Contemporary Mathematics

ALGEBRAS WITH POSITIVE BASES, COMMUTATORS AND COVERING NUMBERS

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Dedicated to Marcel Herzog on the occasion of his retirement.

Abstract. We describe how a general setup called "Algebras with positive bases" can be used for a concurrent study of conjugacy-class covering numbers and character covering numbers in finite groups. This approach yields improvements on the known bounds of the covering numbers.

1. Introduction

Let $Class(G)$ be a the set of conjugacy classes of the finite group G. For a natural number s and $C \in Class(G)$ let $C^s = \{g_1 g_2 \dots g_s \mid g, g_2, \dots, g_s \in C\}$.

The conjugacy-class covering number $cn(G)$ of the finite group G was introduced by Arad, Herzog and Stavi ([**4**]). It is defined as

$$cn(G) = \min \{s \mid C^s = G \text{ for all } C \in Class(G) - \{1\}\} .$$

Clearly $cn(G)$ may not exist. The following is proved in [**4**]:

Theorem 1.1. *([**4**]) Let G be a finite group with exactly k conjugacy classes. Then*

(1) *$cn(G)$ exists if an only if G is a finite nonabelian simple group.*
(2) *If G is a finite nonabelian simple group then $cn(G) \leq \frac{4}{9}k^2$.*

The set of the ordinary irreducible characters of the finite group G will be denoted by $Irr(G)$ and the set of the irreducible constituents of a class function f will be denoted by $Irr(f)$. The value of the class function f on the elements of the conjugacy class C will be denoted by $f(C)$. Other standard notation will be used, they are taken mainly from [**14**].

A "dual" covering number, the character covering number $ccn(G)$ was studied in [**1**]:

$$ccn(G) = \min \{s \mid Irr(\chi^s) = Irr(G) \text{ for all } \chi \in Irr(G) - \{1_G\}\} .$$

Again, $ccn(G)$ may not exists.

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Theorem 1.2. *Let G be a finite group with exactly k conjugacy classes. Then*

(1) *([**1**]) $ccn(G)$ exists if an only if G is a finite nonabelian simple group.*
(2) *([**5**]) If G is a finite nonabelian simple group then $ccn(G) \leq \frac{1}{2}k^2$.*

Covering numbers can be defined for an individual conjugacy class or an individual character. Let $C \in Class(G)$ and θ an ordinary character of G. Set:

$cn(C) = \min\{s \mid C^s = G\}$; $\qquad ccn(\theta) = \min\{s \mid Irr(\theta^s) = Irr(G)\}$.
As before, $cn(C)$ or $ccn(\theta)$ may not exist.

Proposition 1. *Let G be a finite group, $C \in Class(G)$ and θ an ordinary character of G. Then*

(1) *([**4**]) $cn(C)$ exists if an only of $\langle C\rangle = G = G'$.*
(2) *([**8**]) $ccn(\theta)$ exists if and only if $Z(\theta) = 1$.*

Better bounds for the covering numbers are obtained for special types of conjugacy classes and characters.

Theorem 1.3. *([**4**]) Let G be a finite group with exactly k conjugacy classes. Let $C \in Class(G)$. Assume that $\langle C\rangle = G = G'$ and that C is either real or contains commutators. Then $cn(C) \leq 2k - 2$.*

The conjugacy character of the finite group G is the function $c : G \to \mathbb{C}$ satisfying $c(g) = |C_G(g)|$ for every $g \in G$. The second orthogonality relation implies that $c = \sum_{\chi \in Irr(G)} \chi\overline{\chi}$.

Theorem 1.4. *([**8**]) Let G be a finite group with exactly k conjugacy classes. Let θ be a character of G such that $Z(\theta) = 1$. Assume that either θ is real or has a common constituent with the conjugacy character. Then $cnn(\theta) \leq 2k - 2$.*

Let $C \in Class(G)$. The number of distinct numbers in the list (multiset): $\left\{\frac{\chi(C)}{\chi(1)} \mid \chi \in Irr(G)\right\}$ will be denoted by $m(C)$. The number of distinct values of a character θ will be denoted by $m(\theta)$. We mention the following two "dual" results.

Theorem 1.5. *([**14**], chapter 4) Let G be a finite group, $C \in class(G)$ and θ a character of G. Then*

(1) *(Garrison) If $\langle C\rangle = G$ then $1 \cup C \cup C^2 \cup \ldots \cup C^{m(C)-1} = G$.*
(2) *(Brauer-Burnside) If θ is faithful then $Irr(1_G + \theta + \theta^2 + \ldots + \theta^{m(\theta)-1}) = Irr(G)$.*

Here G (respectively $Irr(G)$) is being "covered" by a union (respectively sum) of powers of a class (respectively character), the highest power is in terms of "number of distinct values". Unlike bounds on *cn* and *ccn* which are in terms of the number of conjugacy classes.

In this article we report upper bounds for $cn(C)$ and $ccn(\theta)$ in term $m(C)$ and $m(\theta)$ respectively. Furthermore, we do that in a unified way, namely deduct the bounds from bounds on objects is a general setup (called Algebras with positive bases), of which classes, characters (and Brauer characters) are special cases. While talking on the general setup, we will observe a "connection" between commutators and constituents of the conjugacy character, and between G' and $Irr(G/Z(G))$.

Proofs either appeared or will appear elsewhere (the new results which are: Theorems 3.4 and 4.3, Proposition 2, Corollaries 3 and 4 will appear in [**9**]).

We finish the introduction with a brief summary on other results on the covering numbers. If G is a finite simple group of Lie type, then Ellers, Gordeev and Herzog showed that $cn(G)$ is bounded in terms of the Lie-rank of G ([**12**]). Arad, Fisman and Muzychuk ([**3**]) proved that $cn(C) \leq \frac{|G|}{|C|}$ for a conjugacy class C of the finite simple group G. Liebeck and Shalev ([**16**]) proved that there exists a constant c such that if G is a finite simple group and $C \in class(G) - \{\{1\}\}$, then $cn(C) \leq \frac{c \cdot \log|G|}{\log|C|}$. From this bound, the bounds obtained previously for the finite simple groups of Lie type, can be deduced.

The exact value of $cn(G)$ was found in several cases: $cn(A_n)$ by Y. Dvir ([**10**]), $cn(PSL(n,q))$ by A. Lev ([**15**]), $cn(Sz(q))$ by Arad, chillag and Moran ([**2**]) and $cn(G)$ where G is a sporadic group by I. Zisser ([**20**]).

Much less work has been done on $ccn(G)$. The numbers $ccn(Sz(q))$ and $ccn(G)$ for some of the sporadic groups and small simple groups were found by Arad and Lipman in [**5**] and $ccn(A_n)$ was computed by Zisser ([**21**]).

2. Algebras with positive bases

We start with a few notions from matrix theory. A square matrix is called nonnegative if all its entries are nonnegative real numbers, it is called positive if all its entries are positive real numbers. The Perron-Frobenius theorem (see [**6**], chapter 2), states that the spectral radius $\rho(M)$ of the nonnegative matrix M is an eigenvalue of M and that M has a nonnegative eigenvector corresponding to $\rho(M)$. Clearly, if M is nonnegative, then so is M^t and so M^t has also a nonnegative eigenvector corresponding to $\rho(M^t) = \rho(M)$.

A nonnegative matrix M is called primitive, if M^r is positive for some positive integer r. The smallest r such that M^r is positive is called the primitivity index of M.

Let $\mathbb{F}$ be any subfield of the real number field $\mathbb{R}$ and let $\mathbf{A}$ be a semi-simple, finite-dimensional commutative $\mathbb{F}$-algebra. The identity element of $\mathbf{A}$ (which is known to exist) will be denoted by $1_{\mathbf{A}}$. Let $\mathfrak{B} = \{b_1 = 1_{\mathbf{A}}, b_2, ..., b_n\}$ be a basis of $\mathbf{A}$. The structure constants of $\mathfrak{B}$ are the numbers α_{ijk} defined by the equations: $b_i b_j = \sum_{k=1}^{n} \alpha_{ijk} b_k$. If all the structure constants of $\mathfrak{B}$ are nonnegative real numbers we say that $\mathfrak{B}$ is a nonnegative basis of **A.**

A nonzero element $a = \sum_{i=1}^{n} \alpha_i b_i$ of $\mathbf{A}$ is called a nonnegative element, if each α_i is a nonnegative real number.

For every $a \in \mathbf{A}$, let $M(a, \mathfrak{B}) = (m_{ij}(a, \mathfrak{B}))$ be the matrix whose entries $m_{ij}(a, \mathfrak{B})$ are given by the the equations: $ab_i = \sum_{j=1}^{n} m_{ij}(a, \mathfrak{B}) b_j$. So, $\mathfrak{B}$ is a nonnegative basis, if and only if all entries of all the matrices $M(b_i, \mathfrak{B})$ are nonnegative. Also, if $\mathfrak{B}$ is a nonnegative basis and $a \in \mathbf{A}$ is a nonnegative element, then the matrix $M(a, \mathfrak{B})$ is nonnegative. Hence, in this case both matrices $M(a, \mathfrak{B})$ and $(M(a, \mathfrak{B}))^t$ have nonnegative eigenvector corresponding to $\rho(a) \doteq \rho(M(a, \mathfrak{B}))$.

Definition 2.1. A pair ($\mathbf{A}$,$\mathfrak{B}$) is called an Algebra with positive basis if $\mathfrak{B}$ is a basis of the semi-simple, finite-dimensional commutative $\mathbb{F}$-algebra $\mathbf{A}$ (where $\mathbb{F}$ is a subfield of the real number field), and the following two conditions are satisfied:

(1) The structure constants of $\mathfrak{B}$ are nonnegative.
(2) For every nonnegative element $a \in \mathbf{A}$, each of the the matrices $M(a, \mathfrak{B})$ and $(M(a, \mathfrak{B}))^t$ have a **positive** eigenvector corresponding to $\rho(a)$.

EXAMPLES. Algebras with positive bases are modeled after three algebras associated to finite groups. One is $\mathbb{Q}(Irr(G))$ which is the algebra generated over the rationals by the ordinary irreducible characters of G. Another, $\mathbb{Q}(Ibr(G))$ is the algebra generated over the rationals by the set $Ibr(G)$ of the irreducible Brauer characters of G in a fixed characteristic p. The third is the subalgebra $Z(\mathbb{Q}G)$ of the group algebra $\mathbb{Q}G$ generated over the rationals by the conjugacy-class sums (this is in fact the center of $\mathbb{Q}G$).

In the following, the notation is: $Class(G) = \{C_1 = \{1\}, C_2, \ldots, C_k\}$, $Irr(G) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_k\}$, $Ibr(G) = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ and $\Phi_1, \Phi_2, \ldots, \Phi_m$ are the corresponding principal indecomposable characters of G.

Assumption 2 in the definition of Algebras with positive bases, requires that matrices associated with each nonnegative element of $\mathbf{A}$, would have positive eigenvector corresponding to their leading eigenvalues. In fact it can be proved ([**7**]) that there is a positive eigenvector $\mathbf{r}(\mathbf{A}, \mathfrak{B})$ common to all the matrices in the set $\{M(a, \mathfrak{B}) \mid a \in \mathbf{A},\ a \text{ is nonnegative}\}$, and there is a positive eigenvector $\mathbf{l}(\mathbf{A}, \mathfrak{B})$ common to all the matrices in the set $\left\{(M(a, \mathfrak{B}))^t \mid a \in \mathbf{A},\ a \text{ is nonnegative}\right\}$, corresponding to the leading eigenvalue.

$\mathbf{A}$	$\mathfrak{B}$	$\mathbf{r}(\mathbf{A}, \mathfrak{B})$	$\mathbf{l}(\mathbf{A}, \mathfrak{B})$
$\mathbb{Q}(Irr(G))$	$Irr(G)$	$\begin{pmatrix} \chi_1(1) \\ \chi_2(1) \\ \vdots \\ \chi_k(1) \end{pmatrix}$	$\begin{pmatrix} \chi_1(1) \\ \chi_2(1) \\ \vdots \\ \chi_k(1) \end{pmatrix}$
$Z(\mathbb{Q}G)$	$\left\{ \sum_{x \in C} x \mid C \in class(G) \right\}$	$\begin{pmatrix} \lvert C_1 \rvert \\ \lvert C_2 \rvert \\ \vdots \\ \lvert C_k \rvert \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$
$\mathbb{Q}(Ibr(G))$	$Ibr(G)$	$\begin{pmatrix} \varphi_1(1) \\ \varphi_2(1) \\ \vdots \\ \varphi_m(1) \end{pmatrix}$	$\begin{pmatrix} \Phi_1(1) \\ \Phi_2(1) \\ \vdots \\ \Phi_m(1) \end{pmatrix}$

The nonnegative elements of $\mathbb{Q}(Irr(G))$ are the ordinary characters of G and $\rho(\theta) = \theta(1)$ for a character θ of G. The eigenvalues of $M(\theta, Irr(G))$ are $\theta(C_1)$, $\theta(C_2), \ldots, \theta(C_k)$. Similarly, the nonnegative elements of $\mathbb{Q}(Ibr(G))$ are the Brauer characters of G and $\rho(\theta) = \theta(1)$ for a Brauer character θ of G, and the eigenvalues of $M(\theta, Ibr(G))$ are the values of θ on the p-regular conjugacy classes of G. The nonnegative elements of $Z(\mathbb{Q}G)$ are the nonzero linear combinations with nonnegative coefficients of the conjugacy-class sums, and $\rho(\overline{C}) = |C|$ for the class sum $\overline{C}$ of the conjugacy class C. The eigenvalues of the matrix associated with $\overline{C}$ are $\frac{|C|\chi_1(C)}{\chi_1(1)}, \frac{|C|\chi_2(C)}{\chi_2(1)}, \ldots, \frac{|C|\chi_k(C)}{\chi_k(1)}$.

3. Covering numbers

Let $(\mathbf{A},\mathfrak{B})$ be an algebra with a positive basis , where $\mathfrak{B}=\{b_1=1_{\mathbf{A}},\ b_2,...,b_n\}$. Let $a\in\mathbf{A}$. The the support of a is denoted by $Irr(a)$. Namely, if we write $a=\sum_{i=1}^{n}\alpha_i b_i$, then $Irr(a)=\{b_i\mid\alpha_i\neq 0\}$.

Suppose that $a\in\mathbf{A}$ is nonnegative. Denote by $z(a)$ the number of eigenvalues of $M(a,\mathfrak{B})$ whose absolute value is equal to $\rho(a)$.

The covering number $cn(a)$ of a nonnegative $a\in\mathbf{A}$ is defined as follows:

$$cn(a)=\min\left\{s\mid Irr(a^s)=\mathfrak{B}\right\}.$$

Of course, $cn(a)$ may not exist.

Theorem 3.1. *([**7**]) Let $(\mathbf{A},\mathfrak{B})$ be an algebra with a positive basis, and let $a\in\mathbf{A}$ be nonnegative. Then*

(1) *$cn(a)$ exists if and only if $z(a)=1$.*
(2) *$cn(a)$ exists if and only if the matrix $M(a,\mathfrak{B})$ is primitive, and $cn(a)$ is equal to the primitivity index of $M(a,\mathfrak{B})$.*

EXAMPLE. For a character θ in $\mathbb{Q}\left(Irr(G)\right)$, $z(\theta)=1$ means that $Z(\theta)=1$. The same for a Brauer character. For a sum $\overline{C}$ in $Z(\mathbb{Q}G)$ of the conjugacy class C, $z(\overline{C})=1$ means that $G=G'=\langle C\rangle$. So the two parts of Proposition 1 are special cases of Theorem 3.1(1)

Bounds for the index of primitivity of primitive matrices in terms of the degree m of the minimal polynomial of the matrix were found by Neumann, Hartwig and Shen ([**13**], [**19**]). It is not hard to see that the matrices $M(a,\mathfrak{B})$ for $a\in\mathbf{A}$ can be diagonalized (even simultaneously). So the degree of the minimal polynomial of each $M(a,\mathfrak{B})$ is equal to the number of its distinct eigenvalues.

Theorem 3.2. *([**19**]). Let M be a primitive matrix and m the degree of its minimal polynomial. Then the index of primitivity of M is at most m^2-2m+2.*

Corollary 1. *([**7**]) Let $(\mathbf{A},\mathfrak{B})$ be algebra with a positive basis, and let $a\in\mathbf{A}$ be nonnegative with $z(a)=1$. Let m be the number of distinct eigenvalues of $M(a,\mathfrak{B})$. Then $cn(a)\leq m^2-2m+2$.*

The consequences for covering numbers of groups are:

Corollary 2. *([**7**]). Let G be a finite group, C a conjugacy class of G, θ an ordinary character of G or a Brauer character in some characteristic. Then:*

(1) *If $G=G'=\langle C\rangle$ then $cn(C)\leq m(C)^2-2m(C)+2$.*
(2) *If $Z(\theta)=1$ then $ccn(\theta)\leq m(\theta)^2-2m(\theta)+2$.*

The index of primitivity of a matrix M ($cn(a)$ in our case) is closely related to the adjacency graph of the primitive matrix M. In the case of $M(a,\mathfrak{B})$ this directed graph has the numbers 1,2,3,. . . ,n as vertices, and an edge $i\to j$ exists if an only if $b_j\in Irr(ab_i)$ (here $a\in\mathbf{A}$ and $\mathfrak{B}=\{b_1=1_{\mathbf{A}},b_2,...,b_n\}$). Hartwig and Neumann proved:

THEOREM 3.3. *([**13**]) Let M be a primitive matrix, m the degree of its minimal polynomial and g the length of the smallest circuit of the adjacency graph of M. Then the index of primitivity of M is smaller or equal than $(m-1)(g+1)$.*

We say that a matrix M is of normal type if MM^t and M^tM have the same zero entries.

THEOREM 3.4. *([**9**]) Let M be a primitive $n \times n$ matrix of normal type. Then the adjacency graph of M has a circuit of length at most $\lceil \frac{n}{2} \rceil$.*

COROLLARY 3. *([**9**]) Let $(\mathbf{A},\mathfrak{B})$ be an n- dimensional algebra with a positive basis, and let $a \in \mathbf{A}$ be nonnegative with $z(a) = 1$. Assume that $M(a,\mathfrak{B})$ is of normal type, and let m be the number of distinct eigenvalues of $M(a,\mathfrak{B})$. Then $cn(a) \leq (m-1)(\lceil \frac{n}{2} \rceil + 1)$.*

It so happens that the matrices $M(\theta, Irr(G))$ and $M(\overline{C}, class-sums(G)\ )$ are of normal type (here θ is a character of G and $C \in class(G)$). This is not true for $M(\theta, Ibr(G))$ with θ a Brauer character.

PROPOSITION 2. *([**9**]) Let G be a finite group, C a conjugacy class of G and θ an ordinary character of G. Set $M(\theta) = M(\theta, Irr(G))$ and $M(C) = M(\overline{C}, class-sums(G))$. Then $M(\theta)$ and $M(C)$ are of normal type.*

COROLLARY 4. *([**9**]). Let G be a finite group with exactly k conjugacy classes, C a conjugacy class of G and θ and ordinary character of G. Then:*

1. *If $G = G' = \langle C \rangle$ then $cn(C) \leq (m(C)-1)(\lceil \frac{k}{2} \rceil + 1)$.*
2. *If $Z(\theta) = 1$ then $ccn(\theta) \leq (m(\theta)-1)(\lceil \frac{k}{2} \rceil + 1)$.*

4. Commutators

This section is devoted to observations on so called "commutators" in $(\mathbf{A},\mathfrak{B})$ and on their covering numbers.

Let $(\mathbf{A},\mathfrak{B})$ be an algebra with a positive basis. It can be shown (see [**7**]) that the set of commuting matrices $\{M(a,\mathfrak{B}) \mid a \in \mathbf{A}\}$ can be simultaneously diagonalized, and that a diagonalizing matrix X exists such that for all nonnegative $a \in \mathbf{A}$ the (1,1) entry of $M(a,\mathfrak{B})$ is $\rho(a)$.

Let $a(1) = \rho(a), a(2), \ldots, a(n)$ be the ordering of the eigenvalues of $M(a,\mathfrak{B})$ as prescribed by X, namely:

$$X^{-1}M(a,\mathfrak{B})X = diag(a(1), a(2), \ldots, a(n)) \text{ for all } a \in \mathbf{A}.$$

EXAMPLE. In $(\mathbb{Q}\,(Irr(G))\,, Irr(G))$ the character table is such an X. For a character θ, we have that $\theta(i) = \theta(C_i)$ for all i. In $(\mathbb{Q}\,(Ibr(G))\,, Ibr(G))$ the Brauer character table is such an X. For a Brauer character θ, we have that $\theta(i) = \theta(K_i)$ for all i, where the $K_i's$ are the $p-$ regular conjugacy classes of G. In $(Z(\mathbb{Q}G), class-sums(G))$ the matrix $\left(\frac{|C_i|\chi_j(C_i)}{\chi_j(1)}\right)$ is such an X. If $C \in class(G)$ then $\overline{C}(i) = \frac{|C|\chi_i(C)}{\chi_i(1)}$ for all i.

Let ($\mathbf{A}$,$\mathfrak{B}$) be an algebra with a positive basis where $\mathfrak{B} = \{b_1 = 1_{\mathbf{A}}, b_2, ..., b_n\}$. An index i is called a commutator if $\sum_{b\in\mathcal{B}} \frac{b(i)}{b(1)}$ is not zero.

Our first observation is:

PROPOSITION 3. *Let G be a finite group, $Class(G) = \{C_1 = \{1\}, C_2, \ldots, C_k\}$, and $Irr(G) = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_k\}$. Then*

1. *i is a commutator in $(Z(\mathbb{Q}G), class - sums(G))$ if and only if C_i contains a commutator (in the usual sense) of G.*
2. *i is a commutator in $(\mathbb{Q}\left(Irr(G)\right), Irr(G))$ if and only if χ_i is a constituent of the conjugacy character.*

PROOF. 1. Here $\sum_{b\in\mathcal{B}} \frac{b(i)}{b(1)} = \sum_{\chi\in Irr(G)} \frac{\chi(C_i)}{\chi(1)}$ and the result is known (see e.g. [**14**] problem 3.10).

2. Here

$$\sum_{b\in\mathcal{B}} \frac{b(i)}{b(1)} = \sum_{C\in Class(G)} \frac{\left(\frac{|C|\chi_i(C)}{\chi_i(1)}\right)}{\left(\frac{|C|\chi_1(C)}{\chi_1(1)}\right)} = \sum_{C\in Class(G)} \frac{\chi_i(C)}{\chi_i(1)} =$$

$$\frac{1}{\chi_i(1)} \sum_{C\in Class(G)} \chi_i(C).$$

So $\sum_{b\in\mathcal{B}} \frac{b(i)}{b(1)} \neq 0$ means that $\sum\limits_{C\in Class(G)} \chi_i(C) \neq 0$.

Let c be the conjugacy character, then

$$[\chi_i, c] = \frac{1}{|G|} \sum_{C\in Class(G)} |C|\, \chi_i(C)\, |C_G(C)| = \sum_{C\in Class(G)} \chi_i(C),$$

and the result follows. □

So classes of commutator and constituents of the conjugacy character are special cases of the same object in the algebra.

In a forthcoming paper higher commutators are defined in ($\mathbf{A}$,$\mathfrak{B}$). The set of all higher commutator, A', corresponds to classes of commutators in $\mathbb{Q}\left(Irr(G)\right)$ and to the set of constituents of powers of the conjugacy character in $Z(\mathbb{Q}G)$. Also, linear elements and kernel are defined and the equality $A' = \bigcap\{\ker(b) \mid b \in \mathfrak{B},\, b \text{ is linear}\}$ is proved. This equality corresponds to the knows equality

$$G' = \bigcap\{\ker(\lambda) \mid \lambda \in Irr(G),\, \lambda(1) = 1\ \}$$

in $\mathbb{Q}\left(Irr(G)\right)$, and to the easy fact that $Irr(G/Z(G))$ is equal to the set of constituent of powers of the conjugacy character in $Z(\mathbb{Q}G)$.

Bounds of covering numbers for commutators are better than the general bounds.

THEOREM 4.1. *([**7**]). Let G be a finite group, C a conjugacy class of G and θ an ordinary character of G. Then:*

1. *Assume that $G = G' = \langle C\rangle$. If C is either a real or contains commutators, then $cn(C) \leq 2m(C) - 2$.*
2. *Assume that $Z(\theta) = 1$. If θ is either real or has a common constituent with the conjugacy character, then $cnn(\theta) \leq 2m(\theta) - 2$.*

This improves Theorems 1.3 and 1.4 as $m(C) \leq k$, and $m(\theta) \leq k$.

The significance of better bounds for commutators lies in the Ore's conjecture stating that in a nonnabelian finite simple group every element is a commutator. This conjecture was proved for almost all finite nonabelian simple groups. Ore proved it for the alternating groups ([**18**]), Neubuser, Pahlings and Cleuvers ([**17**]) for the sporadic groups and Ellers and Gordeev ([**11**]) proved it for groups of Lie type over fields with more than 8 elements. In fact Ellers and Gordeev dealt with many cases where the field has 8 elements or less (actually, a stronger conjecture (Thompson's) was verified in [**17**] and [**11**], see later on). Furthermore, the conjecture was verified for some groups of Lie type over all finite fields (see the introduction of [**11**] for a list). From this we get:

Theorem 4.2. *Let G be a finite simple group with exactly k conjugacy classes. Set $m = \max\{m(C) | C \in Class(G)\}$. Then either G is a group of Lie type over a field with less that nine elements or $cn(G) \leq 2m - 2 \leq 2k - 2$.*

As mentioned above, the bounds is known to be true for many cases of Lie groups over fields with less that nine elements as well.

For constituents of the conjugacy character, it is not true that every irreducible character of a finite simple group is a constituent of the conjugacy character, the character of degree 6 of $PSU(3,3)$ is an example (given by Frame).

QUESTION. In which finite nonnabelian simple group each irreducible character is a constituent of the conjugacy character?

The alternating groups are such groups, as shown by A.Mann (unpublished).

A stronger conjecture than Ore's (attributed to Thompson) is that every nonnabelian simple group G contains a conjugacy class C such that $C^2 = G$. It easy to see that Ore's conjecture follows from Thompsons'. A "dual" (equally easy to see) statement is that if a finite simple G group has an irreducible character χ such that $Irr(\chi^2) = Irr(G)$ than every irreducible character of G is a constituent of the conjugacy character.

QUESTION. Which finite nonnabelian simple groups G have irreducible character χ such that $Irr(\chi^2) = Irr(G)$?

An element g of the group G is a commutator if and only $g \in C^{-1}C$ for some conjugacy C of G. The number of such C 's influences $cn(class(g))$.

Similarly, for a character θ to have a common constituent with the conjugacy character, means that θ has a common constituent with $\chi\overline{\chi}$ for some $\chi \in Irr(G)$. The number of such χ's influences $ccn(\theta)$.

Theorem 4.3. *([**9**]). Let G be a finite group with exactly k conjugacy classes.*

1. *Let D be a conjugacy class such that $< C >= G' = G$. Assume that there are d conjugacy classes C such that $D \subseteq C^{-1}C$. Then $cn(D) \leq 2k-1-d$.*
2. *Let θ be an ordinary character such that $Z(\theta) = 1$. Assume that there are d irreducible characters χ such that $[\theta, \chi\overline{\chi}] \neq 0$. Then $ccn(\theta) \leq 2k-1-d$.*

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