Semi-Simple Commutative Algebras with Positive Bases

David Chillag
Department of Mathematics,Technion, Israel Institute of Technology
Haifa 32000, Israel
e-mail address: chillag@techunix.technion.ac.il
Dedicated to the memory of Professor Michio Suzuki

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Abstract

Algebras that serve as models for concurrent studying of certain aspects of both the algebra of ordinary characters and the center of the group algebra have been considered by various authors. In this article we offer another such model. The main differences between our model and the known ones are: 1. Our model includes Brauer characters and principal indecomposable characters as special cases. 2. Our emphasis is on the eigenvalues of the regular representation of the algebra elements, an approach that gives results on values of characters (ordinary, central, Brauer, principal indecomposable) as well as on their products.

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1 Introduction

Values of ordinary characters, central characters, Brauer characters, and principal indecomposable characters are all eigenvalues of the regular representation of some underling algebras over the field $\mathbb{Q}$ of rational numbers. The algebras are semi-simple finite-dimensional commutative algebras with a basis for which the structure constants are nonnegative integers. Studying the various types of characters (and conjugacy classes, via the central characters whose underling algebra in the center of the group algebra), using the above fact and applying nonnegative m atrix theory, was done separately to each type of characters in [6] and [7]. The purpose of this article is to give a unified treatment by studying regular representations of some algebras containing bases with nonnegative structure constants. All the above mentioned examples will be special cases. What we offer here is a model for concurrent study of some aspects of the above mentioned algebras. Models for concurrent studies of the algebra of the ordinary characters and the center of the group algebra over the rationals, were considered by various authors (C-algebras [12], Table Algebras [2],[3], hypergroups [15]). The main differences between our model and the known ones are: 1. Our model includes also Brauer characters and principal indecomposable characters as special cases. 2. Our emphasis is on the eigenvalues of the regular representation of the algebra elements, an approach that gives results on values of characters (ordinary, central, Brauer, principal indecomposable) as well as on their products.

We first set up our basic notation. Let $\mathbb{F}$ be any subfield of the real number field $\mathbb{R}$ and let $\mathbf{A}$ be a semi-simple, finite-dimensional commutative $\mathbb{F}$-algebra (SFCA over $\mathbb{F}$ for short). If $u \in \mathbf{A}$, then $\mathbb{F}[u]$ is defined to be the following subalgebra of $\mathbf{A}$; $\mathbb{F}[u] = \left\{\sum_{i=0}^{m} \alpha_i u^i \mid \alpha_i \in \mathbb{F}, m = 0, 1, 2...\right\}$. Note that our SFCAs are all separable (in the sense of [8]), since our fields are subfields of $\mathbb{R}$. Also, all our fields are obviously infinite. Thus results on SFCAs from [8] can be used. The identity element of $\mathbf{A}$ (which is known to exist) will be denoted by $1_{\mathbf{A}}$. Let $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$ be a basis of $\mathbf{A}$. The *structure constants* of $\mathfrak{B}$ are the numbers α_{ijk} defined by the equations: $b_i b_j = \sum_{k=1}^{n} \alpha_{ijk} b_k$. If all the structure constants of $\mathfrak{B}$ are nonnegative real numbers we say that $\mathfrak{B}$ is a *nonnegative basis* of $\mathbf{A}$.

A nonzero element $a = \sum_{i=1}^{n} \alpha_i b_i$ of $\mathbf{A}$ is called a *nonnegative element*, if each α_i is a nonnegative real number.

A complex square matrix M will be called a *nonnegative* (respectively *positive*) *matrix*, if all its entries are nonnegative real numbers (respectively, positive). This will be denoted by $M \geq 0$ (respectively $M > 0$). According to the Perron-Frobenius Theorem ([4], Theorem 1.1, p.26) a nonnegative matrix M has an eigenvalue which is equal to the spectral radius of M. This eigenvalue is called *the leading eigenvalue of* M and is denoted by $\rho(M)$. It is known ([4], Theorem 1.1, p.26) that a nonnegative matrix has a nonnegative eigenvector corresponding to the leading eigenvalue.

We write $diag(\alpha_1, \alpha_2, ..., \alpha_n)$ for the diagonal matrix whose main diagonal entries are $\alpha_1, \alpha_2, ..., \alpha_n$.

Let $a \in \mathbf{A}$, we denote by T_a the linear transformation $T_a : \mathbf{A} \to \mathbf{A}$, defined by $T_a(c) = ac$ for all $c \in \mathbf{A}$. In fact, T is the regular representation of $\mathbf{A}$. The transpose of the matrix representation of T_a in the basis $\mathfrak{B}$ will be denoted by $M(a, \mathfrak{B}) = (m_{ij}(a, \mathfrak{B}))$; i.e.: the entries $m_{ij}(a, \mathfrak{B})$ are given by the equations: $ab_i = \sum_{j=1}^{n} m_{ij}(a, \mathfrak{B}) b_j$. So, $\mathfrak{B}$ is a nonnegative basis, if and only if all entries of all the matrices $M(b_i, \mathfrak{B})$ are nonnegative. Also, if $\mathfrak{B}$ is a nonnegative basis and $a \in \mathbf{A}$ is a nonnegative element, then the matrix $M(a, \mathfrak{B})$ is nonnegative. We refer to the eigenvalues of $M(a, \mathfrak{B})$ also as the eigenvalues of a, for $a \in \mathbf{A}$. The mapping $a \to M(a, \mathfrak{B})$ is an algebra isomorphism (See [8], Lemma 2.1). It is also known ([8], Lemma 2.1) that there exists a matrix U over some extension field of $\mathbb{F}$, such that $U^{-1} M(a, \mathfrak{B}) U$ is diagonal for every $a \in \mathbf{A}$. Such a U is called a *diagonalizing matrix* of ($\mathbf{A}$,$\mathfrak{B}$).

Let $a \in \mathbf{A}$ be nonnegative. We denote by $\rho(a)$ the leading eigenvalue of the nonnegative matrix $M(a, \mathfrak{B})$ (clearly, $\rho(a)$ does not depend on the basis $\mathfrak{B}$). As remarked above, each of the matrices $M(a, \mathfrak{B})$ and $(M(a, \mathfrak{B}))^t$ has a nonnegative eigenvector corresponding to $\rho(a)$. Our basic assumption is that these matrices will have positive eigenvectors corresponding to $\rho(a)$. This assumption replaces restrictions on the structure constants that are assumed in some of the models mentioned in the first paragraph. One of the advantages of this assumption is that, unlike the assumptions of other models, it holds for Brauer characters.

Definition 1 *A* right-positive basis *of an SFCA* $\mathbf{A}$ *over a subfield* $\mathbb{F}$ *of* $\mathbb{R}$*, is a nonnegative basis* $\mathfrak{B}$*, such that for every nonnegative element* $a \in \mathbf{A}$*, the matrix* $M(a, \mathfrak{B})$ *has a positive eigenvector corresponding to* $\rho(a)$*. If in addition for every nonnegative element* $a \in \mathbf{A}$*, the matrix* $(M(a, \mathfrak{B}))^t$ *has a positive eigenvector corresponding to* $\rho(a)$ *then* $\mathfrak{B}$ *is called* a positive basis.

Some important properties of our algebras are listed in the following theorem.

Theorem 2 *Let* $\mathbb{F}$ *be a subfield of* $\mathbb{R}$ *and* $\mathbf{A}$ *an FCSA over* $\mathbb{F}$ *with a nonnegative basis* $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$*.*

a. There exists an nonnegative element $u \in \mathbf{A}$ *such that* $\mathbf{A} = \mathbb{F}[u]$*. Moreover, every element of* $\mathbf{A}$ *is a polynomial in* u *of degree less than* n*, with coefficients in* $\mathbb{F}$*. Further, assume that* $\mathbb{D}$ *is a subring of* $\mathbb{R}$ *and that* $\mathbb{F}$ *is its field of fractions, then there exists such a* u *which is a linear combination of the elements of* $\mathfrak{B}$ *with nonnegative coefficients which are in* $\mathbb{D}$*.*

b. Suppose that $\mathfrak{B}$ *is a right-positive basis. Then there exists a positive vector* $\mathbf{w} \in \mathbb{R}^n$ *such that* $M(a, \mathfrak{B})\mathbf{w} = \rho(a)\mathbf{w}$ *for every nonnegative element* $a \in \mathbf{A}$*. If* $\mathfrak{B}$ *is a positive basis, then there exists a positive vector* $\mathbf{v} \in \mathbb{R}^n$ *such that* $(M(a, \mathfrak{B}))^t \mathbf{v} = \rho(a)\mathbf{v}$ *for every nonnegative element* $a \in \mathbf{A}$*. Moreover,* $\mathbf{w}$ *and* $\mathbf{v}$ *are unique up to a positive scalar multiple.*

So $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A},\ a \text{ nonnegtaive}\}$ (and $\left\{(M(a, \mathfrak{B}))^t \mid a \in \mathbf{A},\ a \text{ nonnegtaive}\right\}$, if $\mathfrak{B}$ is positive) has a unique (up to a positive scalar multiple) common eigenvector corresponding to the leading eigenvalues. This enable us to arrange the eigenvalues of all the matrices in the set $\{M(b, \mathfrak{B}) \mid b \in \mathfrak{B}\}$ in an $n \times n$ matrix that has some similarities to a character table of a finite group. This will lead to the definition of the table of the algebra (which is modeled after the character table).

Definition 3 *Let* $\mathbf{A}$ *be an FCSA over* $\mathbb{F}$ *with a basis* $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$*. For every* $a \in \mathbf{A}$ *let* $a(1), a(2), ..., a(n)$ *be an ordering of the eigenvalues of* $M(a, \mathfrak{B})$*. We say that this ordering is a* uniform ordering *of the eigenvalues of all the matrices in the set* $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A}\}$ *if the following three conditions are satisfied:*

a. The ordering is prescribed by some diagonalizing matrix U *of* $(\mathbf{A}, \mathfrak{B})$*, that is*

$$U^{-1}M(a, \mathfrak{B})U = diag(a(1), a(2), ..., a(n)) \qquad \text{for all } a \in \mathbf{A}.$$

b. The positive vector $(b_1(1), b_2(1), \ldots, b_n(1))^t$ *is a* ***common*** *eigenvector for* $M(a, \mathfrak{B})$ *corresponding to* $\rho(a)$*, for* ***all*** *nonnegative* $a \in \mathbf{A}$*.*

c. The equality $\rho(a) = a(1)$ *holds for all nonnegative* $a \in \mathbf{A}$*.*

The next result states that a uniform ordering exists for every FCSA with a positive basis, and introduces the "table" $X(\mathbf{A}, \mathfrak{B})$ of the algebra.

Theorem 4 *Let* $\mathbf{A}$ *be an FCSA over* $\mathbb{F}$ *and* $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$ *be a right-positive basis of* $\mathbf{A}$*. Then:*

a. The eigenvalues of all the matrices in the set $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A}\}$ *have a uniform ordering.*

b. For every $a \in \mathbf{A}$*, let* $a(1), a(2), ..., a(n)$ *be the eigenvalues of* $M(a, \mathfrak{B})$*, ordered by this uniform ordering. Set* $X(\mathbf{A}, \mathfrak{B}) = (b_i(j))$*. Then for all* $a \in \mathbf{A}$

$$(X(\mathbf{A}, \mathfrak{B}))^{-1} M(a, \mathfrak{B}) X(\mathbf{A}, \mathfrak{B}) = diag(a(1), a(2), ..., a(n)) \ .$$

c. $X(\mathbf{A}, \mathfrak{B})$ *is unique in the following sense: Let* U *be any matrix that diagonalizes all the matrices in* $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A}\}$*, then every column of* U *is a scalar multiple of some column of* $X(\mathbf{A}, \mathfrak{B})$*.*

Proofs of Theorems 1.2 and 1.4 can be found in section 2.

Definition 5 ***(with comments)*** *Let* $\mathbf{A}$ *be an FCSA over the field* $\mathbb{F}$ *with a right-positive basis* $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$*. Fix a uniform ordering of the eigenvalues of the matrices in* $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A}\}$ *according to which the eigenvalues of each* $a \in \mathbf{A}$ *are denoted by* $a(1), a(2), ..., a(n)$*. We will say that* $(\mathbf{A}, \mathfrak{B}, a(i))$ *is a* right-positive algebra triple*. If* $\mathfrak{B}$ *happens to be a positive basis we use the term* positive algebra triple.

a. The matrix $X(\mathbf{A}, \mathfrak{B}) = (b_i(j))$*, modeled after the character table, will be called the* $\mathfrak{B}$-table *of* $\mathbf{A}$*.*

b. Denote by $\mathbf{r}(\mathbf{A}, \mathfrak{B})$ *be the unique positive vector with smallest component equal to* 1*, which is an eigenvector of* $M(a, \mathfrak{B})$ *corresponding to* $\rho(a)$*, for every nonnegative* $a \in \mathbf{A}$*. In fact, Theorems 1.2 and 1.4 imply that* $\mathbf{r}(\mathbf{A}, \mathfrak{B})$ *is the vector* $(b_1(1), b_2(1), \ldots, b_n(1))^t$ *divided by its smallest component.*

c. If $\mathfrak{B}$ *is a positive basis, let* $\mathbf{l}(\mathbf{A}, \mathfrak{B})$ *be the unique positive vector with smallest component equal to* 1*, which is an eigenvector of* $(M(a, \mathfrak{B}))^t$ *corresponding to* $\rho(a)$*, for every nonnegative* $a \in \mathbf{A}$*. It will be shown that in many cases (including all our examples below)* $\mathbf{l}(\mathbf{A}, \mathfrak{B})$ *is the vector* $(b'_1(1), b'_2(1), \ldots, b'_n(1))^t$ *divided by its component of smallest absolute value, where* $\mathfrak{B}' = \{b'_1, b'_2, ..., b'_n\}$ *is some basis associated with* $\mathfrak{B}$ *(it will be defined in Section 2).*

d. Let $(\mathbf{A}, \mathfrak{B}, a(i))$ *be a right-positive algebra triple and* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}^n$ *a positive n-tuple. For every* $a, b \in \mathbf{A}$ *define* $[a, b]_\alpha = \sum_{i=1}^n \alpha_i a(i) \overline{b(i)}$*, where the bar means complex conjugation. It is easy to see (Lemma 2.1 in Section 2) that* $[\ ,\ ]_\alpha$ *is an inner product.*

So $X(\mathbf{A},\mathfrak{B})$, $\mathbf{r}(\mathbf{A},\mathfrak{B})$, $\mathbf{l}(\mathbf{A},\mathfrak{B})$ are defined for every positive algebra triple, their uniqueness is guaranteed by Theorems 1.2 and 1.4. More on $X(\mathbf{A},\mathfrak{B})$, $\mathbf{r}(\mathbf{A},\mathfrak{B})$, $\mathbf{l}(\mathbf{A},\mathfrak{B})$ and the inner product can be found in Section 2.

We now present the examples after which the positive algebra triples are modeled. The examples will be given names, as we plan to refer to them later. Most of the facts in the examples are easy to see, the ones who are not so obvious will be verified in Section 5. We use standard group theory notation, taken mainly from [11].

Example 6 ***OC*** *(ordinary characters)***.** *Let G be a finite group. The set of ordinary irreducible characters of G is denoted by $Irr(G) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_n\}$ and the collection of its conjugacy classes by $Con(G) = \{C_1 = \{1\}, C_2, \ldots, C_n\}$. The value of a class function φ of G on each element of the class C will be denoted by $\varphi(C)$. Let $\mathbf{A} = \mathbb{Q}(\mathbb{I}rr(\mathbb{G}))$ be the algebra generated by $Irr(G)$ over $\mathbb{Q}$ and let $\alpha = \left(\frac{|C_1|}{|G|}, \frac{|C_2|}{|G|}, \ldots, \frac{|C_n|}{|G|}\right)$. The characters of G are the nonzero linear combinations of $Irr(G)$ with nonnegative integer coefficients. Next, $\mathfrak{B} = Irr(G)$ is a positive basis of the FCSA $\mathbf{A} = \mathbb{Q}(\mathbb{I}rr(\mathbb{G}))$ with $\mathbf{r}(\mathbf{A},\mathfrak{B}) = \mathbf{l}(\mathbf{A},\mathfrak{B}) = (\chi_1(1), \chi_2(1), \ldots, \chi_n(1))^t$. If $\varphi \in \mathbf{A}$ then the eigenvalues of $M(\varphi,\mathfrak{B})$ are the numbers $\varphi(C_1), \varphi(C_2), \ldots, \varphi(C_n)$. The character table matrix $X(\mathbf{A},\mathfrak{B}) = (\chi_i(C_j))_{i,j=1}^n$ is a diagonalizing matrix for $\mathbf{A}$ and for every $\varphi \in \mathbf{A}$ we have*

$$(X(\mathbf{A},\mathfrak{B}))^{-1} M(\varphi,\mathfrak{B}) X(\mathbf{A},\mathfrak{B}) = diag\left(\varphi(C_1), \varphi(C_2), \ldots, \varphi(C_n)\right).$$

So, the above ordering of the eigenvalues of the matrices in $\{M(\varphi,\mathfrak{B}) \mid \varphi \in \mathbf{A}\}$ is uniform. Thus $(\mathbb{Q}(\mathbb{I}rr(\mathbb{G})), Irr(G), \varphi(C_i))$ is a positive algebra triple. Next, $[\ ,\ ]_\alpha$ is the usual inner product in $\mathbf{A}$. Finally, in this example $\mathfrak{B} = \mathfrak{B}'$ and $(M(\varphi,\mathfrak{B}))^t = M(\overline{\varphi},\mathfrak{B})$ for every $\varphi \in \mathbf{A}$.

Example 7 ***CC*** *(conjugacy classes). Staying with the group theory notation of Example OC, let $\mathbf{A} = \mathbf{Z}(\mathbb{Q}G)$ be the center of the group algebra over the rationals. For every $C \in Con(G)$ let $\overline{C}$ be the class sum of C, that is $\overline{C} = \sum_{x \in C} x$. Then $\mathfrak{B} = \left\{\overline{C_1}, \overline{C_2}, \ldots, \overline{C_n}\right\}$ is a positive basis of the FCSA $\mathbf{A}$. It can be seen that $\mathbf{r}(\mathbf{A},\mathfrak{B}) = (|C_1|, |C_2|, \ldots, |C_n|)^t$ and $\mathbf{l}(\mathbf{A},\mathfrak{B}) = (1, 1, \ldots, 1)^t$. For every $C \in Con(G)$ and $i = 1, 2, \ldots, n$, let $\omega_i(C) = \omega_i(\overline{C}) = \frac{|C|\chi_i(C)}{\chi_i(1)}$. We write $M(C,\mathfrak{B}) = M(\overline{C},\mathfrak{B})$ for $C \in Con(G)$. If $a = \sum_{k=1}^n \beta_k \overline{C_k}$ is a general element of $\mathbf{A}$ ($\beta_i \in \mathbb{Q}$), we set $\omega_i(a) = \sum_{k=1}^n \beta_k \omega_i(C_k)$. The numbers $\omega_1(a), \omega_2(a), \ldots, \omega_n(a)$ are the eigenvalues of the matrix $M(a,\mathfrak{B})$ for every $a \in \mathbf{A}$. The matrix $X(\mathbf{A},\mathfrak{B}) = (\omega_j(C_i))_{i,j=1}^n$ is a diagonalizing matrix for $\mathbf{A}$ which provides the uniform ordering of the eigenvalues of the matrices in $\{M(a,\mathfrak{B}) \mid a \in \mathbf{A}\}$. In fact, for every $a \in \mathbf{A}$ we have:*

$$(X(\mathbf{A},\mathfrak{B}))^{-1} M(a,\mathfrak{B}) X(\mathbf{A},\mathfrak{B}) = diag\left(\omega_1(a), \omega_2(a), \ldots, \omega_n(a)\right).$$

Set $\alpha = \left\{\frac{\chi_1^2(1)}{|G|}, \frac{\chi_2^2(1)}{|G|}, \ldots, \frac{\chi_n^2(1)}{|G|}\right\}$ and let $\mathfrak{B}' = \left\{\frac{\overline{C_1}}{|C_1|}, \frac{\overline{C_2}}{|C_2|}, \ldots, \frac{\overline{C_n}}{|C_n|}\right\}$. Then $\left[\frac{\overline{C_i}}{|C_i|}, \overline{C_j}\right]_\alpha = \delta_{ij}$ and $\sum_{k=1}^n \omega_i\left(\frac{\overline{C_k}}{|C_k|}\right)\overline{\omega_j(C_k)} = \frac{\delta_{ij}}{\alpha_i}$. Finally $(M(C,\mathfrak{B}))^t =$

$M(C^{-1}, \mathfrak{B}')$ for $C \in Con(G)$. Here C^{-1} is the conjugacy class consisting of the inverses of the elements of C.

Example 8 ***BC*** *(Brauer characters). With the above notation let p be a prime, $Con_{p'}(G) = \{K_1, K_2, \ldots, K_m\}$ the set of conjugacy classes of p-regular elements of G, and $G_{p'} = \bigcup_{i=1}^{m} K_i$ the set of all p-regular elements of G. Next let $\mathfrak{B} = Ibr(G) = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ be the set of irreducible Brauer characters in characteristic p, and $\mathfrak{B}' = PI(G) = \{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ the set of principal indecomposable characters of G. We regard the principal indecomposable characters as class functions on $G_{p'}$ and ignore their value (which is zero) on p-singular elements. The algebra generated by $Ibr(G)$ over $\mathbb{Q}$ will denoted by $\mathbb{Q}(Ibr(G))$. Then $\mathfrak{B} = Ibr(G)$ is a positive basis for the m-dimensional FCSA $\mathbf{A} = \mathbb{Q}(Ibr(G))$ in which $\mathbf{r}(\mathbf{A}, \mathfrak{B}) = (\varphi_1(1), \varphi_2(1), \ldots, \varphi_m(1))^t$ and $\mathbf{l}(\mathbf{A}, \mathfrak{B}) = \frac{1}{|G|_p} (\Phi_1(1), \Phi_2(1), \ldots, \Phi_m(1))^t$. Set $\alpha = \left(\frac{|K_1|}{|G|}, \frac{|K_2|}{|G|}, \ldots, \frac{|K_m|}{|G|}\right)$, then $[\Phi_i, \varphi_j]_\alpha = \delta_{ij}$ and $\frac{\delta_{ij}}{\alpha_i} = \sum_{k=1}^{m} \Phi_k(K_i)\overline{\varphi_k(K_j)}$. The Brauer characters are the nonzero linear combinations of $Ibr(G)$ with nonnegative integer coefficients. If $\varphi \in \mathbf{A}$, then its eigenvalues are the numbers $\varphi_1(a), \varphi_2(a), \ldots, \varphi_m(a)$ and the diagonalizing matrix (which is the Brauer character table) $X(\mathbf{A}, \mathfrak{B}) = (\varphi_i(K_j))_{i,j=1}^{m}$ provides a uniform ordering. Finally $(M(\varphi, \mathfrak{B}))^t = M(\overline{\varphi}, \mathfrak{B}')$ for every $\varphi \in \mathbf{A}$.*

Example 9 ***PI*** *(principal indecomposables). Here the FCSA is $\mathbf{A} = \mathbb{Q}(PI(G))$, the positive basis is $PI(G)$, $X(\mathbf{A}, \mathfrak{B}) = (\Phi_i(K_j))_{i,j=1}^{m}$, $\mathbf{r}(\mathbf{A}, \mathfrak{B}) = \frac{1}{|G|_p} (\Phi_1(1), \Phi_2(1), \ldots, \Phi_m(1))^t$, and $\mathbf{l}(\mathbf{A}, \mathfrak{B}) = (\varphi_1(1), \varphi_2(1), \ldots, \varphi_m(1))^t$. The rest of the properties can be read from the previous example.*

REMARK. Note that $[\mathbf{r}(\mathbf{A}, \mathfrak{B}), \mathbf{l}(\mathbf{A}, \mathfrak{B})] = |G|$ (respectively $|G|_{p'}$) in examples OC and CC (respectively, BC and IP). Also in all the examples we get that $(M(a, \mathfrak{B}))^t = M(\overline{\varphi}, \mathfrak{B}')$ for every $a \in \mathbf{A}$, a fact that is true in a more general setting (See Section 2). Other common facts of all the four examples are the formulas

$$[b_i', b_j]_\alpha = \delta_{ij} \quad ; \quad \frac{\delta_{ij}}{\alpha_i} = \sum_{k=1}^{n} b_k'(i)\overline{b_k(j)} \quad \text{for all } i, j = 1, 2, \ldots, n.$$

These "orthogonality relations" are also true in a general setup (see Section 2).

The structure of the paper is as follows. In Section 2 we will prove the above two theorems and will give some further basic facts on FSCAs with nonnegative and positive bases. In section 3, we will introduce the notion of the kernel of elements of $\mathbf{A}$, define intersections of kernels (that will be called normal subsets) and multiplication-closed subsets of $\mathfrak{B}$, and discuss quotient structures. In Section 4, notions connected to kernels (like faithfulness) will be related to notions of nonnegative matrix theory (like irreducibility and primitivity). Also in Section 4 we use nonnegative matrix theory to obtain results on powers and eigenvalues of algebra elements, as well as a result connecting multiplication-closed subsets of $\mathfrak{B}$ to normal subsets. In Section 5 we will apply the results obtained to the four special cases presented above and see how to obtain uniformly, results on powers and values of their elements, as a consequence of our general approach. Here are two examples of such applications.

Application. Let G be a finite group.

1. Let $\chi \in Irr(G)$ be faithful. Then $|Z(G)| = g.c.d\{m \mid 1 \in Irr(\chi^m)\}$.
2. Let C be a conjugacy class that generates G. Then $|G:G'| = g.c.d\{m \mid 1 \in C^m\}$.

3. Let φ be an irreducible faithful Brauer character in characteristic p, of G. Then the number of p-regular classes in $Z(\varphi)$ is equal to $g.c.d\{m \mid 1 \in Irr(\varphi^m)\}$ (here $Z(\varphi) = \{x \in G_{p'} \mid |\varphi(x)| = \varphi(1)\}$.

Application. 1. Let χ be an ordinary character of the finite group G having exactly m distinct values. If χ is faithful then $Irr(1+\chi+\chi^2+\ldots+\chi^{m-1}) = Irr(G)$ and if $Z(\chi)=1$, then $Irr(\chi^{m^2-m+2}) = Irr(G)$.

2. Let C be a conjugacy class of the finite group G and let m be the number of elements in the set $\left\{\frac{\chi(C)}{\chi(1)} \mid \chi \in Irr(G)\right\}$. If C generates G, then $G = \{1\} \cup C \cup C^2 \cup \cdots \cup C^{m-1}$, and if C generates G and G is perfect then $G = C^{m^2-2m+2}$. The statement on BC is similar and can be found in Section 5.

2 Basic Properties, Associated Bases and Self-conjugate Bases

In this section we prove Theorems 1.2 and 1.4, and discuss further properties of FCSAs with a positive basis.

Lemma 10 *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and $\mathbf{A}$ an FCSA over $\mathbb{F}$ with a nonnegative basis $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$. Let U be a diagonalizing matrix for $(\mathbf{A},\mathfrak{B})$. For each $a \in \mathbf{A}$ let $a(i),\ i = 1,2,\ldots,n$ be the eigenvalues of a ordered as dictated by U, that is*

$$U^{-1}M(a,\mathfrak{B})U = diag(a(1), a(2), ..., a(n)).$$

Then:

a. For every $c, d \in \mathbf{A}$, $\alpha \in \mathbb{F}$ and $i = 1,2,\ldots,n$ the following holds:

$$(c+d)(i) = c(i) + d(i)\ ;\ (\alpha c)(i) = \alpha c(i)\ ;\ (cd)(i) = c(i)d(i).$$

b. Let $c, d \in \mathbf{A}$. If $c(i) = d(i)$ for all $i = 1,2,\ldots,n$, then $c = d$.

c. Let α be a positive n-tuple, then $[\ ,\ ]_\alpha$ is an inner product.

Proof. a. The map $a \to M(a,\mathfrak{B})$ is linear. Thus

$U^{-1}M(\alpha c + d,\mathfrak{B})U = U^{-1}\left(\alpha M(c,\mathfrak{B}) + M(d,\mathfrak{B})\right)U =$
$\alpha U^{-1}M(c,\mathfrak{B})U + U^{-1}M(d,\mathfrak{B})U = \alpha \cdot diag(c(1), c(2), ..., c(n)) +$
$diag(d(1), d(2), ..., d(n)) = diag(\alpha c(1) + d(1), \alpha c(2) + d(2), \ldots, \alpha c(n) + d(n)).$

Since The map $a \to M(a, \mathfrak{B})$ is multiplicative we obtain

$$U^{-1}M(cd,\mathfrak{B})U = U^{-1}M(c,\mathfrak{B})U \cdot U^{-1}M(d,\mathfrak{B})U = diag(c(1), c(2), ..., c(n))\cdot$$
$$diag(d(1), d(2), ..., d(n)) = diag(c(1)d(1), c(2)d(2), \ldots, c(n)d(n)).$$

b. Here

$$M(c,\mathfrak{B}) = U diag(c(1), c(2), ..., c(n))U^{-1} = U diag(d(1), d(2), ..., d(n))U^{-1} = M(d,\mathfrak{B}).$$

But the mapping $a \to M(a, \mathfrak{B})$ in an isomorphism ([8], Lemma 2.1), so $c = d$.

c. The linearity of $[\ ,\ ]_\alpha$ follows from part a. It is clear that $\overline{[a,b]_\alpha} = [b,a]_\alpha$ and that $[a,a]_\alpha \geq 0$. Finally if $[a,a]_\alpha = 0$, then $a(i) = 0 = 0(i)$ for all $i = 1, 2, \ldots, n$. So $a = 0$ by part b. ■

Proof of Theorem 1.2 Let U be a diagonalizing matrix for $(\mathbf{A},\mathfrak{B})$. For each $a \in \mathbf{A}$ let $a(i),\ i = 1, 2, \ldots, n$ be the eigenvalues of a ordered as dictated by U, that is $U^{-1}M(a,\mathfrak{B})U = diag(a(1), a(2), ..., a(n))$.

By Theorem 1.1 of [8], there exists $v \in \mathbf{A}$ such that $\mathbf{A} = \mathbb{F}[v]$, and the n eigenvalues of v are distinct. Since $v = \sum_{i=1}^{n} \alpha_i b_i$ with $\alpha_i \in \mathbb{F} \subseteq \mathbb{R}$, we can write $v = v_1 - v_2$, where v_1, v_2 are nonnegative elements of $\mathbf{A}$. Chose a natural number m such that $m \neq \frac{v(i)-v(j)}{v_2(j)-v_2(i)}$ for every pair (i, j) for which $v_2(i) \neq v_2(j)$. Then define $v_3 = v + mv_2$. By Lemma 2.1 we get $v_3(k) = v(k) + mv_2(k)$ for $k = 1, 2, \ldots, n$. Clearly, v_3 is nonnegative.

For a general $\mathbb{F}$ we set $u = v_3$. If $\mathbb{F}$ is the field of fractions of $\mathbb{D}$ we can write $v_3 = \sum_{i=1}^{n} \frac{\gamma_i}{\beta_i} b_i$, with γ_i, β_i nonnegative elements of $\mathbb{D}$. Set $\beta = \prod_{i=1}^{n} \beta_i$. Then β is a positive element of $\mathbb{D}$ so that $u = \beta v_3$ is a linear combination of the elements of $\mathfrak{B}$ with nonnegative coefficients in $\mathbb{D}$. We now show that $\mathbf{A} = \mathbb{F}[u]$. By Theorem 1.1 of [8] it is suffices to show that $u(i) \neq u(j)$ for all $i \neq j$. Assume that $u(i) = u(j)$ for some i,j with $i \neq j$. Then $v_3(i) = v_3(j)$. It follows that $v(i) + mv_2(i) = v(j) + mv_2(j)$. Since $i \neq j$ we have that $v(i) \neq v(j)$ and therefore $v_2(i) \neq v_2(j)$ and consequently $m = \frac{v(i)-v(j)}{v_2(j)-v_2(i)}$, contrary to the choice of m. Hence, $u(i) \neq u(j)$ for $i \neq j$ and thus every element of $\mathbf{A}$ is of the form $\sum_{i=0}^{s} \alpha_i u^i$, with $\alpha_i \in \mathbb{F}$ and $\alpha_s \neq 0$. As $1, u, u^2, \ldots, u^n$ are linearly independent over $\mathbb{F}$, we can chose s with $s < n$. This proves part a.

Suppose that $\mathfrak{B}$ is a positive basis, there exist two positive vectors $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{w}$ is an eigenvector of $M(u, \mathfrak{B})$ with respect to $\rho(u)$ and $\mathbf{v}$ is an a eigenvector of $(M(u,\mathfrak{B}))^t$ with respect to $\rho(u)$. Let $a \in \mathbf{A}$, then $a = \sum_{i=0}^{n-1} \alpha_i u^i$ for some $\alpha_i \in \mathbb{F}$. As the mapping $a \to M(a, \mathfrak{B})$ is both linear and multiplicative we get that $M(a,\mathfrak{B}) = \sum_{i=0}^{n-1} \alpha_i M(u,\mathfrak{B})^i$. Hence $\mathbf{w}$ is an eigenvector of $M(a,\mathfrak{B})$ and $\mathbf{v}$ is an a eigenvector of $(M(a,\mathfrak{B}))^t$. Suppose that a is nonnegative, then $M(a,\mathfrak{B})$ and $(M(a,\mathfrak{B}))^t$ are nonnegative matrices. Since $\mathbf{w}$ and $\mathbf{v}$ are positive vectors, it follows from ([4] 1.12 p.28) that $\mathbf{w}$ and $\mathbf{v}$ corresponds as eigenvectors to the eigenvalue $\rho(M(a,\mathfrak{B})) = \rho\left((M(a,\mathfrak{B}))^t\right) = \rho(a)$. If $\mathfrak{B}$ is assumed to be only right-positive, the existence of $\mathbf{w}$ is proved exactly as in the $\mathfrak{B}$ positive case.

Next we show the uniqueness of $\mathbf{w}$ and $\mathbf{v}$. Recall that $\rho(u)$ is a simple eigenvalue (that is, of algebraic multiplicity equal to 1). So the eigenspace of

$\rho(u)$ (with respect to both to $M(u,\mathfrak{B})$ and $(M(u,\mathfrak{B}))^t$) is of dimension equal to 1. Let $\mathbf{w}_1$ (respectively $\mathbf{v}_1$) be a positive common eigenvector corresponding to $\rho(a)$ of $M(a,\mathfrak{B})$ (respectively $(M(a,\mathfrak{B}))^t$) for all nonnegative $a \in \mathbf{A}$. Then $\mathbf{w}_1$ (respectively $\mathbf{v}_1$) is an eigenvector of $M(u,\mathfrak{B})$ (respectively $(M(u,\mathfrak{B}))^t$) corresponding to $\rho(u)$. Thus $\mathbf{w}_1$ (respectively $\mathbf{v}_1$) is a scalar multiple of $\mathbf{w}$ (respectively $\mathbf{v}$). This proves part b. ■

REMARK. Suppose in Theorem 1.2a., that $1_{\mathbf{A}} \in \mathfrak{B}$. Let s be a positive integer with $s > |u(i)|$ for all $i = 1, 2, \ldots, n$. Set $w = u + s \cdot 1_{\mathbf{A}}$. Then w is a nonnegative element and $w(i) \neq 0$ for all i (by Lemma 2.1). As $u(i) \neq u(j)$ for $i \neq j$, we get that $w(i) \neq w(j)$ for $i \neq j$. By Theorem 1.1 of [8], we have that $\mathbf{A} = \mathbb{F}[w]$. This fact will be used later.

Proof of Theorem 1.4. This is a consequence of [8]. Let V be a diagonalizing matrix for $(\mathbf{A}, \mathfrak{B})$. For each $a \in \mathbf{A}$ set

$$V^{-1}M(a,\mathfrak{B})V = diag(a(1), a(2), ..., a(n)).$$

Also let $X = (b_i(j))$. Now Theorem 2.3 of [8] implies that

$$X^{-1}M(a,\mathfrak{B})X = diag(a(1), a(2), ..., a(n)) \text{ for all } a \in \mathbf{A}.$$

Then the positive vector $\mathbf{v} = (b_1(1), b_2(1), \ldots, b_n(1))^t$ is a common eigenvector for all $M(a,\mathfrak{B})$. If $a \in \mathbf{A}$ is an arbitrary nonnegative element of $\mathbf{A}$, then $M(a,\mathfrak{B}) \geq 0$, and so $\mathbf{v}$ must correspond to the eigenvalue $\rho(a)$ (see [4] 1.12 p.28). As $\mathbf{v}$ corresponds to $a(1)$, we get $\rho(a) = a(1)$. This proves parts a. and b. Part c. is corollary 2.7 of [8] . ■

We mention the following (mostly obvious) properties of the $\mathfrak{B}$-table.

Proposition 11 *Let* $(\mathbf{A},\mathfrak{B}, a(i))$ *be a right-positive algebra triple with* $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$*, and let* $X = (b_i(j))$ *the* $\mathfrak{B}$*-table of* $\mathbf{A}$*. Then:*

a. X *is nonsingular.*

b. The sum of the elements in any row of X *is a nonnegative real number.*

c. If $a \in \mathbf{A}$ *is nonnegative then the elementary symmetric functions in* $a(1), a(2), \ldots, a(n)$ *are real numbers.*

d. If $a \in \mathbf{A}$ *is nonnegative, then* $|a(i)| \leq a(1) = \rho(a)$ *for all* $i = 1, 2, ...n$*.*

e. If $\mathfrak{B}$ *is a positive basis, then the first column of* X *is the only nonnegative column of* X*.*

Proof. Part a. follows from ([8], corollary 2.4). The sum s of the elements of the i-th row of X is the sum of the eigenvalues of b_i. So s is the trace of the nonnegative matrix $M(b_i,\mathfrak{B})$, and part b. follows. The elementary symmetric functions in $a(1), a(2), \ldots, a(n)$ are up to a sign, the coefficients of the characteristic polynomial of the real matrix $M(a,\mathfrak{B})$, so part c. follows. Part d. follows from the definition of $\rho(a) = a(1)$. Part e. will proved in Section 4 (it seems to us appropriate to state it here). ■

We record the following property of FCSAs.

Proposition 12 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple. Let $c,d \in \mathbf{A}$ and suppose that whenever $c(i)=c(j)$ for some i,j then $d(i)=d(j)$ also holds. Then d is a polynomial in c with coefficients in $\mathbb{F}$.*

Proof. This is a direct consequence of Theorem 1.2 of [8]. ■

Next we set up some further notation (We are grateful to Harvey Blau, who suggested this).

Definition 13 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple with $\mathfrak{B}=\{b_1,b_2,...,b_n\}$, and $\alpha \in \mathbb{F}^n$ a positive n-tuple. Let $C=C(\alpha)$ be the inverse matrix of the matrix $Y=([b_i,b_j]_\alpha)$. Then $C=(c_{ij}(\alpha))$ is called the* Cartan matrix of $(\mathbf{A},\mathfrak{B},\alpha,a(i))$. *We will show later that in fact $([b_i,b_j]_\alpha)$ is invertible. For every $i=1,2,\ldots,n$ define $b'_i=\sum_{j=1}^n c_{ij}(\alpha)b_j$. Then $\mathfrak{B}'_{(\alpha)}=\{b'_1,b'_2,\ldots,b'_n\}$ will be called the* associated basis of $\mathfrak{B}$. *When α is fixed we write $\mathfrak{B}'=\mathfrak{B}'_{(\alpha)}$.*

We will show next that in fact $([b_i,b_j]_\alpha)$ is invertible and that $\mathfrak{B}'_{(\alpha)}=\{b'_1,b'_2,\ldots,b'_n\}$ is a basis of $\mathbf{A}$ (not necessarily nonnegative) "orthogonal" to $\mathfrak{B}$. By definition, $Y=Y^*$ and so $C=C^*$.

Theorem 14 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple with $\mathfrak{B}=\{b_1,b_2,...,b_n\}$, and $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ a positive n-tuple. Then:*

a. The matrix $Y=([b_i,b_j]_\alpha)$ is invertible, and $\mathfrak{B}'_{(\alpha)}$ is a basis of $\mathbf{A}$.

b. $[b_i,b'_j]_\alpha=\delta_{ij}$ for $1\le i,j\le n$.

c. $\sum_{k=1}^n b_k(i)\overline{b'_k(j)}=\frac{\delta_{ij}}{\alpha_i}$ for $1\le i,j\le n$.

Proof. a. Set $Y=([b_i,b_j]_\alpha)_{i,j=1}^n$ and $X=(b_i(j))_{i,j=1}^n$. Since $[b_i,b_j]_\alpha=\sum_{s=1}^n b_i(s)\alpha_s\overline{b_j(s)}$, we get that $Y=X\cdot D\cdot\overline{X}^t$, where $D=diag(\alpha_1,\alpha_2,\ldots,\alpha_n)$. By Proposition 2.2, X is nonsingular and since each α_i is positive, we get that Y and hence C is nonsingular. Next, $b'_i(j)=\sum_{s=1}^n c_{is}(\alpha)b_s(j)$ (by Lemma 2.1), so that $Z=(b'_i(j))_{i,j=1}^n=CX$ and consequently, $(b'_i(j))_{i,j=1}^n$ is nonsingular. Now, corollary 2.4 of [8] implies that $\mathfrak{B}'_{(\alpha)}$ is a basis of $\mathbf{A}$.

b. With the same notation as in part a, we have

$$([b'_i,b_j]_\alpha)_{i,j=1}^n=Z\cdot D\cdot\overline{X}^t=CXD\overline{X}^t=CY=I, \text{ as claimed.}$$

c. From part b. we get $Z\cdot D\cdot\overline{X}^t=I$ so $D\cdot\overline{X}^t\cdot Z=I$. Taking the (j,i)-th entry from both sides we get $\sum_{s=1}^n \alpha_j\overline{b_s(j)}b'_s(i)=\delta_{ji}$ from which claim c. follows by exchanging i with j and taking complex conjugates. ■

The above formulas translate to the known formulas in our examples of the introduction.

Definition 15 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple. Then $\mathfrak{B}$ is called* self-conjugate *if for every $b\in\mathfrak{B}$ there exists an element $\overline{b}\in\mathfrak{B}$ such that $\overline{b}(i)=\overline{b(i)}$ for all $i=1,2,\ldots,n$, (here $\overline{b(i)}$ is the complex conjugate of $b(i)$). By Lemma 2.1b., $\overline{b}$ is unique.*

Lemma 16 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple, where $\mathfrak{B}$ is self-conjugate. Then:*

a. For every $a \in \mathbf{A}$, there exists a unique element $\overline{a} \in \mathbf{A}$ such that $\overline{a}(i) = \overline{a(i)}$ for all $i = 1, 2, \ldots, n$.

b. If a is a nonnegative element, then so is $\overline{a}$.

Proof. Set $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$. Write $a = \sum_{i=1}^{n} \beta_i b_i$, where $\beta_i \in \mathbb{F}$. Let $\bar{a} = \sum_{i=1}^{n} \beta_i \overline{b_i}$, then by Lemma 2.1a., we have $\overline{a}(i) = \overline{a(i)}$ for all $i = 1, 2, \ldots, n$. Also Lemma 2.1b. implies that $\bar{a}$ is unique. Furthermore, if a is nonnegative, then $\beta_i \geq 0$ for all i and as $\mathfrak{B} = \left\{\overline{b_i} \mid 1 \leq i \leq n\right\}$, we see that $\bar{a}$ is nonnegative as well. ■

Corollary 17 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple in which $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$ is self-conjugate. Let α be a positive n-tuple and set $\mathfrak{B}' = \mathfrak{B}'_{(\alpha)}$ Then:*

a. for every $a \in \mathbf{A}$ we have $\left(\overline{M(a,\mathfrak{B})}\right)^t = M(\overline{a},\mathfrak{B}')$. In particular, if a is nonnegative then $(M(a,\mathfrak{B}))^t = M(\overline{a},\mathfrak{B}')$.

b. Suppose that $\mathfrak{B}$ is a positive basis. Then $\mathbf{l}(\mathbf{A},\mathfrak{B})$ is the vector $(b'_1(1), b'_2(1), \ldots, b'_n(1))^t$ divided by its component of smallest absolute value.

Proof. a. Write $M(a,\mathfrak{B}) = (m_{ij})$, so $ab_i = \sum_{k=1}^{n} m_{ik} b_k$. Theorem 2.5b. implies that $m_{ij} = \left[ab_i, b'_j\right]_\alpha = \sum_{s=1}^{n} \alpha_s a(s) b_i(s) \overline{b'_j(s)}$. So $\overline{m_{ji}} = \sum_{s=1}^{n} \alpha_s \overline{a(s)} b'_i(s) \overline{b_j(s)} = [\bar{a}b'_i, b_j]_\alpha$. Again Theorem 2.5$b$. implies that $\bar{a}b'_i = \sum_{k=1}^{n} \overline{m_{ki}} b'_k$ so $\overline{m_{ji}} = m_{ij}(\bar{a},\mathfrak{B}')$.

b. Let $u \in \mathbf{A}$ be nonnegative such that $\mathbf{A} = \mathbb{F}[u]$. Set $\mathbf{l} = (b'_1(1), b'_2(1), \ldots, b'_n(1))^t$, $X = (b_i(j))$ and $Z = \left(b'_i(j)\right)$. Then $Z = CX$ where C is the Cartan matrix, which is the transpose of the transition matrix from to $\mathfrak{B}$ to $\mathfrak{B}'$. As $(M(\overline{u},\mathfrak{B}))^t$ and $(M(\overline{u},\mathfrak{B}'))^t$ are matrix representations of the same linear transformation with respect to $\mathfrak{B}$ and $\mathfrak{B}'$ respectively, we get that $M(\overline{u},\mathfrak{B}') = CM(\overline{u},\mathfrak{B})C^{-1}$. Now,

$$M(\overline{u},\mathfrak{B}') \cdot \mathbf{l} = M(\overline{u},\mathfrak{B}') \begin{pmatrix} b'_1(1) \\ b'_2(1) \\ \vdots \\ b'_n(1) \end{pmatrix} = CM(\overline{u},\mathfrak{B})C^{-1} \cdot C \begin{pmatrix} b_1(1) \\ b_2(1) \\ \vdots \\ b_n(1) \end{pmatrix} =$$

$$C \cdot \overline{u}(1) \begin{pmatrix} b_1(1) \\ b_2(1) \\ \vdots \\ b_n(1) \end{pmatrix} = \overline{u}(1) \begin{pmatrix} b'_1(1) \\ b'_2(1) \\ \vdots \\ b'_n(1) \end{pmatrix} = \overline{u}(1)\mathbf{l}.$$

Thus $\mathbf{l}$ is an eigenvector of $M = M(\overline{u},\mathfrak{B}') = (M(u,\mathfrak{B}))^t$ (by part a.), with respect to the eigenvalue $\bar{u}(1)$. As $M \geq 0$, $u(1)$ is real so $\bar{u}(1) = u(1)$. The eigenvalues of M are distinct ([8], Theorem 1.1) so the eigenspace of $u(1)$ has dimension 1. It follows that $\mathbf{l}$ is a scalar multiple of the positive vector $\mathbf{l}(\mathbf{A},\mathfrak{B})$ (which is also an eigenvector of M with respect to $u(1)$). Now claim b. follows. ■

3 Kernels, Closed Sets, Normal Subsets and Quotients

We start with following basic definitions and notation.

Definition 18 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple over the field $\mathbb{F}$ with $\mathfrak{B}=\{b_1,b_2,...,b_n\}$. Let $a=\sum_{i=1}^{n}\alpha_i b_i$ be a general element of $\mathbf{A}$. The* set of irreducible constituents of a, *denoted by $Irr(a)$, is defined by $Irr(a)=\{b_i \mid \alpha_i\neq 0\}$. The elements of $\mathfrak{B}$ will be called* irreducible elements. *The set $Ker(a)=\{i \mid a(i)=a(1)\}$ will be called* the kernel of a *and the set $z(a)=\{i \mid |a(i)|=a(1)\}$ will be called* the generalized kernel of a. *A* faithful element *is an element whose kernel is equal to $\{1\}$ and a* superfaithful element *is an element whose generalized kernel is equal to $\{1\}$. A* normal subset *of $\{1,2,\ldots,n\}$ is a subset which is an intersection of kernels of elements of $\mathfrak{B}$. The* covering number of a, *which is denoted by $cn(a)$ is the smallest positive integer m such that $Irr(a^m)=\mathfrak{B}$. If no such m exits, $cn(a)$ is defined to be infinity. Finally a subset $\mathfrak{C}$ of $\mathfrak{B}$ is called* a closed subset *if $Irr(c_1c_2)\subseteq\mathfrak{C}$ for every $c_1,c_2\in\mathfrak{C}$. Note that being faithful or superfaithful is independent of the choice of the basis $\mathfrak{B}$. Note also that $m_{ij}(a,\mathfrak{B})\neq 0$ if and only if $b_j\in ab_i$.*

The properties of kernels, closed sets, faithful and superfaithful elements will be discussed in this section.

Proposition 19 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple over the field $\mathbb{F}$ with $\mathfrak{B}=\{b_1,b_2,...,b_n\}$. Then*

a. Let $a=\sum_{i=1}^{n}\alpha_i b_i$ be a nonnegative element of $\mathbf{A}$, with $\mathbb{F}\ni\alpha_i\geq 0$, then $Ker(a)=\bigcap\{Ker(b_i)\mid \alpha_i>0\}$.

b. Let N be a subset of $\{1,2,\ldots,n\}$. The N is a normal subset if and only if N is the kernel of some nonnegative element of $\mathbf{A}$.

c. $\bigcap\{Ker(b_i)\mid 1\leq i\leq n\}=\{1\}$.

d. Let a_1, a_2, be nonnegative elements of $\mathbf{A}$. Then $Ker(a_1)\cap Ker(a_2)\subseteq Ker(b)$ for every $b\in Irr(a_1a_2)$.

e. Let N be a normal subset of $\{1,2,,\ldots,n\}$. Then the set $\{b\in\mathfrak{B}\mid N\subseteq Ker(b)\}$ is a closed subset of $\mathfrak{B}$.

Proof. Because of the uniform ordering we know that $|a(i)|\leq a(1)$ for all nonnegative $a\in\mathbf{A}$.

a. Let $a=\sum_{j=1}^{k}\beta_j b_{i_j}$ with $\beta_j>0$ for $1\leq j\leq k$. If $s\in\bigcap\left\{Ker(b_{i_j})\mid j=1,2,\ldots,k\right\}$, then $b_{i_j}(s)=b_{i_j}(1)$ for $j=1,2,\ldots,k$ and by Lemma 2.1 we get $a(s)=$

$\sum_{j=1}^{k}\beta_j b_{i_j}(s) = \sum_{j=1}^{k}\beta_j b_{i_j}(1) = a(1)$ and consequently $s \in Ker(a)$. Conversely, let $s \in Ker(a)$, then

$$a(1) = |a(1)| = |a(s)| = \left|\sum_{j=1}^{k}\beta_j b_{i_j}(s)\right| \le \sum_{j=1}^{k}\beta_j \left|b_{i_j}(s)\right| \le \sum_{j=1}^{k}\beta_j b_{i_j}(1) = a(1).$$

So all the above inequalities, are in fact equalities. It follows that the complex numbers $b_{i_j}(s)$, $1 \le j \le k$ all have the same argument, say θ. Thus $b_{i_j}(s) = \exp(\widehat{i}\theta)\cdot\left|b_{i_j}(s)\right|$, for $1 \le j \le k$, $\widehat{i}$ being the square root of -1. So

$$a(1) = a(s) = \sum_{j=1}^{k}\beta_j b_{i_j}(s) = \exp(\widehat{i}\theta)\sum_{j=1}^{k}\beta_j \left|b_{i_j}(s)\right|.$$

As $a(1) \ne 0$, both $a(1)$ and $\sum_{j=1}^{k}\beta_j\left|b_{i_j}(s)\right|$ are real and positive, so that $\exp(\widehat{i}\theta) = 1$. Hence $b_{i_j}(s) = \left|b_{i_j}(s)\right|$, is nonnegative for $1 \le j \le k$. Finally,

$$b_{i_j}(s) \le b_{i_j}(1) \quad \text{and} \quad a(1) = a(s) = \sum_{j=1}^{k}\beta_j b_{i_j}(s) \le \sum_{j=1}^{k}\beta_j b_{i_j}(1) = a(1)$$

imply that $b_{i_j}(s) = b_{i_j}(1)$ for $1 \le j \le k$. Thus $s \in \bigcap\left\{Ker(b_{i_j}) \mid j = 1,2,\ldots,k\right\}$.

b. From part a. it follows that $N = \bigcap_{j=1}^{k} Ker(b_{i_j})$ if and only if $N = Ker(\sum_{j=1}^{k} b_{i_j})$ and the claim follows.

c. By Theorem 1.2 there is a nonnegative $u \in \mathbf{A}$ with $\mathbf{A} = \mathbb{F}[u]$. By Theorem 1.1 of [8] $u(1) \ne u(i)$ for $2 \le i \le n$ so that $Ker(u) = \{1\}$. Writing $u = \sum_{i=1}^{n}\beta_i b_i$ with nonnegative $\beta_i \in \mathbb{F}$ we get, using part a.:

$$\bigcap\{Ker(b_i) \mid 1 \le i \le n\} \subseteq \bigcap\{Ker(b_i) \mid \beta_i > 0\} = Ker(u) = \{1\},$$

as claimed.

d. Let $Irr(a_1a_2) = \left\{b_{i_j} \mid 1 \le j \le k\right\}$ and let $s \in Ker(a_1) \cap Ker(a_2)$. Lemma 2.1 implies that $(a_1a_2)(s) = a_1(s)a_2(s) = a_1(1)a_2(1)$ so that

$$s \in Ker(a_1a_2) = \bigcap\left\{Ker(b_{i_j}) \mid 1 \le j \le k\right\} \subseteq Ker(b_{i_j}) \text{ for all } 1 \le j \le k.$$

e. Set $\mathfrak{C} = \{b \in \mathfrak{B} \mid N \subseteq Ker(b)\}$ and let $a_1, a_2 \in \mathfrak{C}$ and $c \in Irr(a_1a_2)$. Part d. implies that $N \subseteq Ker(a_1) \cap Ker(a_2) \subseteq Ker(c)$. So $\mathfrak{C}$ is a closed set. ■

At this point, the intersection of the kernels of the elements of a proper closed set can be $\{1\}$. We will see (Theorem 4.4), using nonnegative matrix theory, that if $\mathfrak{B}$ is positive and contains $1_{\mathbf{A}}$, the intersection is always not $\{1\}$. This will imply (Theorem 4.4) that there is a one-to-one mapping from proper closed subsets of $\mathfrak{B}$ onto non-$\{1\}$ normal subsets of $\{1,2,\ldots,n\}$.

Definition 20 ***(with remarks)*** *Let* $(\mathbf{A},\mathfrak{B},a(i))$ *be a right-positive algebra triple over the field* $\mathbb{F}$*, where* $\mathfrak{B} = \{b_1, b_2, \ldots, b_n\}$*. Let* N *be a normal subset of*

$\{1,2,\ldots,n\}$. The algebra generated by the closed set $\mathfrak{C} = \{b \in \mathfrak{B} \mid N \subseteq Ker(b)\}$ will be denoted by $\mathbf{A}\backslash N$. Set $\mathfrak{C} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$. We use the notation $\widetilde{\mathbf{A}}$ for $\mathbf{A}\backslash N$ and elements of $\widetilde{\mathbf{A}}$ will be denoted by tildas. If a is a linear combination of the elements of $\mathfrak{C}$, it is both an element of $\mathbf{A}$ and of $\widetilde{\mathbf{A}}$. When considered as an element of $\widetilde{\mathbf{A}}$, a be will be denoted by $\widetilde{a}$. For example the matrix $M(a,\mathfrak{B})$ is an $n \times n$ matrix, while $M(\widetilde{a},\mathfrak{C})$ is a $k \times k$ matrix. $\widetilde{\mathbf{A}}$ will be called a quotient algebra of $\mathbf{A}$.

We will need the following notation for a $n \times n$ matrix $M = (m_{ij})$ and a subset α of $\{1,2,\ldots,n\}$: the submatrix of M based on the indices in α will be denoted by $M(\alpha)$; that is $M(\alpha) = \left(m_{\alpha_i,\alpha_j}\right)$ where $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$.

The next theorem describes the basic properties of quotients.

Theorem 21 *Let $(\mathbf{A},\mathfrak{B},a(i))$ be a right-positive algebra triple over the field $\mathbb{F}$ and let N be a normal subset of $\{1,2,\ldots,n\}$. Set $\mathfrak{C} = \{b \in \mathfrak{B} \mid N \subseteq Ker(b)\}$. Then:*

a. $\widetilde{\mathbf{A}} = \mathbf{A}\backslash N$ is a FCSA over the field $\mathbb{F}$ with $\mathfrak{C}$ a right-positive basis.

b. Let $\widetilde{a} \in \widetilde{\mathbf{A}}$. Then the set of eigenvalues of $M(\widetilde{a},\mathfrak{C})$ is a subset of the set of eigenvalues of $M(a,\mathfrak{B})$. Moreover, every nonzero eigenvalue of $M(a,\mathfrak{B})$ is an eigenvalue of $M(\widetilde{a},\mathfrak{C})$.

c. Assume that $1_{\mathbf{A}} \in \mathfrak{B}$, and let $a \in \mathbf{A}$ be nonnegative. Then $\widetilde{a}$ is a faithful element of $\mathbf{A}/Ker(a)$.

d. If $\mathfrak{B}$ is a positive basis of $\mathbf{A}$ then $\mathfrak{C}$ is a positive basis of $\mathbf{A}/\mathbf{N}$.

Proof. As $\mathfrak{C}$ is a closed set (Proposition 3.2), $\mathfrak{C}$ generate a FCSA subalgebra of $\mathbf{A}$ for which $\mathfrak{C}$ is a nonnegative basis. Write $\mathfrak{B} = \{b_1, b_2, ..., b_n\}$ and $\mathfrak{C} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$. Let $x_1, x_2, \ldots, x_n$ be the columns of the $\mathfrak{B}$-table $X(\mathbf{A},\mathfrak{B})$ of $\mathbf{A}$. Then:

$$x_l = (b_1(l), b_2(l), \ldots, b_n(l))^t \text{ and } M(a,\mathfrak{B})x_l = a(l)x_l \text{ for all } a \in \mathbf{A} \text{ and } 1 \le l \le n.$$

Let a be an element in $\mathbf{A}/N$ and set $M(a,\mathfrak{B}) = (m_{ij})_{i,j=1}^n$. Let $b_{i_j} \in \mathfrak{C}$, then ab_{i_j} is a linear combination of elements of $\mathfrak{C}$ and hence $m_{i_j,s} = 0$ for every $s \notin \{i_1, i_2, \ldots, i_k\}$ and for every $1 \le j \le k$. Set $\widetilde{x_l} = (b_{i_1}(l), b_{i_2}(l), \ldots, b_{i_k}(l))^t$. It follows that $M(\widetilde{a},\mathfrak{C}) = (m_{i_r,i_s})_{r,s=1}^k$ and that $M(\widetilde{a},\mathfrak{C})\widetilde{x_l} = a(l)\widetilde{x_l}$ for $1 \le l \le n$. Moreover $\widetilde{x_1}$, which is a positive vector, is a eigenvector for the matrix $M(\widetilde{a},\mathfrak{C})$ corresponding to $a(1)$. Now, if a is any nonnegative element of $\mathbf{A}\backslash N$, both $M(a,\mathfrak{B})$ and $M(\widetilde{a},\mathfrak{C})$ are nonnegative matrices. Consequently, $\widetilde{x_1}$ must be an eigenvector of $M(\widetilde{a},\mathfrak{C})$ corresponding to $\rho(\widetilde{a}) = \rho\left(M(\widetilde{a},\mathfrak{C})\right)$, so that $\rho(\widetilde{a}) = a(1)$ ([4] 1.12 p.28). It follows that $\mathfrak{C}$ is a right-positive basis of $\mathbf{A}/N$, which proves part a.

Let $l \in \{1,2,\ldots,n\}$. From $M(\widetilde{a},\mathfrak{C})\widetilde{x_l} = a(l)\widetilde{x_l}$ we get that if $a(l)$ is not an eigenvalue of $M(\widetilde{a},\mathfrak{C})$, then $\widetilde{x_l} = 0$. This means that $b(l) = 0$ for all $b \in \mathfrak{C}$. As a is a linear combination of the elements of $\mathfrak{C}$, Lemma 2.1 implies that $a(l) = 0$. Thus, every nonzero eigenvalue of $M(a,\mathfrak{B})$ is an eigenvalue of $M(\widetilde{a},\mathfrak{C})$. This argument also shows that $a(l) \neq 0$ implies that $\widetilde{x_l} \neq 0$. We will need this in the proof of part c.

Let σ be a permutation on $\{1, 2, \ldots, n\}$ such that $\sigma(r) = i_r$ for $1 \leq r \leq k$ and let P_σ be the corresponding permutation matrix. Then $M(a, \mathfrak{B})$ is similar, via P_σ, to a matrix of the form $M = \begin{pmatrix} M(\widetilde{a}, \mathfrak{C}) & 0 \\ M_1 & M_2 \end{pmatrix}$ where 0 is a zero matrix and M_1, M_2 some matrices. Thus the characteristic polynomial of $M(\widetilde{a}, \mathfrak{C})$ divides that of $M(a, \mathfrak{B})$, and consequently the eigenvalues of $M(\widetilde{a}, \mathfrak{C})$ are also eigenvalues of $M(a, \mathfrak{B})$. This proves part b.

To prove claim c., we let $a \in \mathbf{A}$ be nonnegative and $N = Ker(a)$. We use the notation as in the proofs of the first two parts, for this special N. By part a. of the current theorem, Theorem 1.2 and the remark following its proof, we know that there exists a nonnegative $\widetilde{w} \in \mathbf{A}/N$ such that $\mathbf{A}/N = \mathbb{F}[w]$ and such that the matrix $M(\widetilde{w}, \mathfrak{C})$ has k distinct **nonzero** eigenvalues. As the eigenvalues of $M(\widetilde{w}, \mathfrak{C})$ are eigenvalues of $M(w, \mathfrak{B})$, we know that the distinct k eigenvalues of $M(\widetilde{w}, \mathfrak{C})$ are $w(s_1), w(s_2), \ldots, w(s_k)$ for some indices s_i, $i = 1, 2, \ldots, k$. We have seen that $\rho(\widetilde{w}) = w(1)$. So one of the s_i's is equal to 1. As $w(s_i) \neq 0$, we get that $\widetilde{x_{s_i}} \neq 0$ for $i = 1, 2, \ldots, k$ (see remark at the first paragraph of the proof of part b. above). Let Y be the matrix whose columns are $\widetilde{x_{s_1}}, \widetilde{x_{s_2}}, \ldots, \widetilde{x_{s_k}}$. From $M(\widetilde{w}, \mathfrak{C})\widetilde{x_l} = w(l)\widetilde{x_l}$, we get that $\{\widetilde{x_{s_i}} \mid 1 \leq i \leq k\}$ is a set of eigenvectors of $M(\widetilde{w}, \mathfrak{C})$ corresponding to distinct eigenvalues. In particular $Y^{-1}M(\widetilde{w}, \mathfrak{C})Y = diag\,(w(s_1), w(s_2), \ldots w(s_k))$. Since $\mathbf{A}/N = \mathbb{F}[w]$ we get that $Y^{-1}M(\widetilde{a}, \mathfrak{C})Y = diag\,(a(s_1), a(s_2), \ldots a(s_k))$ (recall that $M(\widetilde{a}, \mathfrak{C})\widetilde{x_l} = a(l)\widetilde{x_l}$).

To show that $\widetilde{a}$ is a faithful element of $\mathbf{A}/N$, we assume that $a(s_{i_0}) = a(1)$ for some i_0, and show that $s_{i_0} = 1$. Recall that $\widetilde{w} \in \mathbf{A}/N$ so $w = \sum_{r=1}^{k} \gamma_r b_{i_r}$, with $\gamma_r \in \mathbb{F}$. Note $N = Ker(a) \subseteq Ker(b_{i_r})$, so $b_{i_r}(s_{i_0}) = b_{i_r}(1)$ for all $r = 1, 2, \ldots, k$. It follows that $w(s_{i_0}) = \sum_{r=1}^{k} \gamma_r b_{i_r}(s_{i_0}) = \sum_{r=1}^{k} \gamma_r b_{i_r}(1) = w(1)$. As the $w(s_i)$'s are distinct, we must have that $s_{i_0} = 1$, as desired.

We now prove claim d. We take a nonnegative $\widetilde{a} \in \mathbf{A}/N$ and show that $(M(\widetilde{a}, \mathfrak{C}))^t$ has a positive eigenvector corresponding to $\rho(\widetilde{a}) = a(1)$.

Consider the matrix $M = (m_{ij}) = \begin{pmatrix} M(\widetilde{a}, \mathfrak{C}) & 0 \\ M_1 & M_2 \end{pmatrix}$ described above. As a is nonnegative (both as an element of $\mathbf{A}$ and of $\mathbf{A}/N$), the matrix $M(a, \mathfrak{B})$ is nonnegative. By assumption, $\mathfrak{B}$ is a positive basis and hence both $M(a, \mathfrak{B})$ and $(M(a, \mathfrak{B}))^t$ have positive eigenvectors corresponding to $\rho(a) = a(1)$. As M and $M(a, \mathfrak{B})$ are similar via a permutation matrix, M is also nonnegative and both M and M^t have positive eigenvectors corresponding to $\rho(a) = a(1)$.

Define the graph of M, $G(M)$ as the directed graph on the vertices $1, 2, \ldots, n$ where an edge leads from i to j if and only if $m_{ij} \neq 0$. Following [4] (p.39), we say that *i has an access to j* if there is a path from i to j. Note that a path of length l exists from i to j if and only if the (i, j)-th entry of the matrix M^l is non-zero (Cor. 3.1, p.78 of [13]). For every l the matrix M^l has the form $\begin{pmatrix} D_1 & 0 \\ D_2 & D_3 \end{pmatrix}$, where $D_1 = (M(\widetilde{a}, \mathfrak{C}))^l$, 0 is a zero matrix and D_3 an $n-k \times n-k$ matrix. So if i is between 1 and k, then i can have access only to indices between

1 and k.

Next we say the index i *communicate* with the index j (See [4], p.39) if i has an access to j and j has access to i. Clearly, communication is an equivalence relation. Denote by $[i]$ the equivalence class of the index i, where $1 \leq i \leq k$. By the previous paragraph we have that $[i] \subseteq \{1, 2, \ldots, k\}$. It follows that $\{1, 2, \ldots, k\} = [j_1] \cup [j_2] \cup \ldots \cup [j_s]$ is a union of equivalence classes of indices j_α, where $1 \leq j_\alpha \leq k$. We now use Theorem 3.14, p.41 of [4] (see also pp.39-41) to get that M is similar via a permutation matrix to a diagonal block matrix: $diag\,(A_1, A_2, \ldots, A_s, D)$. Here $A_\alpha = M\,([j_\alpha])$ for $\alpha = 1, 2, \ldots, s$. This means that $M(\widetilde{a}, \mathfrak{C})$ is permutationaly similar to the matrix $L = diag\,(A_1, A_2, \ldots, A_s)$ and M is pemutationaly similar to $\begin{pmatrix} L & 0 \\ 0 & D \end{pmatrix}$.

It follows that M^t is permutationaly similar to $H = \begin{pmatrix} L^t & 0 \\ 0 & D^t \end{pmatrix}$. Since M^t has a positive eigenvector, H has one too, call it $\mathbf{v}$. The first k components of $\mathbf{v}$ form a positive eigenvector of L^t. Finally $(M(\widetilde{a}, \mathfrak{C}))^t$ is similar via a permutation matrix to L^t so $(M(\widetilde{a}, \mathfrak{C}))^t$ has a positive eigenvector which must correspond to $\rho(a)$ ([4], 1.12, p.28). This proves c. ■

4 Nonnegative Matrices, Products and Eigenvalues of Elements

The theory of nonnegative matrices will be used next, we will need the following definitions.

Definition 22 *Let M be a nonnegative matrix. Then M is called* reducible *if it is permutationaly similar to a matrix of the type $\begin{pmatrix} C & 0 \\ D & E \end{pmatrix}$, where C and E are square matrices and 0 is a zero matrix. An* irreducible nonnegative matrix *is one that is not reducible. If M is an irreducible nonnegative matrix then* the index of cyclicity of M *is defined to be the number of eigenvalues of M whose absolute value is equal to $\rho(M)$. The matrix M is called a* primitive matrix *if there exists a positive integer m such that $M^m > 0$. The smallest such m is called* the index of primitivity of M *and is denoted by $\gamma(M)$.*

Theorem 23 *Let $(\mathbf{A}, \mathfrak{B}, a(i))$ be a positive algebra triple over the field $\mathbb{F}$, with $\mathfrak{B} = \{b_1 = 1_{\mathbf{A}}, b_2, ..., b_n\}$. Let a be a nonnegative element of $\mathbf{A}$. Then*

a. If $Irr(a) = \mathfrak{B}$, then $Irr(ab) = \mathfrak{B}$ for all $b \in \mathfrak{B}$.

b. $M(a, \mathfrak{B}) > 0$ if and only if $Irr(a) = \mathfrak{B}$.

c. The element a is faithful if and only if $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix. In this case the cyclicity index of $M(a, \mathfrak{B})$ is equal to $|z(a)|$.

d. The element a is superfaithful if and only if $M(a, \mathfrak{B})$ is a primitive matrix. In this case $\gamma(M(a, \mathfrak{B})) = cn(a)$. So $cn(a)$ is finite if and only if a is superfaithful.

Proof. a. Suppose that $Irr(a) = \mathfrak{B}$. Then $a = \sum_{i=1}^{n} \beta_i b_i$ with $\mathbb{F} \ni \beta_i > 0$ for $1 \leq i \leq n$. Let $b \in \mathfrak{B}$. We need to show that $Irr(ab) = \mathfrak{B}$. Suppose that contrary. Then there exists a $b_k \in \mathfrak{B}$ such that $b_k \notin Irr(ab)$. Since $ab = \sum_{i=1}^{n} \beta_i b_i b$ and $\beta_i > 0$ for all i, we get that $b_k \notin Irr(b_i b) = Irr(bb_i)$ for all $i = 1, 2, \ldots, n$. Recall that $bb_i = \sum_{r=1}^{n} m_{ir}(b, \mathfrak{B}) b_r$, so that $m_{ik}(b, \mathfrak{B}) = 0$ for all i. It follows that the k-th column of the matrix $M(b, \mathfrak{B})$ is zero so that the k-th row of $(M(b, \mathfrak{B}))^t$ is zero. Now, a matrix with a zero row cannot have a positive eigenvector corresponding to a positive eigenvalue. This contradicts the positivity of the basis $\mathfrak{B}$.

b. If $Irr(a) = \mathfrak{B}$, then $Irr(ab_i) = \mathfrak{B}$ for all $b_i \in \mathfrak{B}$ (by part a.) and so $M(a, \mathfrak{B}) > 0$. Conversely, suppose that $M(a, \mathfrak{B}) > 0$. Then $Irr(a) = Irr(a \cdot 1_{\mathbf{A}}) = Irr(ab_1) = \mathfrak{B}$.

c. If a is faithful, $\rho(a)$ is a simple eigenvalue of the matrix $M(a, \mathfrak{B})$. Since $\mathfrak{B}$ is positive, both $M(a, \mathfrak{B})$ and $(M(a, \mathfrak{B}))^t$ have, each, a positive eigenvector corresponding to $\rho(a)$. Now ([4] 3.15 p.42) implies that $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix. Conversely, if $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix then $\rho(a)$ is a simple eigenvalue of $M(a, \mathfrak{B})$ by the Perron-Frobenius theorem ([4] 1.4 p.27), thus a is faithful. The statement on $|z(a)|$ follows from the definitions.

d. Suppose that a is superfaithful. Then a is faithful and part c. implies that $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix. Also $\rho(a) > |a(i)|$ for $i > 1$. By ([4] 1.7, p.28), $M(a, \mathfrak{B})$ is primitive. Conversely, if $M(a, \mathfrak{B})$ is primitive, ([4] 1.7, p.28) implies that $\rho(a) = a(1) > |a(i)|$ for $i > 1$, which means that a is superfaithful. To prove the statement concerning $cn(a)$, let m be a positive integer. The following two statements are equivalent (by part a.): i. $Irr(a^m) = \mathfrak{B}$; ii. $M(a, \mathfrak{B})^m = M(a^m, \mathfrak{B}) > 0$. Now the statement follows. ■

Proof of part e. of Proposition 2.2. We use Theorem 1.2 to obtain a nonnegative element $u \in \mathbf{A}$ such that $\mathbf{A} = \mathbb{F}[u]$. The first column $\mathbf{v}$ of X is a positive eigenvector of u corresponding to the eigenvalue $u(1) = \rho(u)$. By Theorem 1.1 of [8], $u(1)$ is a simple eigenvalue of the nonnegative matrix $M(u, \mathfrak{B})$ so every eigenvector corresponding to $u(1)$ is a scalar multiple of $\mathbf{v}$. The positivity of $\mathfrak{B}$ and Theorem 4.2c. imply that $M(u, \mathfrak{B})$ is an irreducible nonnegative matrix. As X is nonsingular, $\mathbf{v}$ is the only column of X which is an eigenvector of $M(u, \mathfrak{B})$ corresponding to $u(1)$. Since each column of X is an eigenvector of $M(u, \mathfrak{B})$, and as nonnegative eigenvectors must correspond to $u(1)$ (see [4], 1.4b, p.27), $\mathbf{v}$ is the unique nonnegative column of X. ■

The following corollary is the Brauer-Burnside Theorem, generalized for positive algebra triples.

Theorem 24 *Let* $(\mathbf{A}, \mathfrak{B}, a(i))$ *be a positive algebra triple of dimension* n *over the field* $\mathbb{F}$ *with* $b_1 = 1_{\mathbf{A}}$. *Let* a *be a nonnegative faithful element of* $\mathbf{A}$ *and suppose that* a *has exactly* m *distinct eigenvalues. Then*

$$Irr(1 + a + a^2 + \ldots + a^{m-1}) = Irr(a + a + a^2 + \ldots + a^m) = \mathfrak{B}.$$

Proof. Since a is faithful $M = M(a, \mathfrak{B})$ is an irreducible matrix (Theorem 4.2.c.). As M is diagonalizable, m is the degree of the minimal polynomial of M.

Now, ([4] (2.3), p. 29) implies that both $\sum_{i=0}^{m-1} M^i$ and $\sum_{i=1}^{m} M^i$ are positive matrices. Thus,

$$M(1_{\mathbf{A}}+a+a^2+\ldots+a^{m-1},\mathfrak{B}) = I+M(a,\mathfrak{B})+(M(a,\mathfrak{B}))^2+\ldots+(M(a,\mathfrak{B}))^{m-1} > 0.$$

Now Theorem 4.2 b. implies that $\mathfrak{B} = Irr\left(1_{\mathbf{A}} + a + a^2 + \ldots + a^{m-1}\right)$. The second equality follows similarly. ■

REMARK. If a in Theorem 4.3 is not assume to be faithful, then

$$Irr(1+a+a^2+\ldots+a^{m-1}) = Irr(a+a+a^2+\ldots+a^m) = \{b \in \mathfrak{B} \mid Ker(a) \subseteq Ker(b)\}.$$

This follows from Theorem 4.3 and from Theorem 3.4., that implies that $\widetilde{a}$ is a faithful element of $\mathbf{A}/Ker(a)$ for which $\{b \in \mathfrak{B} \mid Ker(a) \subseteq Ker(b)\}$ is a positive basis.

Theorem 25 *Let* $(\mathbf{A},\mathfrak{B}, a(i))$ *be a positive algebra triple of dimension* n *over the field* $\mathbb{F}$ *with* $b_1 = 1_{\mathbf{A}}$*.*

a. If $N \neq \{1\}$ *is a normal subset of* $\{1, 2, \ldots, n\}$*, then* $\{b \in \mathfrak{B} \mid N \subseteq \ker(b)\}$ *is a proper closed subset of* $\mathfrak{B}$*.*

b. Let $\mathfrak{C}$ *be a proper closed subset of* $\mathfrak{B}$*. Then* $N(\mathfrak{C}) = \{\bigcap \ker(c) \mid c \in \mathfrak{C}\} \neq \{1\}$*.*

c. The mapping $N \to \{b \in \mathfrak{B} \mid N \subseteq \ker(b)\}$ *is a one-to-one mapping from the collection of non-*$\{1\}$ *normal subsets of* $\{1, 2, \ldots, n\}$ *onto the collection of proper closed subsets of* $\mathfrak{B}$*.*

d. In the above correspondence $\{1, 2, \ldots, n\}$ *corresponds to* $\{1_{\mathbf{A}}\}$*.*

Proof. a. Set $\mathfrak{C} = \{b \in \mathfrak{B} \mid N \subseteq \ker(b)\}$. Then $\{\bigcap \ker(c) \mid c \in \mathfrak{C}\} \supseteq N \neq \{1\}$. Proposition 3.2c now implies that $\mathfrak{C}$ is cannot be equal to $\mathfrak{B}$.

b. Let $a = \sum\limits_{c\in\mathfrak{C}} c$. By Proposition 3.2a, $\ker(a) = N(\mathfrak{C})$. If $N(\mathfrak{C}) = \{1\}$, then a is faithful. Let $b \in \mathfrak{B} - \mathfrak{C}$, then by Theorem 4.3, $b \in Irr(a^s)$ for some positive integer s. Since $\mathfrak{C}$ is closed, $Irr(a^s) \subseteq \mathfrak{C}$ so that $b \in \mathfrak{C}$, a contradiction.

c. Let $Clos$ be the collection of proper closed subsets of $\mathfrak{B}$, and Nor the collection of all non-$\{1\}$ normal subsets of N. Define $f : Nor \to Clos$ by $f(N) = \{b \in \mathfrak{B} \mid N \subseteq \ker(b)\}$. Part a. implies that the image of f is in $Clos$. We show that f is one-to-one. Let $L \in Nor$. We show that $N(f(L)) = L$ which will imply injectivity of f. Since L is normal, $L = \bigcap\{\ker(d) \mid d \in \mathfrak{D}\}$ for some $\mathfrak{D} \subseteq \mathfrak{B}$. So $\mathfrak{D} \subseteq f(L)$. It follows that $L = \bigcap\{\ker(d) \mid d \in \mathfrak{D}\} \supseteq \bigcap\{\ker(b) \mid b \in f(L)\} = N(f(L))$ so $L \supseteq N(f(L))$. But $L \subseteq \ker(b)$ for all $b \in f(L)$, so $L \subseteq \bigcap\{\ker(b) \mid b \in f(L)\} = N(f(L))$. Hence $N(f(L)) = L$.

Let $\mathfrak{C} \in Clos$, by part b. $N(\mathfrak{C}) \in Nor$. So if we show that $\mathfrak{C} = f(N(\mathfrak{C}))$, we will get that f is onto. Clearly, $N(\mathfrak{C}) \subseteq \ker(b)$ for all $b \in \mathfrak{C}$ so that $\mathfrak{C} \subseteq \{b \in \mathfrak{B} \mid N(\mathfrak{C}) \subseteq \ker(b)\} = f(N(\mathfrak{C}))$. Suppose that $\mathfrak{C} \neq f(N(\mathfrak{C}))$, then $N(\mathfrak{C}) \subseteq \ker(b)$ for some $b \in \mathfrak{B} - \mathfrak{C}$. Let $a = \sum\limits_{c\in\mathfrak{C}} c$. Then $N(\mathfrak{C}) = \ker(a)$ and from the previous corollary and the remark following it, we get that $b \in Irr(a^s)$

for some positive integer s. Since $\mathfrak{C}$ is closed, $Irr(a^s) \subseteq \mathfrak{C}$ so that $b \in \mathfrak{C}$, a contradiction. Thus $\mathfrak{C} = f(N(\mathfrak{C}))$ for $\mathfrak{C} \in Clos$, as claimed.

d. Let $b \in f(\{1, 2, \ldots, n\})$, then $b(i) = b(1)$ for $1 \leq i \leq n$ and so $M(b, \mathfrak{B})$ is similar to the matrix $b(1)I$, where I is an identity matrix. Thus $M(b, \mathfrak{B}) = b(1)I$ and in particular $b = b \cdot 1_{\mathbf{A}} = b \cdot b_1 = b(1) \cdot 1_{\mathbf{A}}$. But the only scalar multiple of $1_{\mathbf{A}}$ in $\mathfrak{B}$ is $1_{\mathbf{A}}$ itself, thus $b = 1_{\mathbf{A}}$. ■

The next series of results deal with consequences on powers and eigenvalues of faithful and superfaithful elements. All were proved before, separately for characters and classes. The proofs are similar to the ones used in [6] and [7] to prove the special cases. However, there are better bounds on $cn(a)$ here than in the special cases (See Corollary 4.7).

Corollary 26 *Let $(\mathbf{A}, \mathfrak{B}, a(i))$ be a positive algebra triple of dimension n over the field $\mathbb{F}$ with $b_1 = 1_{\mathbf{A}}$. Let a be a nonnegative faithful element of $\mathbf{A}$. Then:*

a. For every $b \in \mathfrak{B}$ we have: $|z(a)| = g.c.d\{m \mid b \in Irr(ba^m)\}$.

b. $|z(a)| = g.c.d\{m \mid 1_{\mathbf{A}} \in Irr(a^m)\}$.

c. If $1_{\mathbf{A}} \in Irr(a^m) \cap Irr\left(a^{m+1}\right)$ for some positive integer m, then a is superfaithful.

d. If $b \in Irr(ab)$ for some $b \in \mathfrak{B}$, then a is superfaithful.

Proof. Clearly, claims b.,c. and d. follow from claim a. So we prove claim a. For a nonnegative $c \in \mathbf{A}$ we denote by $G(c) = G(c, \mathfrak{B})$ the directed graph whose vertices are $1, 2, \ldots, n$ where an edge leads from i to j if and only if $m_{ij}(c, \mathfrak{B}) > 0$. By Theorem 4.2 $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix with index of cyclicity equal to $|z(a)|$. A circuit of length m exists through the vertex i if and only if the (i,i)-th entry of $(M(a, \mathfrak{B}))^m = M(a^m, \mathfrak{B})$ is nonzero. This is equivalent to saying that a loop (i.e. a circuit of length one) through i exists in the graph $G(a^m, \mathfrak{B})$. Thus a circuit of length m exists through the vertex i if and only if $b_i \in Irr(a^m b_i)$. Now theorem 2.30 (p.35) of [4] implies that

$$|z(a)| = g.c.d\{m \mid \text{ there is a loop in } G(a^m, \mathfrak{B}) \text{ through } i\} = g.c.d\{m \mid b_i \in Irr(b_i a^m)\}.$$

■.

Corollary 27 *Let $(\mathbf{A}, \mathfrak{B}, a(i))$ be a positive algebra triple of dimension n over the field $\mathbb{F}$ with $b_1 = 1_{\mathbf{A}}$. Let a be a nonnegative faithful element of $\mathbf{A}$ and assume that $h = |z(a)| \neq 1$. Let $s_1(a) = \sum_{i=1}^n a(i)$, $s_2(a) = \sum_{i<j} a(i)a(j)$, $s_3(a) = \sum_{i<j<k} a(i)a(j)a(k)$, ... , $s_n(a) = \prod_{i=1}^n a(i)$ be the elementary symmetric functions in $a(1), a(2), \ldots, a(n)$. Then*

a. For every i such that $i + h - 1 < n$, at least $h - 1$ of the following h numbers are zero: $s_i(a), s_{i+1}(a), \ldots, s_{i+h-1}(a)$.

b. $s_i(a)s_{i+1}(a) = 0$ for all $i < n$.

c. For each l, $1 \leq l \leq n$, there exists j, $1 \leq j \leq n$ such that $e^{\frac{2\pi \hat{i}}{h}} a(l) = a(j)$.

d. The set of the h distinct h-th roots of $(a(1))^h$ is equal to $\{a(i) \mid i \in z(a)\}$.

e. The set of nonzero elements of the sequence $a(1), a(2), \ldots, a(n)$ can be partitioned into subsets of size h each, the sum of the elements of each subset is zero.

Proof. By Theorem 4.2 $M(a, \mathfrak{B})$ is an irreducible nonnegative matrix with index of cyclicity equal to $|z(a)|$. Let $p_a(x)$ be the characteristic polynomial of $M(a, \mathfrak{B})$. The $s_i(a)$'s are, up to a sign, the coefficients of $p_a(x)$. Let x^{m_1} and x^{m_2} be two consecutive terms in $p_a(x)$ with nonzero coefficients, then ([4] 2.27 p.34) implies that $m_1 - m_2$ is a multiple of h. Now part a. follows. Part b. follows from part a. Parts c. and d. follow from ([4] 2.20, p.32). To show part e., let k be an index such that $a(k) \neq 0$. Set $B_k = \left\{ e^{\frac{2\pi \hat{i}}{h} s} a(k) \mid s = 0, 1, 2, \ldots, h-1 \right\}$. By part c. all elements of B_k are nonzero eigenvalues of $M(a, \mathfrak{B})$. Clearly, the sum of the elements of B_k is zero and the collection of the B_k's is the required partition. ■

Corollary 28 *Let $(\mathbf{A}, \mathfrak{B}, a(i))$ be a positive algebra triple of dimension n over the field $\mathbb{F}$ with $b_1 = 1_{\mathbf{A}}$. Let a be a nonnegative superfaithful element of $\mathbf{A}$ with exactly m distinct eigenvalues. Then*

a. $Irr\left(a^{m^2-2m+2}\right) = \mathfrak{B}$.

b. Suppose that for all $i, j = 1, 2, \ldots, n$, $b_i \in Irr(ab_j)$ holds if and only if $b_j \in Irr(ab_i)$ holds. Then $Irr\left(a^{2m-2}\right) = \mathfrak{B}$.

c. If $a(i)$ is real for $i = 1, 2, \ldots, n$, then $Irr\left(a^{3m-3}\right) = \mathfrak{B}$.

d. If $(\mathrm{real}(a(i)))^2 \geq (imaginary(a(i)))^2$ for $i = 1, 2, \ldots, n$, then $Irr\left(a^{3m-3}\right) = \mathfrak{B}$.

Proof. By theorem 4.2d., the matrix $M(a, \mathfrak{B})$ is a primitive matrix with primitivity index equal to $cn(a)$. Note that m is the degree of the minimal polynomial of $M(a, \mathfrak{B})$. Let $G(c)$ be the graph as defined in the proof of Corollary 4.5. Let g be the minimal length of a circuit in $G(a)$. Lemma 2 of [10] implies that $cn(a) \leq (g+1)(m-1)$.

a. This follows from [14].

b. Clearly $M(a, \mathfrak{B})$ has no zero row, so every main diagonal entry of $(M(a, \mathfrak{B}))^2 = M(a^2, \mathfrak{B})$ is positive so that $G(a^2)$ has a loop through every vertex. By lemma 1 of [10], we have that $M(a^{2m-2}, \mathfrak{B}) = \left((M(a, \mathfrak{B}))^2\right)^{m-1}$ is a positive matrix. The result now follows from Theorem 4.2 d.

c. This is a special case of part d.

d. Let $a(1) = \lambda_1, \lambda_2, \ldots, \lambda_s$ be the real eigenvalues of $M = M(a, \mathfrak{B})$. If μ is a complex eigenvalue, then so is $\overline{\mu}$, as M has real entries. Denote by $\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}, \ldots, \mu_l, \overline{\mu_l}$ the complex non-real eigenvalues of M. So the trace of M^2 is $\sum_{i=1}^{s} \lambda_i^2 + \sum_{i=1}^{l} \left(\mu_i^2 + \overline{\mu_i}^2\right)$. Next, for $i = 1, 2, \ldots, n$ we have

$$\mu_i^2 + \overline{\mu_i}^2 = 2(\mathrm{real}(\mu_i^2)) = 2\left((\mathrm{real}(\mu_i))^2 - (imaginary(\mu_i))^2\right) \geq 0.$$

As $\lambda_1 \neq 0$, we get that the trace of M^2 is positive which means that $G(a^2)$ has at least one loop. Thus $g \leq 2$ and the result follows. ■

Corollary 29 *Let* $(\mathbf{A},\mathfrak{B},a(i))$ *be a positive algebra triple of dimension* n *over the field* $\mathbb{F}$ *with* $b_1 = 1_{\mathbf{A}}$. *Let* a *be a nonnegative faithful element of* $\mathbf{A}$ *with exactly* m *distinct eigenvalues. Assume that* $\sum_{i=1}^{n} a(i) > 0$, *then* $Irr\left(a^{2m-2}\right) = \mathfrak{B}$. *(Recall that in general* $\sum_{i=1}^{n} a(i)$ *is always nonnegative).*

Proof. Theorem 4.2 implies that $M(a,\mathfrak{B})$ is an irreducible nonnegative matrix and our assumption is that the trace of $M(a,\mathfrak{B})$ is positive. By ([4] 2.28 p.34), $M(a,\mathfrak{B})$ is primitive. Using the notation of the previous proof we have here that $g = 1$ and that $cn(a) \le 2(m-1)$. ■

Corollary 30 *Let* $(\mathbf{A},\mathfrak{B},a(i))$ *be a positive algebra triple of dimension* n *over the field* $\mathbb{F}$ *with* $b_1 = 1_{\mathbf{A}}$. *Let* a *be a nonnegative faithful element of* $\mathbf{A}$ *and assume that* $h = |z(a)| \neq 1$. *Then there is a partition* $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \ldots \cup \mathfrak{B}_h$ *into pairwise disjoint subsets* $\mathfrak{B}_i$, $1 \le i \le h$, *with the following properties:*

a. If $b \in \mathfrak{B}_i$, *then*

$$Irr(ab) \subseteq \begin{cases} \mathfrak{B}_{i+1} & if \;\; i < h \\ \mathfrak{B}_1 & if \;\; i = h \end{cases} \qquad and\ Irr(a^h b) \subseteq \mathfrak{B}_i.$$

b. For every $i = 1,2,\ldots,h$ *there exists a positive integer* $m(i)$ *such that for every* $b \in \mathfrak{B}_i$ *we have that* $Irr(a^{hm(i)}b) = \mathfrak{B}_i$.

c. Set $\mu = \min_i \{|\mathfrak{B}_i|\}$. *Then the number of* i'*s such that* $a(i) = 0$ *is at least* $n - h\mu$.

Proof. The proof is rather technical and since it is similar to the proofs of the special cases in [7], we will omit it. ■

5 Applications

We next relate the notation and results of the previous sections to the four examples of the introduction. Then we state the applications which are direct translations of the results of the previous section to the special cases.

In the examples below, we use the group theoretical notation introduced in the examples of the introduction (**OC**, **CC**, **BC**, **PI**). To distinguish between kernels of algebra elements ($Ker(\theta)$) and of characters of groups (they are not the same) we will denote the kernel of a character θ by $\ker(\theta)$; Similarly, the generalized kernel of θ will be denoted by $Z(\theta)$.

RMARKS on the facts in the examples 1.6-1.9.

OC and **CC (**Examples 1.6 an 1.7). The fact that

$$(X(\mathbf{A},\mathfrak{B}))^{-1} M(\varphi,\mathfrak{B}) X(\mathbf{A},\mathfrak{B}) = diag\left(\varphi(C_1),\varphi(C_2),\ldots,\varphi(C_n)\right),$$

is proved in [5]. As here $M(\theta, Irr(G)) = ([\theta\chi_i,\chi_j])$ it is clear that $(M(\theta, Irr(G)))^t = M(\overline{\theta}, Irr(G))$ for every $\theta \in \mathbf{A}$. The rest of the facts mentioned in example 1.6 now follow. All the facts mentioned in Example 1.7 are either shown in ([7],

Proposition 3.3) and its proof, or are easy consequences of it and the orthogonality relations.
BC and **PI** (Examples 1.8 and 1.9). Let φ be a Brauer character and X the Brauer character table. The fact

$$X^{-1}M(\varphi,\mathfrak{B})X = diag\left(\varphi(K_1),\varphi(K_2),\ldots,\varphi(K_m)\right)$$

is proved in [7], Proposition 2.5. The formulas $[\Phi_i,\varphi_j]_\alpha = \delta_{ij}$ and $\frac{\delta_{ij}}{\alpha_i} = \sum_{k=1}^{m}\Phi_k(K_i)\overline{\varphi_k(K_j)}$ are known ([9], Theorems 60.3 and 60.4, pp 367-368). From this it follows that $m_{ij}(\varphi,\mathfrak{B}) = [\varphi\varphi_i,\Phi_j]_\alpha = [\overline{\varphi}\Phi_j,\varphi_i]_\alpha = m_{ji}(\overline{\varphi},\mathfrak{B}')$ (note that all inner products in this equalities are real), so $(M(\varphi,\mathfrak{B}))^t = M(\overline{\varphi},\mathfrak{B}')$. The rest of the facts of Example 1.8 now follow. The ones in Example 1.9 are shown in the same way.

Example 31 ***OC***. *The nonnegative elements of $\mathbb{Q}(Irr(G))$ includes all characters. So let θ be a character of G. $Irr(\theta)$ is the set of the irreducible constituents of θ. Next $Ker(\theta)=\{i_1,i_2,\ldots,i_k\}$ if and only if $\ker(\theta)=C_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_k}$. A set $\{j_1,j_2,\ldots,j_l\}$ is a normal set if an only if $C_{j_1}\cup C_{j_2}\cup\ldots\cup C_{j_l}$ is a normal subgroup of G. A set is closed if and only if it is the collection of all irreducible characters of some factor group (this fact is a consequence of the Brauer-Burnside theorem, it is not used in our proof of the Brauer-Burnside theorem). Further $z(\theta)=\{i_1,i_2,\ldots,i_k\}$ if and only if $Z(\theta)=C_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_k}$. Clearly, θ is faithful as an element of $\mathbb{Q}(Irr(G))$ if and only if it is faithful in the usual sense. Similarly, θ is superfaithful as an element of $\mathbb{Q}(Irr(G))$ if and only $Z(\theta)=\{1\}$. If θ if faithful, then its index of cyclicity is equal to $|Z(\theta)|$.*

Example 32 ***CC***. *The nonnegative elements of $\mathbf{Z}(\mathbb{Q}G)$ are the linear combinations with nonnegative coefficients of the class sums. Let $U = C_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_k}$ be a normal subset of G. Denote $\overline{U}=\sum_{j=1}^{k}\overline{C_{i_j}}$. So $Irr(\overline{U}) = \left\{\overline{C_{i_1}},\overline{C_{i_2}},\ldots,\overline{C_{i_k}}\right\}$. Let C be a conjugacy class and $\overline{C}$ its class sum. Then $Ker(\overline{C})=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ if and only if $Irr\left(G/\langle C\rangle\right)=\{\chi_{\alpha_1},\chi_{\alpha_2},\ldots,\chi_{\alpha_m}\}$. In particular $\overline{C}$ is faithful if and only if $G=\langle C\rangle$. Also, the set $\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ is normal if and only if $\{\chi_{\alpha_1},\chi_{\alpha_2},\ldots,\chi_{\alpha_m}\}=Irr(G/K)$ for some normal subgroup K of G. A set of class sums $\left\{\overline{C_{i_1}},\overline{C_{i_2}},\ldots,\overline{C_{i_k}}\right\}$ is closed if and only if $C_{i_1}\cup C_{i_2}\cup\ldots\cup C_{i_k}$ is a normal subgroup of G. Suppose now that $\overline{C}$ is faithful, namely that $G=\langle C\rangle$. The index of cyclicity $\left|z(\overline{C})\right|$ of $\overline{C}$ is the number of $\chi\in Irr(G)$ such that $|\chi(C)|=\chi(1)$. So*

$$\left|z(\overline{C})\right| = |\{\chi\in Irr(G)\mid C\subseteq Z(\chi)\}| = |\{\chi\in Irr(G)\mid G=\langle C\rangle\subseteq Z(\chi)\}| = \\ |\{\chi\in Irr(G)\mid \chi(1)=1\}|\,.$$

So $\left|z(\overline{C})\right|=|G:G'|$ for a faithful $\overline{C}$. In particular $\overline{C}$ is superfaithful if and only if $\langle C\rangle=G=G'$.

Example 33 ***BC.*** *The nonnegative elements of $\mathbb{Q}(Ibr(G))$ includes all Brauer characters. So let $\phi\neq 1$ be a Brauer character of G. $Irr(\phi)$ is the set of the* *irreducible constituents of ϕ. Clearly ϕ is faithful if and only if $\phi(K) \neq \phi(1)$ for all $K \in Con_{p'}(G)$ and in this case the index of cyclicity of ϕ is equal to $|\{K \in Con_{p'}(G) \mid |\phi(K)| = \phi(1)\}|$.*

The **PI** example and applications is left to the reader.

We now state some straight-forward consequences of the results of the previous section.

Corollary 34 *Let G be a finite group and χ be an ordinary character of G, having exactly m distinct values.*
a. (Brauer-Burnside) If χ is faithful then $Irr(1+\chi+\chi^2+\ldots+\chi^{m-1}) = Irr(G)$.

b. If $Z(\chi) = 1$, then $Irr(\chi^{m^2-m+2}) = Irr(G)$.
c. If $Z(\chi) = 1$ and ***either*** *χ is real* ***or*** *$\sum_{C \in Con(G)} \chi(C) > 0$, then $Irr(\chi^{2m-2}) = Irr(G)$.*

Proof. Follows from Theorem 4.3 and Corollaries 4.7 and 4.8. We comment that if χ is real then $M(\chi, Irr(G))$ is symmetric, so the assumption of Corollary 4.7b. is satisfied, so that $Irr(\chi^{2m-2}) = Irr(G)$. ■

Corollary 35 *Let G be a finite group and C be a conjugacy class of G. Let m be the number of elements in the set $\left\{\frac{\chi(C)}{\chi(1)} \mid \chi \in Irr(G)\right\}$.*
a. (Garrison) If C generates G, then $G = \{1\} \cup C \cup C^2 \cup \cdots \cup C^{m-1}$.
b. If C generates G and G is perfect then $G = C^{m^2-2m+2}$.
c. Assume that C is generates G and G is perfect. If either C is a real class or contains a commutator, then $G = C^{2m-2}$.

Proof. Parts a. and b. follow from Theorem 4.3 and Corollary 4.7. If C is real then $C = C^{-1}$ and from the last two sentences Example 1.7 (**CC**) we get that $m_{ij}(C, \overline{Con(G)}) \neq 0$ if and only if $m_{ji}(C, \overline{Con(G)}) \neq 0$ (here $\overline{Con(G)}$ is the set of the class sums of the conjugacy classes of G). Now the assumption of Corollary 4.7b. is satisfied, so $G = C^{2m-2}$. If C contains a commutator then $\sum \left\{\frac{\chi(C)}{\chi(1)} \mid \chi \in Irr(G)\right\} > 0$ (See [11], p.45) so again $G = C^{2m-2}$ (by Corollary 4.8). ■

Corollary 36 *Let G be a finite group, p a prime and ϕ be a Brauer character of G in characteristic p. Let m be the number of distinct values of ϕ.*
a. (Brauer-Burnside) If ϕ is faithful then $Irr(1+\phi+\phi^2+\ldots+\phi^{m-1}) = Ibr(G)$.

b. If $Z(\phi) = 1$ (namely, $|\phi(g)| \neq \phi(1)$ for all p-regular elements of G), then $Irr(\phi^{m^2-m+2}) = Ibr(G)$.

Proof. Follows from the results of the previous section. ■

REMARK. The smallest positive integer n such that $C^n = G$ for all non-$\{1\}$ conjugacy classes of G, is denoted in the literature by $cn(G)$ (if such n

exists). The smallest positive integer n such that $Irr(\chi^n) = Irr(G)$ for all $\chi \in Irr(G) - \{1\}$ is denoted in the literature by $ccn(G)$ (if such n exists). It is known that $cn(G)$ (respectively, $cc(G)$) exists if and only if G is a finite nonabelian simple group. The best known general upper bounds for a simple group G are $cn(G) \leq \min\left\{0.5k(k-1), \frac{4}{9}k^2\right\}$, and $ccn(G) \leq 0.5(k^2-1)$, where k is the number of conjugacy classes of G (see [1], p. 4). Clearly, $m \leq k$ and in most cases $m < k$. If m is sufficiently smaller than k (e.g. : $m \leq 0.6k$), then a better bounds follow from Corollaries 5.4 and 5.5.

The following three applications are direct consequences of Corollaries 4.5 and 4.6. They are all were proved for each case in [6] and [7]. Here they have a unified proof.

Corollary 37 *Let G be a finite group and χ a faithful character of G. Let $C_1 = \{1\}$, $C_2, \ldots, C_k$ be the conjugacy classes of G. Then:*

a. For every $\theta \in Irr(G)$ we have that $|Z(\chi)| = g.c.d\{s \mid \theta \in Irr(\theta\chi^s)\}$, and $|Z(G)| = g.c.d\{s \mid \theta \in Irr(\theta\chi^s)\}$ if χ is irreducible.

b. $|Z(\chi)| = g.c.d\{s \mid 1_G \in Irr(\chi^s)\}$, and $|Z(G)| = g.c.d\{s \mid 1_G \in Irr(\chi^s)\}$ if χ is irreducible.

c. If $1_G \in Irr(\chi^s) \cap Irr(\chi^{s+1})$ for some positive integer s, then $Z(\chi) = 1$ ($Z(G) = 1$ if χ is irreducible).

d. If $\theta \in Irr(\chi\theta)$ for some $\theta \in Irr(G)$, then $Z(\chi) = 1$ ($Z(G) = 1$ if χ is irreducible).

e. Assume that $h = |Z(\chi)| > 1$ and set $s_1(\chi) = \sum_{i=1}^{k} \chi(C_i)$, $s_2(\chi) = \sum_{i<j} \chi(C_i)\chi(C_j)$, $s_3(\chi) = \sum_{i<j<l} \chi(C_i)\chi(C_j)\chi(C_l)$, $\ldots$, $s_k(\chi) = \prod_{i=1}^{k} \chi(C_i)$. Then for every i such that $i + h - 1 < k$, at least $h - 1$ of the following h numbers are zero: $s_i(\chi), s_{i+1}(\chi), \ldots, s_{i+h-1}(\chi)$. In particular $s_i(\chi)s_{i+1}(\chi) = 0$ for all $i < k$.

Corollary 38 *Let G be a finite group and C a conjugacy class that generates G. Set $Irr(G) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_k\}$ Then:*

a. For every conjugacy class D of G we have: $|G : G'| = g.c.d\{s \mid D \subseteq DC^s\}$

b. $|G : G'| = g.c.d\{s \mid 1 \in C^s\}$.

c. If $1 \in C^s \cap C^{s+1}$ for some positive integer s then G is perfect.

d. If $D \subseteq DC$ for some conjugacy class D, then G is perfect.

e. Let $s_1(C), s_2(C), \ldots, s_k(C)$ be the elementary symmetric functions in the numbers $\frac{|C|\chi_i(C)}{\chi_i(1)}$, $i = 1, 2, \ldots, k$ and assume that $h = |G : G'| > 1$. Then for every i such that $i+h-1 < k$, at least $h-1$ of the following h numbers are zero: $s_i(C), s_{i+1}(C), \ldots, s_{i+h-1}(C)$. In particular $s_i(C)s_{i+1}(C) = 0$ for all $i < k$.

Corollary 39 *Let G be a finite group, p a prime and ϕ a faithful Brauer character of G in characteristic p. Let $K_1 = \{1\}$, $K_2, \ldots, K_m$ be the p-regular conjugacy classes of G and $K_{p'} = \bigcup_{i=1}^{m} K_i$ (the set of all p-regular elements of G). Further, let $\mathfrak{z}(\phi)$ be the number of p-regular classes in the set $\{g \in K_{p'} \mid |\phi(g)| = \phi(1)\}$. Then:*

a. For every $\theta \in Ibr(G)$ we have $\mathfrak{z}(\phi) = g.c.d\{s \mid \theta \in Irr(\theta\phi^s)\}$.

b. $\mathfrak{z}(\phi) = g.c.d\{s \mid 1_G \in Irr(\phi^s)\}$.

c. If $1_G \in Irr(\phi^s) \cap Irr(\phi^{s+1})$ for some positive integer s, then $\mathfrak{z}(\phi) = 1$.

d. If $\theta \in Irr(\phi\theta)$ for some $\theta \in Ibr(G)$, then $\mathfrak{z}(\phi) = 1$

e. Assume that $h = \mathfrak{z}(\phi) > 1$ and set $s_1(\phi) = \sum_{i=1}^{m} \phi(K_i)$, $s_2(\phi) = \sum_{i<j} \phi(K_i)\phi(K_j)$, $s_3(\phi) = \sum_{i<j<l} \phi(K_i)\phi(K_j)\phi(K_l)$, ..., $s_k(\phi) = \prod_{i=1}^{m} \phi(K_i)$. Then for every i such that $i + h - 1 < m$, at least $h - 1$ of the following h numbers are zero: $s_i(\phi), s_{i+1}(\phi), \ldots, s_{i+h-1}(\phi)$. In particular $s_i(\phi)s_{i+1}(\phi) = 0$ for all $i < m$.

Few others results from [6] and [7] that were proved separately for **OC**, **CC** and **BC** can be deduced from the results of Section 4, we will not restate them here.

Concluding Remark. There are other issues concerning algebras like the ones above that are worth studying. Examples of such issues are: One can assume that the structure constants are integers, one can study linear elements (these are elements whose leading eigenvalue is equal to 1) and the one can study duality in the form discussed in other such models.

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