

A NOTE ON FINITE GROUPS WITH FEW VALUES IN A COLUMN OF THE CHARACTER TABLE

MARIAGRAZIA BIANCHI, DAVID CHILLAG, AND EMANUELE PACIFICI

Dedicated to Professor Guido Zappa on the occasion of his 90th birthday

ABSTRACT. Many structural properties of a finite group G are encoded in the set of irreducible character degrees of G . This is the set of (distinct) values appearing in the “first” column of the character table of G . In the current article, we study groups whose character table has a “non-first” column satisfying one particular condition. Namely, we describe groups having a nonidentity element on which all nonlinear irreducible characters take the same value.

As many results in the literature show, the structure of a finite group G is deeply reflected and influenced by certain arithmetical properties of the set of irreducible character degrees of G . This is in fact the set of (distinct) values which the irreducible characters of G take on the identity element. Our aim in this note is to describe groups having a *nonidentity* element on which all nonlinear irreducible characters take the same value, although in the Theorem below we shall consider a situation which is (in principle, and from one point of view) more general. Therefore, our results can be compared with studies on finite groups whose nonlinear irreducible characters have all the same degree (see for instance [3, Chapter 12], and [1]).

In what follows, every group is tacitly assumed to be finite. We shall denote by $\text{Lin}(G)$ the set of linear characters of the group G ; also, if g is an element of G , the symbol g^G will denote the conjugacy class of g .

Theorem. *Let G be a group with centre Z , and let g be a nonidentity element of the derived subgroup G' . Assume that all the values of the nonlinear irreducible characters of G on g are rational and negative. Then $g^G = G' \setminus \{1\}$ (so that G' is an elementary abelian minimal normal subgroup of G , say of order p^n for a suitable prime p), and one of the following holds.*

- (a) G is a direct product of a 2-group with an abelian group, and $G' = \langle g \rangle$ has order 2 (whence it lies in Z).
- (b) $G' \cap Z = 1$, and G/Z is a Frobenius group with elementary abelian kernel $G'Z/Z$ of order $p^n > 2$ and cyclic complement of order $p^n - 1$.

Conversely, we have the following.

- (a') *Let G be a group as in (a). Then G has exactly $|Z|/2$ nonlinear irreducible characters, all of the same degree d . Here $d^2 = |G : Z|$, so d is a power of 2*

2000 *Mathematics Subject Classification.* 20C15.

different from 1. Also, every nonlinear irreducible character of G takes value $-d$ on the nonidentity element of G' .

(b') Let G be a group as in (b). Then G has exactly $|Z|$ nonlinear irreducible characters, all of the same degree $d = |G : G'Z| = p^n - 1$. Also, every nonlinear irreducible character of G takes value -1 on every nonidentity element of G' .

It may be worth stressing that, assuming the hypotheses of the Theorem, the values which the nonlinear irreducible characters of G take on g turn out to be the same value v , and we have $v = -2^s$ for some $s \geq 0$. Moreover, G is of type (a) precisely when $s \neq 0$, whereas it is of type (b) precisely when $s = 0$.

As an immediate consequence of the Theorem, we obtain the following.

Corollary. *Let G be a nonabelian group, and g a nonidentity element of G such that every nonlinear irreducible character of G takes the same value v on g . Then either $v = 0$, which occurs if and only if g does not lie in the derived subgroup G' , or $g^G = G' \setminus \{1\}$, and G is as in (a) or (b) of the Theorem.*

We present next a proof of the main result of this note.

Proof of the Theorem. By the second orthogonality relation ([3, 2.18]), we get

$$0 = \sum_{\chi \in \text{Irr}(G)} \chi(g)\chi(1) = |G : G'| + \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi(g)\chi(1).$$

Suppose now that there exists $y \in G' \setminus \{1\}$ which is not conjugate to g . Then, as above,

$$0 = \sum_{\chi \in \text{Irr}(G)} \chi(y)\chi(g) = |G : G'| + \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi(y)\chi(g).$$

Let us define

$$\Gamma := \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} -\chi(g)\chi.$$

As the $-\chi(g)$ are positive integers, Γ is a character of G .

The two equations above imply $\Gamma(y) = \Gamma(1)$. Thus y lies in $\ker \Gamma$, which is given by $\bigcap \{\ker \chi \mid \chi \in \text{Irr}(G) \setminus \text{Lin}(G)\}$. On the other hand y is in G' , so it lies in the kernel of every linear character of G as well. Since the intersection of all the kernels of irreducible characters of a finite group is trivial, this yields a contradiction.

The conclusion so far is that $G' \setminus \{1\}$ is in fact a conjugacy class, and now G' is a minimal normal subgroup of G in which all nonidentity elements have the same order. Therefore G' is an elementary abelian p -group for some prime p , say of order p^n . It is clear that either $G' \leq Z$, or $G' \cap Z = 1$.

If G' lies in Z , then G is nilpotent (of class 2). Also, as $|G'|$ is forced to be 2, all the Sylow subgroups of G are abelian except the Sylow 2-subgroup. In other words, G is as in (a) of the statement. It is well known that a group of this type satisfies the properties listed in (a') (see for example 7.5 and 7.6(a) in [2]).

In the case $G' \cap Z = 1$, we use an argument as in [1, Theorem 1]. Namely, there exist a prime $q \neq p$ and a Sylow q -subgroup of G (call it Q) such that Q does not centralize G' . In particular Q is not normal, so $T := N_G(Q)$ is a proper subgroup

of G . An application of the Frattini argument yields now $G = TG'$ and, as G' is abelian, we see that $T \cap G'$ is a normal subgroup of G . By the minimality of G' , being T a proper subgroup of G , we get $T \cap G' = 1$. It follows that T is abelian, whence $C_T(G') = Z$. Now, G' can be viewed as a faithful simple module for T/Z over the prime field $GF(p)$ (the action being defined by conjugation); the fact that there exists such a module for T/Z implies that T/Z is cyclic. The order of T/Z equals the length of an orbit in $G' \setminus \{1\}$, that is $p^n - 1$. It is now clear that G is a group of the type described in (b) of the statement.

It remains to show that every group as in (b) satisfies the properties listed in (b') of the statement. Let us consider the abelian normal subgroup $A := Z \times G'$ of G , whose irreducible characters are of the form $\lambda \times \mu$ for λ in $\text{Irr}(Z)$ and μ in $\text{Irr}(G')$. Since G acts transitively on the elements of $G' \setminus \{1\}$, it acts transitively on the set of nonprincipal irreducible characters of G' as well (see [2, 18.5(c)]). Thus, fixed $\bar{\mu} \neq 1_{G'}$ in $\text{Irr}(G')$, we get $I_G(\lambda \times \bar{\mu}) = A$ for all λ in $\text{Irr}(Z)$. We conclude that the map $\lambda \mapsto \chi_\lambda := (\lambda \times \bar{\mu}) \uparrow^G$, defined on $\text{Irr}(Z)$, has image in $\text{Irr}(G) \setminus \text{Lin}(G)$; taking into account that all the χ_λ have the same degree $d = p^n - 1$, it is easy to check that such map is indeed a bijection. This tells us that all the nonlinear irreducible characters of G have the same degree $p^n - 1$; moreover, we get $\chi_\lambda \downarrow_{G'} = \varrho_{G'} - 1_{G'}$ for all λ in $\text{Irr}(Z)$ (here $\varrho_{G'}$ denotes the regular character of G'), whence $\chi_\lambda(g) = -1$ for all g in $G' \setminus \{1\}$. ■

As for the Corollary, let us consider first the case in which the element g is not in G' . Then there exists a linear character σ of G such that $\sigma(g) \neq 1$. Now, if χ is in $\text{Irr}(G) \setminus \text{Lin}(G)$, our assumption implies $(\sigma\chi)(g) = \chi(g)$, which forces $\chi(g)$ to be 0.

We shall henceforth assume that g lies in G' . The second orthogonality relation yields

$$0 = \sum_{\chi \in \text{Irr}(G)} \chi(g)\chi(1) = |G : G'| + v \sum_{\chi \in \text{Irr}(G) \setminus \text{Lin}(G)} \chi(1),$$

so v is a negative rational number (note that this proves $\chi(g) \neq 0$). We are now in a position to apply the Theorem, and the claim follows.

ACKNOWLEDGMENTS

The authors wish to thank the Referee for his valuable advice.

REFERENCES

- [1] M. Bianchi, A. Gillio, M. Herzog, G. Qian, W. Shi, *Characterization of non-nilpotent groups with two irreducible character degrees*, J. Algebra 284 (2005), 326–332.
- [2] B. Huppert, *Character Theory of Finite Groups*, De Gruyter, Berlin, 1988.
- [3] I.M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

MARIAGRAZIA BIANCHI, DIPARTIMENTO DI MATEMATICA “FEDERIGO ENRIQUES”
UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO, ITALY
E-mail address: `bianchi@mat.unimi.it`

DAVID CHILLAG, DEPARTMENT OF MATHEMATICS
TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL
E-mail address: `chillag@techunix.technion.ac.il`

EMANUELE PACIFICI, DIPARTIMENTO DI MATEMATICA “FEDERIGO ENRIQUES”
UNIVERSITÀ DEGLI STUDI DI MILANO, VIA C. SALDINI 50, 20133 MILANO, ITALY
E-mail address: `pacifici@mat.unimi.it`