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Existence of global solutions of some ordinary differential equations

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Abstract

Existence of globally defined solutions of ordinary differential equations is considered. The article studies the situation when most of the solutions run away to infinity in a finite time interval, but between them there exists at least one solution which is defined at all times.

Keywords: ODE; Global solution; Runaway solution; Escape time; Funnel

Given a first-order differential equation

$$x'(t) = f(t,x), \quad (t,x) \in R^2, \tag{1}$$

which satisfies some local existence and uniqueness theorem. It is an interesting question whether Eq. (1) has a global solution $x(t)$, which is defined for every value of t. Usually this is not the case. Linear equations and some sublinear equations are fortunate but exceptional examples, with all their solutions existing globally. Here we consider the opposite situation, when among infinitely many solutions of Eq. (1) which exist only locally, there is at least one global solution.

An efficient method for establishing the existence of a global solution on a given interval is the funnel method. The set

$$\ell(t) < x < u(t), \quad a < t < b, \tag{2}$$

is called a *funnel* for Eq. (1) on the interval (a,b), $-\infty \leqslant a < b \leqslant \infty$, if

$$\begin{aligned} &\ell'(t) < f\big(t,\ell(t)\big),\\ &u'(t) > f\big(t,u(t)\big), \end{aligned} \tag{3}$$

for $a < t < b$. The set (2) is called an *antifunnel* if the inequalities (3) are reversed, namely

$$\begin{aligned} &\ell'(t) > f\big(t,\ell(t)\big),\\ &u'(t) < f\big(t,u(t)\big). \end{aligned} \tag{4}$$

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It is known that if a funnel (an antifunnel) exists, it contains a solution $x(t)$ which is defined for all t in (a,b). For a detailed discussion and more general results see [1]. For an elegant application, see [2]. The concepts of funnel and antifunnel are closely related, since a change of variables $s=-t$ transforms the funnel (2) for Eq. (1) on (a,b) into an antifunnel for the equation $dx/ds=-f(-s,x)$ on $(-b,-a)$.

While the application of funnels is highly efficient and intuitive, it has a practical drawback: the functions $\ell(t), u(t)$ must be explicitly given and they must be found on an ad hoc basis for each equation and this may be a nontrivial task. We suggest to evade this difficulty and to replace the idea of funnels by some more diffuse concept.

The geometric interpretation of (3) is that the direction field associated with Eq. (1) intersects the boundaries $x=\ell(t)$, $x=u(t)$ of the funnel transversally and the trajectories of solutions enter the funnel as t increases. Between these solutions there exists a solution (or more) which stays in the funnel for all t. (4) means the opposite: through every upper or lower boundary point, a solution leaves the antifunnel as t increases and between them there exists a solution which is trapped for every t.

We suggest to generalize this idea and to consider the whole plane as a funnel (an antifunnel), whose boundary lines lay at $x=+\infty$ and $x=-\infty$, solutions "enter" (or "leave") it and bound between them some global solution. Hence, we shall assume that either for each value of t,

$$\begin{aligned} &f(t,x)>0 \quad \text{as } x\to+\infty,\\ &f(t,x)<0 \quad \text{as } x\to-\infty \end{aligned} \tag{5}$$

(an "antifunnel") or

$$\begin{aligned} &f(t,x)<0 \quad \text{as } x\to+\infty,\\ &f(t,x)>0 \quad \text{as } x\to-\infty \end{aligned} \tag{6}$$

(a "funnel").

Unfortunately, assumption (5) (or (6)) alone is not sufficient. In the following example we show an equation of type (1) that satisfies (5) and nevertheless each of its solutions escapes to infinity in a finite time interval and none of them is defined for all t.

Example 1. Consider the equation

$$x'(t)=g(t,x)=\begin{cases} -tx^3+x^{1/2}, & \text{I: } x>0,\ t\leqslant 0,\\ (x-t^4)^{1/2}, & \text{II: } x>t^4,\ t>0,\\ (x-t^4)^3, & \text{III: } x\leqslant t^4,\ t>0,\\ x^3, & \text{IV: } x\leqslant 0,\ t\leqslant 0. \end{cases} \tag{7}$$

$g(t,x)$ is continuous everywhere. It satisfies local Lipshitz conditions that ensure uniqueness of initial value problems except for $x=0$, $t\leqslant 0$. Any solution that starts in I, increases as long as it stays in I. It satisfies there $x'(t)\geqslant -tx^3$, so for any $t_0\leqslant t\leqslant 0$, integrating the inequality $x^{-3}x'\geqslant -t$ from t_0 to t yields

$$x(t)\geqslant x_0/\big(1-x^2(t_0)\big(t_0^2-t^2\big)\big)^{1/2}.$$

Hence $x(t)$ either escapes to $+\infty$ in I or crosses into II.

In II, $x'(t)=(x-t^4)^{1/2}\leqslant x^{1/2}$, so for $t>t_1\geqslant 0$, integrating $x^{-1/2}x'\leqslant 1$ from t_1 to t leads to

$$x(t)\leqslant\big(x^{1/2}(t_1)+(t-t_1)/2\big)^2.$$

Consequently each such trajectory crosses the curve $x=t^4$ and enters III.

In III every solution $x(t)$ decreases. It cannot be bounded from below for all t by any x_1, otherwise we would have $x'(t)\to 0$ as $t\to\infty$, which is impossible in III. Hence it must cross into $x<0$, $t>0$. But for $x<0$, we have $x'(t)=(x-t^4)^3\leqslant x^3<0$, and integration of $x^{-3}x'\geqslant 1$ from t_2 to t, $t>t_2$, results in

$$x(t)\leqslant -\big(x^{-2}(t_2)-2(t-t_2)\big)^{-1/2},$$

as long as $x(t)$ exists. Hence $x(t)$ escapes to $-\infty$ in a finite interval of time. The same argument holds for solutions which pass through IV. See Fig. 1. This verifies that in spite of inequalities (5), no solution of Eq. (7) is defined for all values of t.

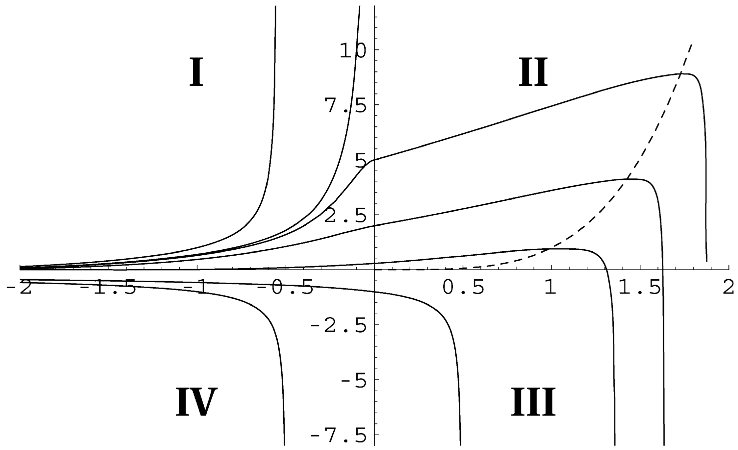

Fig. 1. Trajectories of Eq. (7).

Theorem 1. *Let Eq.* (1) *satisfy a local existence and uniqueness theorem near every point of the* (t, x) *plane. Suppose that for every* t*-interval* $[a, b]$ *there exists a function* $h_{a,b}(x)$ *defined for* $|x| > X_{a,b} > 0$ *such that*

$$f(t,x) > h_{a,b}(x) > 0 \quad \text{for } a \leqslant t \leqslant b,\ x > X_{a,b},$$
$$f(t,x) < h_{a,b}(x) < 0 \quad \text{for } a \leqslant t \leqslant b,\ x < -X_{a,b}, \tag{8}$$

and

$$\int^{\infty} \frac{dx}{h_{a,b}(x)} < \infty,$$
$$\int_{-\infty} \frac{dx}{|h_{a,b}(x)|} < \infty. \tag{9}$$

Then Eq. (1) *has a global solution* $x(t)$ *that is defined for every* t, $-\infty < t < \infty$.

Proof. Let $x_{\alpha,\beta}(t)$ denote the solution of (1) that is defined by the initial value condition $x(\alpha) = \beta$ and let U (respectively L) be the set of the points (α, β) such that $x_{\alpha,\beta}(t)$ escapes to $+\infty$ (to $-\infty$) at some finite time τ, $\tau > \alpha$.

First we show that U is an open and nonempty set. Without loss of generality it may be assumed that if $[c, d] \supset [a, b]$, then $h_{c,d}(x) \leqslant h_{a,b}(x)$ for $x > \max(X_{a,b}, X_{c,d})$. Indeed, the given $h_{a,b}(x)$ may be replaced by $\min_{a\leqslant t\leqslant b} f(t, x)$ for $x > X_{a,b}$. Let $(\alpha, \beta) \in U$, i.e., $x_{\alpha,\beta}(t) \to +\infty$ as $t \to \tau^-$ for some finite τ and take a point (γ, δ) close enough to (α, β). Consider Eq. (1) on the interval $[\alpha, \tau + 1]$ and choose ξ such that

$$h_{\alpha,\tau+1}(x) > 0 \quad \text{for } x > \xi$$

and

$$\int_{\xi}^{\infty} \frac{dx}{h_{\alpha,\tau+1}(x)} < 1.$$

Since $x_{\alpha,\beta}(t) \to +\infty$ as $t \to \tau^-$, there exists t_1, $\alpha < t_1 < \tau$, such that $x_{\alpha,\beta}(t_1) > \xi$. Uniqueness of solutions of initial value problems implies their continuous dependence on initial value conditions. So, if (γ, δ) is sufficiently close to (α, β), the solution $x_{\gamma,\delta}(t)$ is defined on $[\alpha, t_1]$ and on this interval it is as close to $x_{\alpha,\beta}(t)$ as we wish. In particular, $x_{\gamma,\delta}(t_1) > \xi$. The solution $x_{\gamma,\delta}(t)$ satisfies for values of t which are both in its domain of definition and also in $[t_1, \tau + 1]$,

$$x'(t) = f(t,x) \geqslant h_{t_1,\tau+1}(x) \geqslant h_{\alpha,\tau+1}(x),$$

$$1 > \int_{\xi}^{\infty} \frac{dx}{h_{\alpha,\tau+1}(x)} > \int_{t_1}^{t} dt = t - t_1.$$

Thus $x_{\gamma,\delta}(t)$ escapes to infinity for some t, $t < t_1 + 1 < \tau + 1$, and $(\gamma,\delta) \in U$. Consequently, the set U is open. The same argument shows that for a given interval $a \leqslant t \leqslant b$, any solution $x_{a,\eta}(t)$ escapes to $+\infty$ at some τ, $a \leqslant \tau \leqslant b$, provided that η is sufficiently large. So, U is nonempty.

An analogous argument verifies that the set L is open and nonempty as well. Take now any point (α,β) not in $U \cup L$. By the definitions of U and L, the whole trajectory of $x_{\alpha,\beta}(t)$ stays out of U and L and it cannot escape either to $+\infty$ or to $-\infty$ in any finite time interval. $x_{\alpha,\beta}(t)$ can neither terminate at any finite τ while remaining bounded. If $\lim_{t\to\tau^-} x_{\alpha,\beta}(t)$ exists and has a finite value η, the solution of the initial value condition $x(\tau) = \eta$ extends beyond τ. And if the limit does not exist, the bounded $x_{\alpha,\beta}(t)$ oscillates, as $t \to \tau^-$, between finite lim sup and lim inf, which is impossible with a bounded slope $x' = f(t,x)$. So the corresponding solution $x_{\alpha,\beta}(t)$ is defined globally for every t, $-\infty < t < \infty$.

We conclude the proof with some remarks.

(1) The claim of the theorem holds if the inequalities (8) are replaced by

$$\begin{aligned} &f(t,x) < h_{a,b}(x) < 0 \quad \text{for } a \leqslant t \leqslant b,\ x > X_{a,b}, \\ &f(t,x) > h_{a,b}(x) > 0 \quad \text{for } a \leqslant t \leqslant b,\ x < -X_{a,b}, \end{aligned} \tag{10}$$

and (9) by

$$\begin{aligned} &\int^{\infty} \frac{dx}{|h_{a,b}(x)|} < \infty, \\ &\int_{-\infty} \frac{dx}{h_{a,b}(x)} < \infty. \end{aligned} \tag{11}$$

It is equivalent to replacing t by $-t$ in Eq. (1).

(2) The proof above explains why Example 1 has no global solution: it happens since its set of solutions that escape to $+\infty$ in a finite time is not open. To verify this, consider the solutions of (7) with the initial values $x(0) = \beta$, $\beta > 1$, i.e., solutions which cross from I to II. For $-1 \leqslant t \leqslant 0$, $x \geqslant 1$, one has $x'(t) = -tx^3 + x^{1/2} \leqslant 2x^3$. The integration $\int_x^{\beta} x^{-3}\,dx \leqslant \int_t^0 2\,dt$ yields

$$x^2(t) \geqslant \beta^2/\bigl(1 - 4t\beta^2\bigr),$$

so $\lim_{\beta\to\infty} x(-1/8) > 1$. Let now $B = \sup_{\beta>1} x(-1/8) > 1$. It follows that the solution of the initial value condition $x(-1/8) = B$ never crosses the x-axis and it is the last (rightmost) solution which escapes to $+\infty$, while all solutions with $x(-1/8) < B$ cross into II and III and eventually escape to $-\infty$.

(3) A lower bound $f(t,x) \geqslant h(x) > 0$ for $a \leqslant t \leqslant b$ with $\int^{\infty} dx/h(x) < \infty$, which hints to superlinear behavior of f, is but one possible sufficient condition for the existence of a global solution. An assumption of the opposite type, an upper bound

$$\bigl|f(t,x)\bigr| \leqslant h(x)g(t), \quad h(x), g(t) > 0, \tag{12}$$

with $\int^{\infty} dx/h(x) = \infty$, implies that $\int_{x_0}^{x} dx/h(x) \leqslant \int_{t_0}^{t} g(t)\,dt$, i.e., every solution $x(t)$ is defined globally. (12) happens for sublinear equations and it represents, of course, a completely different situation. □

Example 2. Theorem 1 applies to all polynomial differential equations

$$x'(t) = a_0(t)x^n + a_1(t)x^{n-1} + \cdots + a_n(t), \tag{13}$$

where n is an odd integer, $n \geqslant 3$, $a_i(t)$ are continuous and $a_0(t) \neq 0$ for all t. In particular, when $n = 3$, this is an Abel differential equation of the first type. For an application see [3].

The following result is a hybrid of Theorem 1 and the antifunnel method.

Theorem 2. *Let $\ell(t) \in C^1(-\infty,\infty)$ and let Eq.* (1) *satisfy a local existence and uniqueness theorem in $W = \{(t,x) \mid x \geqslant \ell(t)\}$. Suppose that*

(a) $\ell'(t) > f(t,\ell(t))$ *for all t,*
(b) *for every t-interval $[a,b]$ there exists a function $h_{a,b}(x)$ such that*

$$f(t,x) > h_{a,b}(x) > 0 \quad \text{for } a \leqslant t \leqslant b,\ x > X_{a,b}, \tag{14}$$

and

$$\int^{\infty} \frac{dx}{h_{a,b}(x)} < \infty. \tag{15}$$

Then Eq. (1) *has a global solution $x(t)$ such that $x(t) > \ell(t)$ for every t.*

Proof. U is defined and treated as in the proof of Theorem 1. L is defined as the set of points (α,β) in W such that the corresponding solution $x_{\alpha,\beta}(t)$ meets the boundary $x = \ell(t)$ of W for some finite value of t, $t \geqslant \alpha$. In order to verify that L is open relative to W, let $(\alpha,\beta) \in L$, i.e., there exists some finite τ, $\tau \geqslant \alpha$, so that $x_{\alpha,\beta}(\tau) = \ell(\tau)$. Our aim is to show that for any point (γ,δ) sufficiently close to (α,β), $x_{\gamma,\delta}(t)$ meets $\ell(t)$.

According to assumption (a), we denote $m \equiv \ell'(\tau) - f(t,\ell(\tau)) > 0$. Let $\delta > 0$ be such that $|f(t,x) - f(\tau,\ell(\tau))| < m/3$ holds when $|t-\tau| < \delta$, $|x - \ell(\tau)| < \delta$ and $(t,x) \in W$, and that also $|\ell'(t) - \ell'(\tau)| < m/3$ when $|t-\tau| < \delta$. Take t_1, $\tau - \delta < t_1 < \tau$ such that $0 < x_{\alpha,\beta}(t_1) - \ell(\tau) < \min\{\delta, \delta m/3\}$ and fix this t_1. By the continuous dependence of solutions on the initial value conditions, $x_{\gamma,\delta}(t)$ exists on $[\gamma,t_1]$ and we also have

$$0 < x_{\gamma,\delta}(t_1) - \ell(\tau) < \min\{\delta, \delta m/3\}, \tag{16}$$

provided that (γ,δ) is sufficiently close to (α,β).

On the other hand, for $|t-\tau| < \delta$ and as long as $x_{\gamma,\delta}(t)$ stays in W,

$$\begin{aligned}\big(x_{\gamma,\delta}(t) - \ell(t)\big)' &= f\big(t, x_{\gamma,\delta}(t)\big) - \ell'(t) \\ &= \big(f\big(t,x_{\gamma,\delta}(t)\big) - f\big(\tau,\ell(\tau)\big)\big) + \big(f\big(\tau,\ell(\tau)\big) - \ell'(\tau)\big) + \big(\ell'(\tau) - \ell'(t)\big) \\ &< -m/3. \end{aligned} \tag{17}$$

By (16) and (17), $x_{\gamma,\delta}(t)) - \ell(t)$ must vanish for some t, $t_1 < t < t_1 + \delta$. Consequently $(\gamma,\delta) \in L$ and L is open relative to W. The rest of the proof follows the arguments of Theorem 1. □

Example 3. Theorem 2 with $\ell(t) \equiv 0$ is useful when one looks for a positive global solution. For example, Eq. (13) has a positive global solution if $a_0(t)a_n(t) < 0$.

The work in [4] is a step towards extending the method of funnels to higher dimensions. The method of Theorem 1 may be generalized also to some equations of higher order. Take for example a second-order equation

$$x''(t) = f(t,x,x'). \tag{18}$$

Theorem 3. *Let Eq.* (18) *satisfy a local existence and uniqueness theorem near every point in R^3. Suppose that for every interval $[a,b]$ and for every number X_0 there exist a function $h_{a,b}(x')$ and a number $X_{a,b} > 0$ such that for $a \leqslant t \leqslant b$,*

$$\begin{aligned} &f(t,x,x') \geqslant h_{a,b}(x') > 0 \quad \text{for } x' > X_{a,b},\ x > X_0, \\ &f(t,x,x') \leqslant h_{a,b}(x') < 0 \quad \text{for } x' < -X_{a,b},\ x < -X_0, \end{aligned} \tag{19}$$

and

$$\int^{\infty} \frac{dx_1}{h_{a,b}(x_1)} < \infty,$$
$$\int_{-\infty} \frac{dx_1}{|h_{a,b}(x_1)|} < \infty. \tag{20}$$

Then Eq. (18) *has a global solution $x(t)$ that is defined for every t, $-\infty < t < \infty$.*

Proof. If $|x(t)| + |x'(t)|$ is unbounded as $t \to \tau^-$ for some finite τ, $x'(t)$ is necessarily unbounded near τ, otherwise $x(t)$ would be bounded, too. However, $x(t)$ may be bounded near τ. For example, the equation $x'' = 2x'^3$ has solutions $x(t) = \pm(c_1 - t)^{1/2} + c_2$, where $|x'(t)| \to \infty$ as $t \to c_1^-$ while $x(t)$ is bounded. This explains the critical role of $x'(t)$. Assumptions (19) restrict the behavior of runaway solutions. Consider a solution on some $[a, b] \supset [t_0, \tau]$. The unbounded $x'(t)$ accepts arbitrary large values, say $x'(t_1) > X_{a,b} > 0$ for some t_1. Choose $X_0 = x(t_1)$. Then the inequality $x'' \geqslant h_{a,b}(x') > 0$ implies that $x'(t)$ increases for $t > t_1$. Hence $x'(t)$ is not only unbounded but it increases to $+\infty$ and $x(t) > x(t_1)$ for $t > t_1$. Essentially, the solution is eventually monotone.

We denote by $x_{\alpha,\beta_0,\beta_1}(t)$ the solution of (18) with the initial value conditions $x(\alpha) = \beta_0$, $x'(\alpha) = \beta_1$. Let U be the set of the points $(\alpha, \beta_0, \beta_1)$ such that $x'_{\alpha,\beta_0,\beta_1}(t) \to +\infty$ as $t \to \tau^-$ for some finite τ. The corresponding $x_{\alpha,\beta_0,\beta_1}(t)$ is, of course, bounded from below for $t > \alpha$. An argument as in the proof of Theorem 1 shows that U is an open, nonempty set. It is only necessary to replace $x_{\alpha,\beta}(t_1) > \xi$ by $x'_{\alpha,\beta_0,\beta_1}(t_1) > \xi$ and to note that the corresponding $x_{\alpha,\beta_0,\beta_1}(t)$ is bounded from below. The set L is defined and treated analogously and the existence of a global solution follows.

Before we try to apply Theorem 3 to a given equation, a change of the variable t to $-t$ may be helpful.

A drawback of Theorem 3 is the restrictive nature of our assumption that inequalities (19) hold for every $|x| > |X_0|$. For example, the method is applicable to $x'' = x'^3 + x^3$ but not to $x'' = x'^3 - x^3$ and $x'' = x'^3 + x^2$, though each of them has a global solution $x(t) \equiv 0$.

Note also that inequalities (19) are opposite to the Nagumo conditions. □

Example 4. Theorem 3 applies to polynomial differential equations of the type

$$x''(t) = a(t)p(x)x'^{2m+1} + \sum_{j=0}^{2m} a_j(t)p_j(x)x'^j + c(t), \quad m \geqslant 1, \tag{21}$$

provided that $a(t) > 0$, $a_j(t)$, $c(t)$ are continuous, $p(x)$ is a positive definite polynomial and $p_j(x)$ are polynomials such that $\deg[p_j] \leqslant \deg[p]$. The assumptions of Theorem 3 are satisfied since the first term of (21) dominates the other terms when $|x'| \to \infty$, either if x is bounded or unbounded.

Equations of the type (21) are admissible also when $a(t) > 0, a_j(t) \geqslant 0, c(t)$ are continuous, $p(x)$ is a positive definite polynomial, and $p_j(x)$ are monic polynomials such that $j + \deg[p_j]$ are arbitrary odd integers. In this case the first term dominates the other terms when $|x'| \to \infty$ and $x(t)$ is bounded; while if both $x'(t), x(t)$ tend to $+\infty$ (to $-\infty$), then all terms, except perhaps $c(t)$, have the same sign.

The equations $x'' = (x^2 + 1)x'^3 - x^2x'^2 - xx' + 1$ and $x'' = x'^3 + x^3x'^2 + x^4x' + 1$ satisfy, respectively, the two sets of assumptions stated above, but not the other way.

Theorem 3 does not apply to the Rayleigh equation $x'' - (x' - x'^3) + x = p(t)$, but it applies to the equation $x'' - (x' - x'^3) - x = p(t)$ (after replacing t by $-t$). This is not a coincidence. Many equations of applied mathematics are perturbations of periodic or stable equations. Our results are related to the opposite situation, when most solutions explode in a finite time and perhaps only one solution survives for every t.

A similar result may be stated for equations $x^{(n)}(t) = f(t, x, \dots, x^{(n-1)})$ of any order. For example, the first of inequalities (19) will be replaced by

$$f\bigl(t, x, \dots, x^{(n-1)}\bigr) \geqslant h_{a,b}\bigl(x^{(n-1)}\bigr) > 0$$

and U will be the set of initial values $(\alpha, \beta_0, \dots, \beta_{n-1}) \in R^{n+1}$ for which $x^{(n-1)}(t) \to +\infty$ as $t \to \tau^-$, etc.

References

[1] J.H. Hubbard, B.H. West, Differential Equations, a Dynamical System Approach, vol. 1, Springer, New York, 1991.
[2] J.H. Hubbard, J.M. McDill, A. Noonburg, B.H. West, A new look at the Airy equation with fences and funnels, in: Organic Mathematics, Burnaby, BC, 1995, in: CMS Conf. Proc., vol. 20, Amer. Math. Soc., Providence, RI, 1997, pp. 277–303.
[3] A.M. Ilin, B.I. Suleimanov, Asymptotic behaviour of a special solution of Abel's equation relating to a cusp catastrophe, Sb. Math. 197 (1) (2006) 53–67.
[4] V.A. Noonburg, A separating surface for the Painlevé differential equation $x'' = x^2 - t$, J. Math. Anal. Appl. 193 (1995) 817–831.