations

(i) when the coefficient p is unbounded near the endpoints $a, b \in \mathbb{R}$ of the interval,
(ii) when $a = -\infty$ or $b = \infty$ so that the interval (a, b) is unbounded.

In both cases, the conventional assumption that a solution u vanishes twice is replaced by the two boundary conditions

$$\int_a \frac{\mathrm{d}t}{u^2(t)} = \infty \quad \text{and} \quad \int^b \frac{\mathrm{d}t}{u^2(t)} = \infty.$$

In this work, we denote $\mathbb{Z}$ as the set of integers and formulate a discrete analogue of case (ii) for sequences $u := \{u_k\}_{k\in\mathbb{N}_0}$ and for Sturm–Liouville difference equations of the form

$$\Delta^2 u_k + p_k u_{k+1} = 0 \quad \text{for } k \in \mathbb{Z}, \tag{1}$$

where Δ denotes the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$. Together with (1), we consider another Sturm–Liouville difference equation of the form

$$\Delta^2 v_k + P_k v_{k+1} = 0 \quad \text{for } k \in \mathbb{Z}, \tag{2}$$

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and we assume throughout that

$$P_k \geq p_k \quad \text{for all } k \in \mathbb{Z}. \tag{3}$$

The Sturm comparison theorem for second-order difference equations on a finite set $[M,N]_{\mathbb{Z}} := [M,N] \cap \mathbb{Z}$ with $M, N \in \mathbb{Z}$ and $M < N$ is usually formulated and proved using the concept of disconjugacy (see [1,5,8] and [9, Theorem 6.19]). Here, we want to give a formulation that relies rather on generalized zeros and a proof that is elementary and more straightforward without the detour through disconjugacy. In Theorem 1.2 below and throughout, we use the following well-known concept of a generalized zero at $m \in \mathbb{Z}$.

Definition 1.1 (Generalized zero at an integer). *A solution u of* (1) (*or* (2)) *is said to have a* generalized zero *at* $m \in \mathbb{Z}$ *provided that either* $u_m = 0$ *or* $u_m u_{m+1} < 0$.

Theorem 1.2 (Sturm comparison theorem on a finite interval). *Let* $M, N \in \mathbb{Z}$ *with* $M < N$. *Assume* (3). *Suppose u is a nontrivial solution of* (1) *with generalized zeros at M and at N but without any generalized zeros in* $[M+1, N-1]_{\mathbb{Z}}$. *Then any solution v of* (2) *has at least one generalized zero in* $[M,N]_{\mathbb{Z}}$.

The main result of this paper is the following extension of Theorem 1.2 to infinite intervals. In Theorem 1.4 below and throughout, we use the following concept of a recessive solution at ∞.

Definition 1.3 (Recessive solution at ∞). *A solution u of* (1) (*or* (2)) *is said to be* recessive *at* ∞ *provided that there exists* $m \in \mathbb{Z}$ *such that*

$$u_k u_{k+1} > 0 \quad \text{for all } k \in [m,\infty)_{\mathbb{Z}} \quad \text{and} \quad \sum_{k=m}^{\infty} \frac{1}{u_k u_{k+1}} = \infty.$$

Theorem 1.4 (Sturm comparison theorem on an infinite interval). *Let* $M \in \mathbb{Z}$. *Assume* (3). *Suppose u is a solution of* (1) *which is recessive at* ∞ *and has a generalized zero at M but has no generalized zero in* $[M+1,\infty)_{\mathbb{Z}}$. *Then any solution v of* (2) *is recessive at* ∞ *or has at least one generalized zero in* $[M,\infty)_{\mathbb{Z}}$.

Note that both Sturm comparison theorems (Theorems 1.2 and 1.4) also provide Sturm separation theorems in the case when we have equality in all inequalities of (3). A 'proper' version of Theorem 1.4 that does not contain a Sturm separation theorem is as follows.

Theorem 1.5 ('Proper' Sturm comparison theorem on an infinite interval). *Let* $M \in \mathbb{Z}$. *Assume* (3) *and*

$$\text{there exists} \quad \ell \in [M,\infty)_{\mathbb{Z}} \quad \text{such that } P_\ell > p_\ell. \tag{4}$$

Suppose u is a solution of (1) *which is recessive at* ∞ *and has a generalized zero at M but has no generalized zero in* $[M+1,\infty)_{\mathbb{Z}}$. *Then any solution v of* (2) *has at least one generalized zero in* $[M,\infty)_{\mathbb{Z}}$.

In Section 2, we prove the above results and supply some examples. Section 3 deals with the corresponding results in the double-infinite case.

2. Auxiliary results, proofs and examples

In Lemma 2.1 below, we give several auxiliary formulae that are needed in the proof of Theorem 1.4 below. These auxiliary results may be checked by simple calculations. As these results also serve to supply a short proof of Theorem 1.2, we will present this proof below as well.

LEMMA 2.1. *Let u and v be solutions of* (1) *and* (2), *respectively, and define the sequence w by*

$$w_k := v_k \Delta u_k - u_k \Delta v_k = u_{k+1} v_k - u_k v_{k+1}.$$

(i) *We have*

$$\Delta w_k = (P_k - p_k) u_{k+1} v_{k+1}.$$

(ii) *If* $u_k u_{k+1} \neq 0$ *at* $k \in \mathbb{Z}$, *then*

$$\Delta \frac{v_k}{u_k} = -\frac{w_k}{u_k u_{k+1}}.$$

(iii) *If* $v_k v_{k+1} \neq 0$ *at* $k \in \mathbb{Z}$, *then*

$$\Delta \frac{u_k}{v_k} = \frac{w_k}{v_k v_{k+1}}.$$

(iv) *If* $u \neq 0$ *has a generalized zero at* $k \in \mathbb{Z}$ *but v does not and* $v_{k+1} \neq 0$, *then*

$$\frac{u_{k+1}}{v_{k+1}} w_k > 0.$$

(v) *If u has a generalized zero at* $k \in \mathbb{Z}$ *but v does not, then*

$$\frac{u_k}{v_k} w_k \leq 0.$$

Proof of Theorem 1.2. Suppose the claim is wrong, i.e. assume that v has no generalized zero in $[M,N]_{\mathbb{Z}}$, i.e. $v_k v_{k+1} > 0$ for all $k \in [M, N-1]_{\mathbb{Z}}$ and $v_N v_{N+1} \geq 0$. First note that by Lemma 2.1(iv)

$$\frac{u_{M+1}}{v_{M+1}} w_M > 0, \quad \text{so} \quad \frac{u_{k+1}}{v_{k+1}} w_M \geq 0 \quad \text{for all } k \in [M, N-1]_{\mathbb{Z}}$$

so that $w_M \neq 0$ and, due to Lemma 2.1(i), the sequence w is on $[M,N]_{\mathbb{Z}}$ increasing if $w_M > 0$ and decreasing if $w_M < 0$. Therefore,

$$w_k w_M \geq w_M^2 > 0 \quad \text{for all } k \in [M,N]_{\mathbb{Z}}.$$

By Lemma 2.1(iii), we thus have

$$w_M\left(\frac{u_N}{v_N}-\frac{u_M}{v_M}\right)=w_M\sum_{k=M}^{N-1}\frac{w_k}{v_kv_{k+1}}\geq w_M^2\sum_{k=M}^{N-1}\frac{1}{v_kv_{k+1}}\geq\frac{w_M^2}{v_Mv_{M+1}}=w_M\frac{u_{M+1}}{v_{M+1}}-w_M\frac{u_M}{v_M}$$

and therefore, using Lemma 2.1(v) for the last inequality,

$$0<\frac{u_{M+1}}{v_{M+1}}w_M\leq\frac{u_N}{v_N}w_M\leq\frac{u_N}{v_N}w_N\leq 0,$$

a contradiction. □

Now we prove the main result of this paper.

Proof of Theorem 1.4. Suppose the claim is wrong, i.e. assume that v is not recessive at ∞ and has no generalized zero in $[M,\infty)_{\mathbb{Z}}$, i.e. $v_kv_{k+1}>0$ for all $k\in[M,\infty)_{\mathbb{Z}}$ and

$$\sum_{k=M}^{\infty}\frac{1}{v_kv_{k+1}}<\infty.$$

First note that by Lemma 2.1(iv)

$$\frac{u_{M+1}}{v_{M+1}}w_M>0,\quad\text{so}\quad\frac{u_{k+1}}{v_{k+1}}w_M\geq 0\quad\text{for all }\ k\in[M,\infty)_{\mathbb{Z}}$$

so that $w_M\neq 0$ and, due to Lemma 2.1(i), the sequence w is on $[M,\infty)_{\mathbb{Z}}$ increasing if $w_M>0$ and decreasing if $w_M<0$. Therefore,

$$w_kw_M\geq w_M^2>0\quad\text{for all }\ k\in[M,\infty)_{\mathbb{Z}}.$$

By Lemma 2.1(ii), we thus have for $n\in[M+1,\infty)_{\mathbb{Z}}$

$$w_M\left(\frac{v_n}{u_n}-\frac{v_{M+1}}{u_{M+1}}\right)=-w_M\sum_{k=M+1}^{n-1}\frac{w_k}{u_ku_{k+1}}\leq -w_M^2\sum_{k=M+1}^{n-1}\frac{1}{u_ku_{k+1}},$$

which tends to $-\infty$ as $n\to\infty$ so that

$$\frac{u_n}{v_n}\to 0\quad\text{as }\ n\to\infty.$$

Moreover, by Lemma 2.1(iii)

$$w_M\left(\frac{u_n}{v_n}-\frac{u_M}{v_M}\right)=w_M\sum_{k=M}^{n-1}\frac{w_k}{v_kv_{k+1}}\geq w_M^2\sum_{k=M}^{n-1}\frac{1}{v_kv_{k+1}}\geq\frac{w_M^2}{v_Mv_{M+1}}=w_M\frac{u_{M+1}}{v_{M+1}}-w_M\frac{u_M}{v_M}.$$

Hence, by letting $n\to\infty$, we obtain

$$0\geq w_M\frac{u_{M+1}}{v_{M+1}}>0,$$

a contradiction. □

We next supply an example concerning Theorem 1.4.

Example 2.2. Let

$$P_k \equiv 2 \quad \text{and} \quad p_k = \begin{cases} \frac{3}{2} & \text{for } k \in (-\infty, -1]_{\mathbb{Z}}, \\ -\frac{1}{2} & \text{for } k \in [0, \infty)_{\mathbb{Z}}. \end{cases}$$

Thus, (3) holds. Then (1) has a solution u satisfying

$$u_{-1} = 0 \quad \text{and} \quad u_k = 2^{-k} \quad \text{for all } k \in \mathbb{N}_0.$$

This solution u is recessive at ∞ and has a generalized zero at -1 but has no generalized zero in $[0, \infty)_{\mathbb{Z}}$. By Theorem 1.4, any solution v of (2) has at least one generalized zero in $[-1, \infty)_{\mathbb{Z}}$. Since (2) is equivalent to the equation $u_{k+2} + u_k = 0$, this statement can also easily be verified directly.

We now provide an example of how to construct a solution with two generalized zeros in $[M, \infty)_{\mathbb{Z}}$ provided the assumptions of Theorem 1.4 hold under the additional condition that strict inequality holds in (3) for at least one integer greater than or equal to M.

Theorem 2.3. *Let $M \in \mathbb{Z}$. Assume* (3) *and* (4). *Suppose u is a solution of* (1) *which is recessive at ∞ and has a generalized zero at M but has no generalized zero in $[M+1, \infty)_{\mathbb{Z}}$. Then the solution v of* (2) *satisfying*

$$v_\ell = u_\ell \quad \text{and} \quad v_{\ell+1} = u_{\ell+1}$$

has at least one generalized zero in $[M, \ell]_{\mathbb{Z}}$ and at least one generalized zero in $[\ell+1, \infty)_{\mathbb{Z}}$.

Proof. First suppose that the first half of the claim is wrong, i.e. $\ell > M$, $v_k v_{k+1} > 0$ for all $k \in [M, \ell-1]_{\mathbb{Z}}$ and $v_\ell v_{\ell+1} \geq 0$. Then $u_{M+1} v_M > 0$ and $u_M v_{M+1} \leq 0$ so that $w_M > 0$. Hence, by Lemma 2.1(i),

$$0 > -w_M = w_\ell - w_M = \sum_{k=M}^{\ell-1} (P_k - p_k) u_{k+1} v_{k+1} \geq 0,$$

a contradiction. Next suppose that the second half of the claim is wrong, i.e. $v_k v_{k+1} > 0$ for all $k \in [\ell+1, \infty)_{\mathbb{Z}}$. Then for $k \in [\ell+1, \infty)_{\mathbb{Z}}$, using Lemma 2.1(i),

$$w_k = w_k - w_\ell = \sum_{i=\ell}^{k-1} (P_i - p_i) u_{i+1} v_{i+1} \geq (P_\ell - p_\ell) u_{\ell+1}^2 =: C > 0.$$

By Lemma 2.1(ii), for all $n \in [\ell+1, \infty)_{\mathbb{Z}}$,

$$1 \geq \frac{v_\ell}{u_\ell} - \frac{v_n}{u_n} = \sum_{k=\ell}^{n-1} \frac{w_k}{u_k u_{k+1}} \geq C \sum_{k=\ell}^{n-1} \frac{1}{u_k u_{k+1}},$$

which tends to ∞ as $n \to \infty$, a contradiction. □

Using Theorems 2.3 and 1.4, we may now prove Theorem 1.5.

Proof of Theorem 1.5. Since the assumptions are the same as in Theorem 2.3, the solution of (2) defined in Theorem 2.3 has at least two generalized zeros in $[M,\infty)_{\mathbb{Z}}$. Thus, by Theorem 1.2 in the case of $P=p$, i.e. the Sturm separation theorem on a finite interval, any solution v must have at least one generalized zero in $[M,\infty)_{\mathbb{Z}}$. □

3. The double-infinite case

In this final section, we establish the Sturm comparison theorem on the double-infinite case $\mathbb{Z}$. Our result contains a corresponding Sturm separation theorem. In Theorem 3.2 below and throughout, we use the following concept of a recessive solution at $-\infty$.

Definition 3.1 (Recessive solution at $-\infty$). *A solution u of* (1) (*or* (2)) *is said to be* recessive *at $-\infty$ provided that there exists $m\in\mathbb{Z}$ such that*

$$u_k u_{k+1} > 0 \quad \textit{for all } k\in(-\infty,m]_{\mathbb{Z}} \quad \textit{and} \quad \sum_{k=-\infty}^{m}\frac{1}{u_k u_{k+1}}=\infty.$$

Theorem 3.2 (Sturm comparison theorem on a double-infinite interval). *Suppose u is a solution of* (1) *which is recessive at $-\infty$ and at ∞ and has no generalized zero in $\mathbb{Z}$. Then any solution v of* (2) *is recessive at $-\infty$ or at ∞ or has at least one generalized zero in $\mathbb{Z}$.*

Proof. Suppose the claim is wrong, i.e. $v_k v_{k+1}>0$ for all $k\in\mathbb{Z}$ and

$$\sum_{k=0}^{\infty}\frac{1}{v_k v_{k+1}}<\infty \quad \text{and} \quad \sum_{k=-\infty}^{0}\frac{1}{v_k v_{k+1}}<\infty.$$

Without loss of generality, suppose that $u_k v_k>0$ for all $k\in\mathbb{Z}$ (otherwise, we consider $\tilde{v}=-v$). Thus, by Lemma 2.1(i), w is increasing on $\mathbb{Z}$. So there are only the following three possibilities:

1. There exists $M\in\mathbb{Z}$ such that $w_M>0$. Then $w_k\ge w_M>0$ for all $k\in[M,\infty)_{\mathbb{Z}}$. Thus, for all $n\in[M,\infty)_{\mathbb{Z}}$, using Lemma 2.1(ii),

$$\frac{v_n}{u_n}-\frac{v_M}{u_M}=-\sum_{k=M}^{n-1}\frac{w_k}{u_k u_{k+1}}\le -w_M\sum_{k=M}^{n-1}\frac{1}{u_k u_{k+1}},$$

which tends to $-\infty$ as $n\to\infty$, and thus

$$\frac{u_n}{v_n}\to\infty \quad \text{as } n\to\infty.$$

Therefore, for all $n\in[M,\infty)_{\mathbb{Z}}$, using Lemma 2.1(iii),

$$\frac{u_n}{v_n}-\frac{u_M}{v_M}=\sum_{k=M}^{n-1}\frac{w_k}{v_k v_{k+1}}\ge w_M\sum_{k=M}^{n-1}\frac{1}{v_k v_{k+1}}>\frac{w_M}{v_M v_{M+1}}=\frac{u_{M+1}}{v_{M+1}}-\frac{u_M}{v_M}$$

and hence, by letting $n\to\infty$,

$$0\ge\frac{u_{M+1}}{v_{M+1}},$$

a contradiction.

2. There exists $M \in \mathbb{Z}$ such that $w_M < 0$. Then $w_k \leq w_M < 0$ for all $k \in (-\infty, M]_{\mathbb{Z}}$. Thus, for all $n \in (-\infty, M]_{\mathbb{Z}}$, using Lemma 2.1(ii),

$$\frac{v_{M+1}}{u_{M+1}} - \frac{v_n}{u_n} = -\sum_{k=n}^{M} \frac{w_k}{u_k u_{k+1}} \geq -w_M \sum_{k=n}^{M} \frac{1}{u_k u_{k+1}},$$

which tends to ∞ as $n \to -\infty$, and thus

$$\frac{u_n}{v_n} \to \infty \quad \text{as} \quad n \to -\infty.$$

Therefore, for all $n \in (-\infty, M]_{\mathbb{Z}}$, using Lemma 2.1(iii),

$$\frac{u_{M+1}}{v_{M+1}} - \frac{u_n}{v_n} = \sum_{k=n}^{M} \frac{w_k}{v_k v_{k+1}} \leq w_M \sum_{k=n}^{M} \frac{1}{v_k v_{k+1}} < \frac{w_M}{v_M v_{M+1}} = \frac{u_{M+1}}{v_{M+1}} - \frac{u_M}{v_M}$$

and hence, by letting $n \to -\infty$,

$$0 \leq -\frac{u_M}{v_M},$$

a contradiction.

3. $w_k = 0$ for all $k \in \mathbb{Z}$. Then we have $u_{k+1} v_k = v_{k+1} u_k$ for all $k \in \mathbb{Z}$ and hence

$$\frac{u_{k+1}}{v_{k+1}} = \frac{u_k}{v_k} \equiv \frac{u_0}{v_0} =: \alpha \quad \text{for all} \ \ k \in \mathbb{Z}$$

so that $u_k = \alpha v_k$ for all $k \in \mathbb{Z}$, where $\alpha > 0$, a contradiction.

Since no other possibilities exist, the proof is complete. □

As in Theorems 2.3 and 1.5, we may also prove the following two results.

THEOREM 3.3. *Assume* (3) *and*

$$\textit{there exists} \quad \ell \in \mathbb{Z} \quad \textit{such that} \quad P_\ell > p_\ell. \tag{5}$$

Suppose u is a solution of (1) *which is recessive at* $-\infty$ *and at* ∞ *and has no generalized zero in* $\mathbb{Z}$. *Then the solution v of* (2) *satisfying*

$$v_\ell = u_\ell \quad \textit{and} \quad v_{\ell+1} = u_{\ell+1}$$

has at least one generalized zero in $(-\infty, \ell]_{\mathbb{Z}}$ *and at least one generalized zero in* $[\ell+1, \infty)_{\mathbb{Z}}$.

THEOREM 3.4 ('PROPER' STURM COMPARISON THEOREM ON A DOUBLE-INFINITE INTERVAL). *Assume* (3) *and* (5). *Suppose u is a solution of* (1) *which is recessive at* $-\infty$ *and at* ∞ *and has no generalized zero in* $\mathbb{Z}$. *Then any solution v of* (2) *has at least one generalized zero in* $\mathbb{Z}$.

Example 3.5. Let

$$P_k = \begin{cases} -\frac{1}{2} & \text{for } k \in \mathbb{Z}\backslash\{-1\}, \\ 2 & \text{for } k = -1, \end{cases} \quad \text{and} \quad p_k = \begin{cases} -\frac{1}{2} & \text{for } k \in \mathbb{Z}\backslash\{-1\}, \\ 1 & \text{for } k = -1. \end{cases}$$

Thus, (3) holds and (5) holds with $\ell = -1$. Then (1) has a solution u satisfying

$$u_{-k} = u_k = 2^{-k} \quad \text{for all } k \in \mathbb{N}_0.$$

This solution u is recessive at $-\infty$ and at ∞ and has no generalized zero in $\mathbb{Z}$. By Theorem 3.3, the solution v of (2) satisfying

$$v_{-1} = u_{-1} \quad \text{and} \quad v_0 = u_0$$

has at least two generalized zeros in $\mathbb{Z}$. Since v is explicitly given by

$$v_{-1} = \frac{1}{2}, \quad v_0 = 1, \quad v_1 = -\frac{1}{2}, \quad v_{-k} = v_k = -2^{-|k|} \quad \text{for all } k \in \mathbb{N}\backslash\{1\},$$

this statement can also easily be verified directly as v obviously has on $\mathbb{Z}$ exactly the two generalized zeros -2 and 0.

We conclude this paper by mentioning three more possibilities of extensions, which may become the subjects of future work.

Remark 3.6. One could consider more general self-adjoint difference equations of the form

$$\Delta(r_k \Delta u_k) + p_k u_{k+1} = 0, \tag{6}$$

where r_k is assumed to be different from zero but may have both positive and negative values. See [3, Section 1.5] (and [9, Example 6.7]), where the Fibonacci difference equation is written as

$$\Delta\big((-1)^k \Delta u_k\big) + (-1)^k u_{k+1} = 0.$$

A solution of an equation (6) is said to have a generalized zero at $k \in \mathbb{Z}$ if $u_k = 0$ or $r_k u_k u_{k+1} < 0$.

Remark 3.7. A generalization of all the presented results to time scales (see [6]) is also a natural step.

Remark 3.8. The Sturmian comparison method has been extended in various directions. We are grateful to the referee who drew our attention to the paper [7] (see also [4]), where a Sturmian theory is developed for pairs of difference equations of a special type, namely

$$x(i+1) - a(i)x(i) + \sum_{k=1}^{N} b^{(k)}(i)x(i-k) + \sum_{\ell=1}^{M} c^{(\ell)}(i)x(i+\ell) = 0,$$
$$y(i-1) - a(i)y(i) + \sum_{k=1}^{N} b^{(k)}(i+k)y(i+k) + \sum_{\ell=1}^{M} c^{(\ell)}(i-\ell)y(i-\ell) = 0,$$

where $b^{(k)}(i), c^{(\ell)}(i) \geq 0$.

Notes

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