JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **97**, 277–290 (1983)

Generalizations of an Inequality of Kiguradze

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1. INTRODUCTION

A very useful, though simple, result about two term ordinary differential equations is the following proposition of Kiguradze [11]:

(a) *Given the equation*

$$\frac{d^n y}{dx^n} + yF(x, y) = 0 \tag{1.1}$$

where $F(x, y)$ *has a fixed sign, positive or negative, for* $0 \leqslant x < \infty$, $-\infty < y < \infty$. *If* y *is a nonoscillatory solution of* (1.1) *on* $[0, \infty)$, *there exists an integer* k, $0 \leqslant k \leqslant n$ *such that*

$$\begin{aligned} y^{(i)} &\geqslant 0, \qquad i = 0,\ldots, k,\\ (-1)^{j-k} y^{(j)} &\geqslant 0, \qquad j = k+1,\ldots, n, \end{aligned} \tag{1.2}$$

on a certain ray (a, ∞). *Moreover, the parity of* k *is such that*

$$(-1)^{n-k} F(x, y) \leqslant 0.$$

(b) *If*

$$\begin{aligned} y^{(i)} &\geqslant 0, \qquad i = 0,\ldots, k,\\ y^{(k+1)} &\leqslant 0, \end{aligned} \tag{1.3}$$

on (a, ∞) *then*

$$(x - a)\, y^{(t+1)} \leqslant (k - t)\, y^{(t)}, \qquad t = 0,\ldots, k, \tag{1.4}$$

on (a, ∞).

Proposition (a) is widely used in the study of linear, nonlinear, and delay

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differential equations. Inequalities (1.4) are used to estimate nonoscillatory solutions and to obtain criteria for the existence (or nonexistence) of various nonoscillatory solutions. See [4–9, 11, 12] and Lemma 3.2 of [13].

The aim of this note is to generalize inequalities (1.4) in various directions.

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In (1.3) Kiguradze utilized only the inequalities $y,\dots,y^{(k)} \geqslant 0$, $y^{(k+1)} \leqslant 0$ while in (1.2) we have information about the signs of other derivatives too. The next theorem generalizes (1.4) in this direction.

THEOREM 1. *If the function y satisfies*

$$\begin{aligned} &y(0), y'(0),\dots, y^{(k-1)}(0) \geqslant 0, \\ &(-1)^{j-k} y^{(j)}(x) \geqslant 0, \qquad 0 \leqslant x \leqslant L, \end{aligned} \tag{2.1}$$

where j is a certain integer, $j \geqslant k$, then

$$(-1)^{j-k} \frac{d^{j-k}}{dx^{j-k}} \left(\frac{y^{(t)}}{x^{k-t}}\right) \geqslant 0, \qquad 0 < x \leqslant L, \quad \text{for each } t = 0,\dots,k. \tag{2.2}$$

Equivalently,

$$(-1)^{j-k} \Delta_t^{j-k} \left(\frac{y^{(t)}}{x^{j-t-1}/(j-t-1)!}\right) \geqslant 0,$$

$$0 \leqslant t \leqslant k, \qquad \text{for each fixed} \quad x \in (0, L], \tag{2.3}$$

where Δ_t^{j-k} denotes the $(j-k)$th *forward difference with respect to the variable t.*

For example, if $j = k + 1$, inequalities (2.2), (2.3) are

$$\left(\frac{y^{(t)}}{x^{k-t}}\right)' \leqslant 0, \tag{2.4}$$

$$\frac{y^{(t+1)}}{x^{k-t-1}/(k-t-1)!} - \frac{y^{(t)}}{x^{k-t}/(k-t)!} \leqslant 0, \qquad t = 0,\dots,k, \tag{2.5}$$

both of which are identical with (1.4). Thus each of the $k + 1$ functions y/x^k, $y'/x^{k-1},\dots, y^{(k)}$ decreases on $(0, L]$ and for each fixed $x \in (0, L]$, we have a decreasing sequence $y(x)/(x^k/k!) \geqslant y'(x)/(x^{k-1}/(k-1)!) \geqslant \cdots \geqslant y^{(k)}(x)$. For $j = k + 2$ each of the functions $y/x^k,\dots, y^{(k)}$ is convex on $(0, L]$ and for each fixed $x \in (0, L]$, the sequence $y(x)/(x^{k+1}/(k+1)!)$, $y'(x)/(x^k/k!), \ldots, y^{(k)}(x)/x$ is convex.

Here we prove the theorem by a method which can be easily generalized. First we present some simple facts which will be used later on.

LEMMA. *The operators* $L_1 = x^p(d^{p-q}/dx^{p-q})\,x^{-q}$ *and* $L_2 = x^r(d^{r-s}/dx^{r-s})\,x^{-s}$, $p > q$, $r > s$, *are permutable. At* $x = 0$,

$$L_1 u|_{x=0} = (-1)^{p-q}\, q(q+1) \cdots (p-1)\, u(0), \tag{2.6}$$

$$L_1 L_2 u|_{x=0} = (-1)^{p-q+r=s}\, q(q+1) \cdots (p-1)\, s(s+1) \cdots (r-1)\, u(0) \tag{2.7}$$

Proof. In order to prove that $L_1 L_2 = L_2 L_1$ it is sufficient to show that $L_1 L_2 u = L_2 L_1 u$ for $p - q + r - s + 1$ functions with nonvanishing Wronskian. We shall show this for the functions $u = x^\alpha$ for any α. Indeed,

$$\begin{aligned} L_1[x^\alpha] &= (-q+\alpha)(-q+\alpha-1) \cdots (-p+\alpha+1)\, x^\alpha, \\ L_1 L_2[x^\alpha] = L_2 L_1[x^\alpha] &= \{(-q+\alpha)(-q+\alpha-1) \cdots (-p+\alpha+1)\} \\ &\quad \times \{(-s+\alpha)(-s+\alpha-1) \cdots (-r+\alpha+1)\}\, x^\alpha. \end{aligned}$$

Consequently $L_1 L_2 = L_2 L_1$.

By the Leibnitz rule

$$\begin{aligned} L_1 u &= x^p(x^{-q}u)^{(p-q)} \\ &= \sum_{l=0}^{p-q} \binom{p-q}{l} (-q)(-q-1) \cdots (-q-l+1)\, x^{p-q-l} u^{(p-q-l)} \end{aligned}$$

and for $x = 0$ we obtain (2.6). To show (2.7), let

$$L_1 L_2 u = \sum_{l=0}^{r-s} \binom{r-s}{l} (-s)(-s-1) \cdots (-s-l+1)\, L_1[x^{r-s-l} u^{(r-s-l)}].$$

Applying L_1 to the functions $x^{r-s-l} u^{(r-s-l)}$ and putting $x = 0$, we may have, by (2.6), a nonvanishing term only for $l = r - s$. Thus

$$L_1 L_2 u|_{x=0} = (-s)(-s-1) \cdots (-r+1)\, L_1 u|_{x=0}$$

and (2.7) follows.

Proof of Theorem 1. To prove (2.2) we define $H(x) = x^j(y/x^k)^{(j-k)}$, that is $H(x) = (x^j D^{j-k} x^{-k})\,y$, where $D = d/dx$.

By the lemma

$$\begin{aligned} H^{(t)}(x) &= x^{-t}(x^t D^t)(x^j D^{j-k} x^{-k})\,y = x^{-t}(x^j D^{j-k} x^{-k})(x^t D^t)\,y \\ &= (x^{j-t} D^{(j-t)-(k-t)} x^{-(k-t)})\,y^{(t)}. \end{aligned} \tag{2.8}$$

If we use (2.6) with $p = j - t$, $q = k - t$ we have for $t = 0, \dots, k-1$

$$H^{(t)}(0) = (-1)^{j-k} (k-t)(k-t+1) \cdots (j-t-1) y^{(t)}(0)$$

and thus, by (2.1)

$$(-1)^{j-k} H^{(t)}(0) \geqslant 0, \qquad t = 0, \dots, k-1. \tag{2.9}$$

For $t = k$ (2.8) gives $H^{(k)}(x) = x^{j-k} y^{(j)}$ and again by (2.1),

$$(-1)^{j-k} H^{(k)}(x) \geqslant 0 \qquad \text{for} \quad x \geqslant 0. \tag{2.10}$$

Inequalities (2.9), (2.10) imply that $(-1)^{j-k} H^{(t)}(x) \geqslant 0$ on $[0, L]$ for $t = 0, \dots, k$, which means by (2.8)

$$(-1)^{j-k} x^{j-t} \frac{d^{j-k}}{dx^{j-k}} \left(\frac{y^{(t)}}{x^{k-t}} \right) \geqslant 0, \qquad t = 0, \dots, k,$$

and (2.2) is proved.

To verify (2.3) we rewrite (2.2) as

$$\begin{aligned} 0 &\leqslant (-1)^{j-k} [x^{-(k-t)} y^{(t)}]^{(j-k)} \\ &= (-1)^{j-k} \sum_{l=0}^{j-k} \binom{j-k}{l} (-k+t)(-k+t-1) \cdots \\ &\quad \times (-k+t-l+1) x^{-k+t-l} y^{(t+j-k-l)} \\ &= (-1)^{j-k} \sum_{l=0}^{j-k} \binom{j-k}{l} (-1)^l \frac{(k-t+l-1)!}{(k-t-1)!} x^{-k+t-l} y^{(t+j-k-l)}. \end{aligned}$$

Multiplying by $x(k-t-1)!$ we obtain

$$(-1)^{j-k} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l} \frac{y^{(t+j+k-l)}}{x^{k-t+l-1}/(k-t+l-1)!} \geqslant 0$$

which is exactly (2.3).

Remark. (1) Kiguradze assumed originally that $y(x)$, $y'(x), \dots, y^{(k)}(x) \geqslant 0$, which is indeed the natural thing to do in context of Eq. (1.1). However only $y(0)$, $y'(0), \dots, y^{(k-1)}(0) \geqslant 0$ is really needed and y is not necessarily positive. For example, $y = x^{3/2} - x^2$ satisfies (2.1) and (2.2) with $k = 2, j = 3$ while y is not positive on $(0, \infty)$.

(2) It is not necessary to utilize all the assumptions of (2.1). If, for example, we use only

$$y(0), y'(0), \ldots, y^{(k-3)}(0) \geqslant 0,$$
$$(-1)^{j-k+2} y^{(j)}(x) = (-1)^{j-k} y^{(j)}(x) \geqslant 0,$$

that is, we replace k by $k-2$, we obtain the additional inequalities

$$(-1)^{j-k} \left(\frac{y^{(t)}}{x^{k-t-2}}\right)^{(j-k+2)} \geqslant 0, \qquad t = 0, \ldots, k-2 \tag{2.11}$$

AN EXAMPLE. Let $F(x) = 1/x \int_0^x f(s)\, ds$. If $f^{(n)} \geqslant 0$ for $x \geqslant 0$ then also $F^{(n)} \geqslant 0$, $x \geqslant 0$.

Indeed, put $y(x) = (-1)^n \int_0^x f(s)\, ds$. Then $y(0) = 0$, $(-1)^n y^{(n+1)} = f^{(n)} \geqslant 0$ and the result follows by (2.2) with $k = 1$, $j = n+1$.

This example belongs naturally to the subject of convexity of order n. See also [1].

The next theorem presents differential inequalities which correspond to other sign patterns of the initial values.

THEOREM 2. *If the function* y *satisfies*

$$\begin{aligned} &\varepsilon_i y^{(i)}(0) \geqslant 0, && i = 0, \ldots, n-1, \\ &\varepsilon_n y^{(n)}(x) \geqslant 0 && \text{on} \quad [0, L], \end{aligned} \tag{2.12}$$

(with $\varepsilon_i = \pm 1$*) then*

$$\varepsilon_0 \pi_{n-1} \pi_{n-2} \cdots \pi_1 \pi_0 y \geqslant 0 \tag{2.13}$$

where

$$\begin{aligned} \pi_t &= \frac{1}{x} && \text{if} \quad \varepsilon_i = \varepsilon_{i+1} \\ &= -\frac{d}{dx} && \text{if} \quad \varepsilon_i = -\varepsilon_{i+1}. \end{aligned} \tag{2.14}$$

Moreover, if $0 \leqslant k_1 \leqslant h_2 \leqslant k_2 \leqslant \cdots \leqslant h_r \leqslant k_r < h_{r+1}$ *are integers and*

$$\begin{aligned} y^{(i)}(0) &\geqslant 0, && 0 \leqslant i \leqslant k_1 - 1, \\ (-1)^{h_2 - k_1} y^{(i)}(0) &\geqslant 0, && h_2 \leqslant i \leqslant k_2 - 1, \\ &\vdots \\ (-1)^{(h_r - k_{r-1}) + \cdots + (h_2 - k_1)} y^{(i)}(0) &\geqslant 0, && h_r \leqslant i \leqslant k_r - 1, \\ (-1)^{(h_{r+1} - k_r) + \cdots + (h_2 - k_1)} y^{(h_{r+1})}(x) &\geqslant 0 && \text{on} \quad [0, L], \end{aligned} \tag{2.15}$$

then

$$\left(-\frac{d}{dx}\right)^{h_{r+1}-k_r}\left[\frac{1}{x^{k_r-h_r}}\cdots\left[\frac{1}{x^{k_2-h_2}}\left(-\frac{d}{dx}\right)^{h_2-k_1}\left[\frac{y^{(t)}}{x^{k_1-t}}\right]\right]\cdots\right]\geqslant 0 \quad (2.16)$$

for every $t=0,\ldots,k_1$, $0<x\leqslant L$.

Proof. First we show that every set of conditions of the form (2.12) can be reduced to the form (2.15) so that (2.13) will be concluded from (2.16). Indeed, given a sequence ε_i, $i=0,1,\ldots$, we choose $h_l\leqslant k_l\leqslant h_{l+1}$ so that

$$\begin{aligned}\varepsilon_{h_l}&=\ \ \varepsilon_{h_l+1}&&=\cdots&&=\varepsilon_{k_l},\\ \varepsilon_{k_l}&=-\varepsilon_{k_l+1}&&=\cdots=(-1)^{h_{l+1}-k_l}&&\varepsilon_{h_{l+1}},\\ \varepsilon_{h_{l+1}}&=\ \ \varepsilon_{h_{l+1}+1}&&=\cdots.\end{aligned}$$

Assuming without loss of generality that $y(0)\geqslant 0$ ($\varepsilon_0=1$), the corresponding condition in (2.15) will be

$$(-1)^{(h_l-k_{l-1})+\cdots+(h_2-k_1)}\,y^{(i)}(0)\geqslant 0,\qquad h_l\leqslant i\leqslant k_l-1.$$

In this case we have in (2.13) $\pi_{h_{l+1}}\cdots\pi_{k_l}\pi_{k_l-1}\cdots\pi_{h_l}=(-d/dx)^{h_{l+1}-k_{l(1/x)}k_l-h_l}$ and those are exactly the corresponding terms in (2.16). Two cases deserve some attention. If $\varepsilon_0=-\varepsilon_1=\cdots=(-1)^p\,\varepsilon_p$, $\varepsilon_p=\varepsilon_{p+1}$, we choose $k_1=0$, $h_2=p$ and both (2.13) and (2.16) begin with $\pi_{p-1}\cdots\pi_0 y=(-d/dx)^p y$. If $\varepsilon_{p-1}\neq\varepsilon_p=\cdots=\varepsilon_n$, $p<n$, we choose $h_r=p$, $k_r=h_{r+1}=n$ and both (2.13) and (2.16) terminate with $\pi_{n-1}\cdots\pi_p=(1/x)^{n-p}$. Thus it is possible to put (2.12) in the form (2.15) and to deduce (2.13) from (2.16). Moreover, assumption (2.15) is more flexible than (2.12): in (2.15) it is not necessary to know the signature of $y^{(i)}(0)$ for $i\notin[0,k_1)\cup[h_1,k_2)\cup\cdots$ and we only require that the common sign of the $y^{(i)}(0)-s$ in the lth group ($h_l\leqslant i\leqslant k_l-1$) differs from the sign of those in the $(l-1)$th group by $(-1)^{h_l-k_{l-1}}$, where h_l-k_{l-1} is the length of the gap between the groups. Therefore, we shall deal only with (2.15).

To keep the notation simple we shall prove the proposition only for $r=2$; the proof of the general case is similar. So, let $p\geqslant q\geqslant r\geqslant s\geqslant 0$,

$$\begin{aligned}y^{(i)}(0)&\geqslant 0, && i=0,\ldots,s-1,\\ (-1)^{r-s}\,y^{(i)}(0)&\geqslant 0, && i=r,\ldots,q-1,\\ (-1)^{p-q+r-s}\,y^{(p)}(x)&\geqslant 0, && \text{on}\quad[0,L]. \qquad (2.17)\end{aligned}$$

We shall prove that

$$(-1)^{p-q+r-s}\frac{d^{p-q}}{dx^{p-q}}\left(x^{-(q-r)}\frac{d^{r-s}}{dx^{r-s}}\left(x^{-(s-t)}y^{(t)}\right)\right)\geqslant 0, \qquad t=0,\dots,s, \tag{2.18}$$

on $(0, L]$.

First, the function $u=(-1)^{r-s}y^{(r)}$ satisfies $u^{(i)}(0)\geqslant 0$, $i=0,\dots,q-r-1$ and $(-1)^{p-q}u^{(p-r)}(x)\geqslant 0$. Consequently, by Theorem 1, $(-1)^{p-q}(u/x^{q-r})^{(p-q)}\geqslant 0$, that is,

$$(-1)^{p-q+r-s}\left(\frac{y^{(r)}}{x^{q-r}}\right)^{(p-q)}\geqslant 0 \quad \text{on} \quad (0, L]. \tag{2.19}$$

When $s=0$ (and $t=0$), (2.18) is identical with (2.19) and the proof is completed. When $s>0$ we put

$$H(x)=(x^pD^{p-q}x^{-q})(x^rD^{r-s}x^{-s})\,y\equiv L_1L_2y$$

Then, by the lemma,

$$\begin{aligned}H^{(t)}(x)&=x^{-t}(x^tD^t)\,L_1L_2y=x^{-t}L_1L_2(x^tD^t)\,y\\&=x^{-t}(x^pD^{p-q}x^{-q})(x^rD^{r-s}x^{-s})\,x^ty^{(t)}\\&=(x^{p-t}D^{(p-t)-(q-t)}x^{-(q-t)})(x^{r-t}D^{(r-t)-(s-t)}x^{-(s-t)})\,y^{(t)}.\end{aligned} \tag{2.20}$$

and by (2.7),

$$\begin{aligned}H^{(t)}(0)=(-1)^{p-q+r-s}\,&(q-t)(q-t+1)\cdots\\&\times(p-t-1)(s-t)\cdots(r-t-1)\,y^{(t)}(0).\end{aligned}$$

Combining this equality with (2.17) we have

$$(-1)^{p-q+r-s}H^{(t)}(0)\geqslant 0, \qquad t=0,\dots,s-1.$$

For $t=s$ we have by (2.20) $H^{(s)}(x)=x^{p-s}(y^{(r)}/x^{q-r})^{(p-q)}$, so that according to (2.19),

$$(-1)^{p-q+r-s}H^{(s)}(x)\geqslant 0 \quad \text{on} \quad [0, L].$$

From the last inequalities it follows that $(-1)^{p-q+r-s}H^{(t)}(x)\geqslant 0$ on $[0, L]$ for $t=0,\dots,s$ and (2.18) follows if we substitute for $H^{(t)}$ the explicit expression which was obtained in (2.20). The proof of the general case (2.15) is based on the same argument and it is completed by mathematical induction.

In (2.15), as in (2.1), we may disregard pairs of initial values to obtain additional inequalities. For example, if $y^{(i)} \geqslant 0$, $i = 0,..., q-1$, then (2.18) still holds with $0 < s < r < q$ such that $r - s$ is even.

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The differential inequalities of the previous section can be reproduced when the derivatives are replaced by the *generalized derivatives*

$$\begin{aligned} &L_0 y = \rho_0 y \\ &L_i y = \rho_i (L_{i-1} y)', \qquad i = 1, 2, \ldots. \end{aligned}$$

Here the ρ_i's are positive and sufficiently smooth. We define

$$\phi_\nu(x) = \rho_0^{-1}(x) \int_a^x \rho_1^{-1}(x_1) \int_a^{x_1} \cdots \int_a^{x_{\nu-1}} \rho_\nu^{-1}(x_\nu)\, dx_\nu \cdots dx_1, \qquad \nu = 0, 1, \ldots.$$

Note that

$$\begin{aligned} &L_\mu \phi_\nu(x) \equiv 0 \qquad \text{for} \quad \mu > \nu, \\ &L_\mu \phi_\mu(x) \equiv 1, \end{aligned} \tag{3.1}$$

and $L_\mu \phi_\nu(x) > 0$ on (a, ∞) for $\mu \leqslant \nu$.

THEOREM 3. *If the function y satisfies*

$$\begin{aligned} &L_0 y(a), \ldots, L_{k-1} y(a) \geqslant 0, \\ &(-1)^{j-k} L_j y(x) \geqslant 0, \qquad a \leqslant x \leqslant b, \end{aligned} \tag{3.2}$$

where j is a fixed integer, $j \geqslant k$, then

$$\begin{vmatrix} L_t y, & L_t \phi_k, \ldots, L_t \phi_{j-1} \\ \vdots & \vdots \\ L_{t+j-k} y, & L_{t+j-k} \phi_k, \ldots, L_{t+j-k} \phi_{j-1} \end{vmatrix} \geqslant 0, \qquad t = 0, \ldots, k \tag{3.3}$$

on $[a, b]$. Equivalently,

$$W(\phi_0, \phi_1, \ldots, \phi_{t-1}, y, \phi_k, \phi_{k+1}, \ldots, \phi_{j-1}) \geqslant 0, \qquad t = 0, \ldots, k, \tag{3.4}$$

where W denotes the Wronskian determinant.

Proof. First we remark that (3.3) is a generalization of (2.2). Indeed, for ordinary derivatives ($\rho_i \equiv 1$) and $a = 0$ we have $\phi_\nu(x) = x^\nu / \nu!$ and (3.3) is

$$(-1)^{j-k}\, W(x^{k-t}/(k-t)!, \ldots, x^{j-1-t}/(j-1-t)!, y^{(t)}) \geqslant 0.$$

By the identity

$$W(ry_1,\dots,ry_l)=r^l W(y_1,\dots,y_l), \tag{3.5}$$

[14, 2], with $r=x^{k-t}$, $l=j-k+1$ the last Wronskian equals

$$(-1)^{j-k}[(k-t)!\cdots(j-1-t)!]^{-1}\,x^{(k-t)(j-k+1)}(y^{(t)}/x^{k-t})^{(j-k)}$$

and so (3.3) implies (2.2). Nevertheless we think that the particular case deserves a separate proof because of its own interesting features.

It is sufficient to prove (3.3) for $t=0$. For $1\leqslant t\leqslant k-1$ the result follows by the same argument if we look at the conditions

$$L_t y(a),\dots,L_{k-1}y(a)\geqslant 0, \qquad (-1)^{j-k}L_j y(x)\geqslant 0$$

and consider them as generalized derivatives of the function $L_t y$. For $t=k$ the determinant in (3.3) is identical with $(-1)^{j-k}L_j y$ by (3.1), so there is nothing to prove.

In addition to the Wronskian $W(y_1,\dots,y_i)=\det(y_\nu^{(\mu-1)})_{\nu,\mu=1,\dots,i}$, it is convenient to introduce the generalized Wronskian $\tilde{W}(y_1\cdots y_i)=\det(L_{\mu-1}y_\nu)_{\nu,\mu=1,\dots,i}$. Those two are related by

$$\tilde{W}(y_1,\dots,y_i)=\rho_0^i\rho_1^{i-1}\cdots\rho_{i-1}W(y_1,\dots,y_i). \tag{3.6}$$

Equation (3.6) is verified by expanding the terms $L_{\mu-1}y_\nu$ and some row operations. According to this notation, (3.3) (with $t=0$) is written as

$$D(x)\overset{\text{def}}{\equiv}\tilde{W}(y,\phi_k,\dots,\phi_{j-1})\geqslant 0. \tag{3.7}$$

First we prove (3.7) under the more restrictive assumption

$$L_0 y(a)=\cdots=L_{k-1}y(a)=0, \qquad (-1)^{j-k}L_j y\geqslant 0. \tag{3.8}$$

If $D(c)=0$ for some $c>a$, there exists a nontrivial linear combination $u=c_0y+c_1\phi_k+\cdots+c_{j-k}\phi_{j-1}$ such that

$$\begin{aligned} L_t u(c)&=0, \qquad t=0,\dots,j-k,\\ L_t u(a)&=0, \qquad t=0,\dots,k-1. \end{aligned} \tag{3.9}$$

Apply Rolle's theorem to the generalized derivatives $L_0u,\dots,L_ju$. By the $j+1$ boundary conditions (3.9) we obtain that if $u\not\equiv 0$ on (a,c) L_ju must change its sign there. This is impossible since $L_ju\equiv c_0L_jy$ has a fixed sign and so $u\equiv 0$ on $[a,c]$. Hence, either $D(x)\equiv 0$ on $[a,c]$ for some c, $a<c<b$ and $D(x)\neq 0$ on $(c,b]$ or $D(x)\neq 0$ on $(a,b]$. In either case $D(x)$ has a fixed sign on $[a,b]$.

The sign of $\tilde{W}(y, \phi_k, \ldots, \phi_{j-1})$ is the same for every y which satisfies (3.8). Otherwise, if $\tilde{W}(y_1, \phi_k, \ldots, \phi_{j-1})$, $\tilde{W}(y_2, \phi_k, \ldots, \phi_{j-1})$ have opposite signs on $(a, b]$, then for some $0 < \lambda < 1$, $\tilde{W}(\lambda y_1 + (1-\lambda) y_2, \phi_k, \ldots, \phi_{j-1})$ would change its sign, which was shown above to be impossible. Hence it is sufficient to show (3.7) for one function y which satisfies (3.8). We choose $y = (-1)^{j-k} \phi_j$ and show that

$$\tilde{W}((-1)^{j-k} \phi_j, \phi_k, \ldots, \phi_{j-1}) = \tilde{W}(\phi_k, \ldots, \phi_{j-1}, \phi_j) > 0 \tag{3.10}$$

on $(a, b]$. This inequality is well known; see [10, p. 279]. Nevertheless we complete its proof here. By the above argument it is sufficient to show that (3.10) holds in a right neighborhood of $x = a$. Now, for values of x close to a,

$$L_\mu \phi_\nu = (x-a)^{\nu-\mu} \left[(\nu-\mu)!\, \rho_{\mu+1}(a) \cdots \rho_\nu(a)\right]^{-1} [1 + o(1)],$$

so

$$\tilde{W}(\phi_k, \ldots, \phi_j) = A \det((x-a)^{k+\nu-\mu}/(k+\nu-\mu)!)_{\nu,\mu=0,\ldots,j-k}[1 + o(1)]$$

where $A > 0$ and we agree that $1/m! = 0$ for a negative integer m. The last determinant is exactly $W((x-a)^k/k!, \ldots, (x-a)^j/j!)$ and by (3.5) we obtain immediately that

$$\tilde{W}(\phi_k, \ldots, \phi_j) = B(x-a)^{k(j-k+1)} [1 + o(1)]$$

with $B > 0$. This establishes (3.10) in a right neighborhood of a and (3.7) is proved under the assumption (3.8).

If $L_0 y(a), \ldots, L_{k-1} y(a) \geqslant 0$ then $u = y - \sum_{i=0}^{k-1} L_i y(a) \phi_i$ satisfies (3.8), hence $\tilde{W}(u, \phi_k, \ldots, \phi_{j-1}) \geqslant 0$. To prove

$$\tilde{W}(y, \phi_k, \ldots, \phi_{j-1}) = \tilde{W}(u, \phi_k, \ldots, \phi_{j-1}) + \sum_{i=0}^{k-1} L_i y(a)\, \tilde{W}(\phi_i, \phi_k, \ldots, \phi_{j-1}) \geqslant 0$$

we need only to show that

$$\tilde{W}(\phi_i, \phi_k, \ldots, \phi_{j-1}) > 0 \qquad \text{on} \quad (a, b]. \tag{3.11}$$

This known inequality follows again as (3.10). Indeed, if $\tilde{W}(\phi_i, \phi_k, \ldots, \phi_{j-1})(c) = 0$, then there exists $u = c_0 \phi_i + c_1 \phi_k + \cdots + c_{j-k} \phi_{j-1}$ such that

$$L_t u(c) = 0, \qquad t = 0, \ldots, j-k,$$
$$L_t u(a) = 0, \qquad t = 0, \ldots, i-1, i+1, \ldots, k-1.$$

By these j boundary conditions it follows that $L_{j-1} u$ must change its sign in (a, c), in contradiction with $L_{j-1} u \equiv c_{j-k}$. The sign of our determinant is determined by $\tilde{W}(\phi_i, \phi_k, \ldots, \phi_{j-1}) = B(x-a)^{i+(k-1)(j-k)} [1 + o(1)]$, $B > 0$. This completes the proof of (3.3).

Recalling (3.6), we see that instead of (3.4) we may prove

$$\tilde{W}(\phi_0, ..., \phi_{t-1}, y, \phi_k, ..., \phi_{j-1}) \geqslant 0, \qquad t = 0, ..., k.$$

But by (3.1), this determinant is identical with the determinant in (3.3); thus (3.4) is proved too.

Remarks. (3) A careful review of the proof of Theorem 1 shows that we use implicitly the identity

$$(D^k x^j D^{j-k} x^{-k}) y \equiv x^{j-k} D^j y, \qquad \left(D = \frac{d}{dx}\right). \tag{3.12}$$

An analogous proof of Theorem 3 may be given if (3.12) is replaced by the Frobenius–Polya factorization of the disconjugate operator L_j,

$$L_j y = \frac{W_j}{W_{j-1}} D \frac{W_{j-1}^2}{W_j W_{j-2}} \cdots \frac{W_2^2}{W_3 W_1} D \frac{W_1^2}{W_2} D \frac{1}{W_1} y,$$

with $W_i = W(y_1, ..., y_i)$ and the solutions $y_1, ..., y_j$ of $L_j y = 0$ are choosen as $\phi_k, ..., \phi_{j-1}, \phi_0, \phi_1, ..., \phi_{k-1}$ in *this order*.

(4) Theorem 2 may be generalized too. If, for example

$$\begin{aligned} L_i y(a) &\geqslant 0, & i &= 0, ..., s-1, \\ (-1)^{r-s} L_i y(a) &\geqslant 0, & i &= r, ..., q-1, \\ (-1)^{p-q+r-s} L_p y(x) &\geqslant 0, & \text{on}\ & [a, b], \end{aligned} \tag{3.13}$$

$s \leqslant r \leqslant q \leqslant p$, then

$$W(\phi_0, ..., \phi_{t-1}, y, \phi_s, ..., \phi_{r-1}, \phi_q, ..., \phi_{p-1}) \geqslant 0, \qquad t = 0, ..., s, \tag{3.14}$$

on $[a, b]$. The proof is similar to that of Thorem 3.

An Example. For $j = k+1$, (3.3) is

$$\begin{vmatrix} L_t y & L_t \phi_k \\ L_{t+1} y & L_{t+1} \phi_k \end{vmatrix} \geqslant 0, \qquad t = 0, ..., k. \tag{3.15}$$

This is equivalent to each of the inequalities

$$\begin{aligned} &\left(\frac{L_t y}{L_t \phi_k}\right)' \leqslant 0, \\ &\frac{L_{t+1} y}{L_{t+1} \phi_k} \leqslant \frac{L_t y}{L_t \phi_k}, \qquad t = 0, ..., k. \end{aligned} \tag{3.16}$$

which are the analogous of (2.4), (2.5). This is the direct generalization of the inequalities of Kiguradze and in the next theorem we suggest a simple application.

In Theorem 6 of [4] we proved that if the equation

$$y^{(n)}+p(x)y=0$$

is disconjugate on (a, ∞), so is the mth order equation, $2 \leqslant m < n$,

$$u^{(m)}+[(m-1)!/(n-1)!](x-a)^{n-m}p(x)u=0.$$

In fact, in [4] $m!/n!$ appears instead of $(m-1)!/(n-1)!$, but the bigger constant is obtained without any change of the proof. For $m=2$, see [8]. This result may be immediately generalized by using (3.16). Recall the definition

$$L_0 y=\rho_0 y, \qquad L_i y=\rho_i(L_{i-1}y)',$$

that is,

$$L_n y=\rho_n(\cdots(\rho_1(\rho_0 y)')' \cdots)',$$

and define the mth order $(2 \leqslant m < n)$ operator M_m by

$$\begin{aligned} &M_0 u=\rho_{n-m}u,\\ &M_i u=\rho_{n-m+i}(M_{i-1}u)', \qquad i=1,\ldots,m, \end{aligned}$$

that is

$$M_m u=\rho_n(\cdots(\rho_{n-m+1}(\rho_{n-m}u)')\cdots)'.$$

THEOREM 4. *Suppose that*

$$\int^{\infty}\rho_i^{-1}\,dx=\infty, \qquad i=0,\ldots,n-1. \tag{3.17}$$

If

$$L_n y+p(x)y=0 \tag{3.18}$$

is disconjugate on (a, ∞), *so is the* mth *order equation* $(2 \leqslant m < n)$

$$M_m u+\left\{\frac{\phi_{n-1}(x)\,\rho_{n-m}(x)}{L_{n-m}\phi_{n-1}(x)}p(x)\right\}u=0. \tag{3.19}$$

Proof. The proof is identical with that of [2], except the modifications of the inequalities of Kiguradze. First, by (3.17), disconjugacy and disfocality are equivalent for equations (3.18), (3.19) [3]. It suffices to prove that (3.19) is $(q, m-q)$-disfocal on (a, ∞) for every q, $1 \leqslant q \leqslant m-1$, such that

$(-1)^{m-q} p(x) \leqslant 0$. Let q be such an integer. Since (3.18) is disconjugate on (a, ∞), it is $(n-m+q, m-q)$-disfocal and it has [3, Lemma 3] a solution y such that

$$\begin{aligned} L_i y &> 0, \qquad i = 0, \ldots, n-m+q, \\ (-1)^{j-(n-m+q)} L_j y &> 0, \qquad j = n-m+q+1, \ldots, n, \quad x > a. \end{aligned}$$

Put $u = \rho_{n-m}^{-1} L_{n-m} y = (L_{n-m-1} y)'$. Then

$$\begin{aligned} M_i u &> 0, \qquad i = 0, \ldots, q, \\ (-1)^{j-q} M_j u &> 0, \qquad j = q+1, \ldots, m, \quad x > a. \end{aligned} \tag{3.20}$$

Moreover,

$$M_m u/u = \rho_{n-m} L_n y / L_{n-m} y = -p\rho_{n-m} y / L_{n-m} y. \tag{3.21}$$

Applying (3.16) with $k = n-m+q$ and $t = 0, \ldots, n-m-1$, we have

$$\frac{L_0 y}{L_{n-m} y} \geqslant \frac{L_0 \phi_{n-m+q}}{L_{n-m} \phi_{n-m+q}}. \tag{3.22}$$

If we take in Theorem 3, $y = -\phi_{n-1}$, $k = n-m+q$, $j = k+1$, we obtain from (3.15) for $t = 0, \ldots, n-m-1$ (here $L_i y < 0$!)

$$\frac{L_0 \phi_{n-m+q}}{L_{n-m} \phi_{n-m+q}} \geqslant \frac{L_0 \phi_{n-1}}{L_{n-m} \phi_{n-1}} \tag{3.23}$$

(which is a particular case of (3.11)). Combining (3.21)–(3.23) and using $L_0 y = \rho_0 y$ we have

$$(-1)^{m-q} M_m u/u \geqslant (-1)^{m-q-1} p(x)\, \rho_{n-m} \phi_{n-1}(x) / L_{n-m} \phi_{n-1}(x) > 0.$$

Thus

$$(-1)^{m-q} \left[M_m u + p(x) \frac{\phi_{n-1}(x)\, \rho_{n-m}(x)}{L_{n-m} \phi_{n-1}(x)} u \right] \geqslant 0, \qquad x > a. \tag{3.24}$$

In Theorem 3 of [3] we proved that inequalities (3.20), (3.24) with ordinary derivatives imply the $(q, m-q)$-disfocality of the appropriate equation. The same is true when the derivatives are replaced by generalized derivatives. Thus (3.19) is $(q, m-q)$-disfocal and Theorem 4 is proved.

Note that if we put in the last proof $k = n-m+q$ and skip inequality (3.23), we obtain a better result about disfocality:

THEOREM 5. *If* $L_n y + p(x)y = 0$ *is* $(k, n-k)$*-disfocal on* (a, b) *and* $m < n$ *then*

$$M_m u + \left\{ \frac{\phi_k(x)\,\rho_{n-m}(x)}{L_{n-m}\phi_k(x)}\, p(x) \right\} u = 0$$

is $(m-n+k,\ n-k)$*-disfocal on* (a, b).

EXAMPLE. If the nth order equation

$$(x \cdots (x(xy)')' \cdots)' + p(x)\,y = 0,$$

where $p(x)$ is one signed, is disconjugate on $[1, \infty)$, so is the mth order equation, $2 \leqslant m < n$,

$$(x \cdots (x(xu)')' \cdots)' + \frac{(m-1)!}{(n-1)!}(\log x)^{n-m}\, p(x)\, u = 0$$

Here, $\phi_{n-1}(x) = (\log x)^{n-1}/(n-1)!\, x$.

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