Finite groups with extremal conditions on sizes of conjugacy classes and on degrees of irreducible characters

David Chillag
Department of Mathematics
Technion, Israel Institute of Technology
Haifa 32000, Israel
and
Marcel Herzog
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Tel-Aviv, Israel

I Introduction

In this survey G denotes a finite group of order g, with k conjugacy classes and center $Z(G)$ of order z. Denote the order of G', the commutator subgroup of G, by g' and assume that $g > 1$. Denote by $Cls(G) = \{c_1 = 1, c_2, \ldots, c_k\}$ the multiset composed of the sizes of the conjugacy classes of G (with $c_1 = |\{1\}|$) and by $Chd(G) = \{x_1 = 1, x_2, \ldots, x_k\}$ the multiset composed of the degrees of the irreducible characters of G (with $x_1 = 1_G(1)$). The influence of the arithmetical structure of the c_i's and the x_i's on the group-theoretical structure of G has been investigated in many papers. For example, the following results concerning the class sizes in G were proved in [4]. Here, and in the sequel, by "a class" we mean "a conjugacy class" and by "a prime" we mean "a non-necessarily fixed prime". For additional information, see [4], [5], [6] and [8].

Theorem 1 *The following statements hold:*

1. *If c_i equals 1 or a prime for each $c_i \in Cls(G)$, then either G is nilpotent of class ≤ 2 or $G/Z(G)$ is a Frobenius group of order pq, where p and q are distinct primes.*

2. *If c_i equals 1 or a prime power for each $c_i \in Cls(G)$, then either G is nilpotent or $G/Z(G)$ is a solvable Frobenius group.*

3. *If c_i is a squarefree number for each $c_i \in Cls(G)$, then G is supersolvable and $dl(G) \leq 3$.*

4. *If p is a fixed prime and $p \nmid c_i$ for each $c_i \in Cls(G)$, then the Sylow p-subgroup of G is central in G.*

5. *If p is a fixed prime and $p \mid c_i$ for each $c_i \in Cls(G)$ satisfying $c_i \neq 1$, then $C_G(P) \leq Z(G)$ for each Sylow p-subgroup P of G.*

6. *If $4 \nmid c_i$ for each $c_i \in Cls(G)$, then G is solvable.*

Similar results were obtained by many authors concerning the irreducible character degrees of G. We summarize the best known ones in the following omnibus theorem. The names of the authors and the references can be found in [4], [8] and [9].

Theorem 2 *The following statements hold:*

1. *If x_i equals 1 or a prime for each $x_i \in Chd(G)$, then G is solvable and $dl(G) \leq 3$.*

2. *If x_i equals 1 or a prime power for each $x_i \in Chd(G)$, then G is not necessarily solvable. If G is assumed to be solvable, then $dl(G) \leq 5$.*

3. *If x_i is a squarefree number for each $x_i \in Chd(G)$, then again G is not necessarily solvable. If G is assumed to be solvable, then $dl(G) \leq 4$.*

4. *If p is a fixed prime and $p \nmid x_i$ for each $x_i \in Chd(G)$, then G has a normal abelian Sylow p-subgroup.*

5. *If p is a fixed prime and $p \mid x_i$ for each $x_i \in Chd(G)$ satisfying $x_i \neq 1$, then G has a normal p-complement.*

6. *If $4 \nmid x_i$ for each $x_i \in Chd(G)$, then either G is solvable or $G/N \cong A_7$ for some solvable $N \lhd G$.*

II Extremal conditions

In this note we wish to consider groups satisfying some extremal conditions with respect to the sizes of $Cls(G)$ and $Chd(G)$ viewed **as sets**. It is clear that either of the conditions $|Cls(G)| = 1$ (i.e. all classes of G are of size 1) and $|Chd(G)| = 1$ (i.e. all irreducible characters of G are linear) is equivalent to G being abelian. Therefore we shall deal with nonabelian groups G satisfying one of the following conditions:

(1) $|Cls(G)| = 2$ (i.e. all noncentral classes are of the same size);

(2) $|Chd(G)| = 2$ (i.e. all nonlinear irreducible characters are of the same degree);

(3) $|Cls(G)| = k$ (i.e. all classes are of distinct sizes);

(4) $|Chd(G)| = k$ (i.e. all irreducible characters are of distinct degrees).

We shall consider two additional conditions with a similar flavor. We first define $Cls^*(G)$ as the set of sizes of the noncentral classes of G and $Chd^*(G)$ as the set of degrees of the nonlinear irreducible characters of G. The remaining conditions are:

(5) $|Cls^*(G)| = k - z$ (i.e. all noncentral classes are of distinct sizes); and

(6) $|Chd^*(G)| = k - g/g'$ (i.e. all nonlinear irreducible characters are of distinct degrees).

Some additional conditions will be considered in the next section, which deals with groups of odd order.
We shall consider now each case separately.

(1) $|Cls(G)| = 2$.

In this case, it was shown by N.Ito [11] that G is nilpotent with a unique non-abelian Sylow p-subgroup, which implies that the class sizes are 1 and p^r for some integer $r > 0$. Thus the problem reduces to a p-group problem and is **still open**, as far as a complete classification is considered. L.Verardi [14] showed that $G/Z(G)$ is of exponent p and recently K.Ishikawa [10] proved that $cl(G) \leq 3$. For Isaacs' simpler proof of Ishikawa's result see [1] and for Mann's generalization see [13].

(2) $|Chd(G)| = 2$.

This problem is also **still open**. Let the character degrees of G be 1 and m. Isaacs and Passman proved that G' is abelian and either G is nilpotent, with $m = p^a$ for some prime p, or there exists an abelian normal subgroup A of G with $[G : A] = m$ (see [9, Chapter 12]). Recently, the nonnilpotent groups were completely classified by Bianchi, Gillio, Herzog, Qian and Shi in [3]. They proved:

Theorem 3 *Let G be a nonnilpotent group. Then the irreducible character degrees of G are 1 and $m > 1$ if and only if G' is abelian and one of the following holds.*

1. *$m = p$, a prime, and there exists an abelian normal subgroup A of G with $[G : A] = p$.*

2. *$G' \cap Z(G) = 1$ and $G/Z(G)$ is a Frobenius group with the kernel $G' \times Z(G)/Z(G)$ and a cyclic complement of order $[G : G' \times Z(G)] = m$.*

(3) $|Cls(G)| = k$.

This problem is **still open** in the nonsolvable case. In the solvable case it was proved by Zhang [15] and independently by Knörr, Lempken and Thielcke [12] that $G \cong S_3$. It is generally believed that no nonsolvable group satisfies this condition.

(4) $|Chd(G)| = k$.

No such group $G \neq 1$ exists, since it was proved in [2] that groups with nonlinear irreducible characters of distinct degrees are solvable (see Theorem 5 below) and hence contain more than one linear character.

(5) $|Cls^*(G)| = k - z$.
This condition implies that $z = 1$ by the following result of Herzog and Schönheim in [7]:

Theorem 4 *Let G be a nonabelian finite group and suppose that G contains at most two noncentral classes of each size. Moreover, suppose that if $x, y \in G - Z(G)$ and the classes of x and y are of equal sizes, then x and y are of the same order. Then $Z(G) = 1$.*

Hence conditions (3) and (5) are identical.
(6) $|Chd^*(G)| = k - g/g'$.
In this case, Berkovich, Chillag and Herzog proved in [2] the following classification theorem.

Theorem 5 *Suppose that G is a nonabelian group with nonlinear irreducible characters of distinct degrees. Then G is one of the following groups:*

1. *An extraspecial 2-group of order 2^{2m+1}, with a unique nonlinear irreducible character of degree 2^m.*

2. *A Frobenius group of order $p^n(p^n - 1)$ for some prime p, with an elementary abelian kernel G' of order p^n, a cyclic complement and a unique nonlinear irreducible character of degree $p^n - 1$.*

3. *The Frobenius group of order 72, with a complement isomorphic to the quaternion group of order 8 and two nonlinear irreducible characters of degrees 2 and 8.*

III Nonabelian groups of odd order

If G is a nonabelian group of odd order g, then it has at least two nonidentity classes of each size and at least two nonprincipal irreducible characters of each degree. Therefore the extremal conditions (3)-(6) correspond to the following conditions:

(7) $|Cls(G)| = \frac{k+1}{2}$ (i.e. there are exactly two nonidentity classes of each size);

(8) $|Chd(G)| = \frac{k+1}{2}$ (i.e. there are exactly two nonprincipal irreducible characters of each degree);

(9) $|Cls^*(G)| = \frac{k-z}{2}$ (i.e. there are exactly two noncentral classes of each size); and

(10) $|Chd^*(G)| = \frac{kg'-g}{2g'}$ (i.e. there are exactly two nonlinear irreducible characters of each degree).

We shall consider now each case separately. First we notice that cases (7) and (9) are identical. Indeed, if G satisfies (7), then it clearly satisfies (9) and if G satisfies (9), then by Theorem 4 $z = 1$ and G satisfies (7). Thus it suffices to consider G satisfying

(9) $|Cls^*(G)| = \frac{k-z}{2}$.

This problem was solved in [7]. They proved:

Theorem 6 *Let G be a nonabelian group of odd order. Then G has exactly two noncentral classes of each size if and only if G is the nonabelian group of order* 21.

It remains to deal with conditions (8) and (10). We begin with

(10) $|Chd^*(G)| = \frac{kg'-g}{2g'}$.

Also this problem was solved recently. In [4], Chillag and Herzog proved:

Theorem 7 *Let G be a nonabelian group of odd order. Then G has exactly two nonlinear ireducible characters of each degree if and only if G is one of the following groups:*

1. *An extraspecial* 3*-group, with exactly two nonlinear irreducible characters of degree* $\sqrt{\frac{|G|}{3}}$.

2. *A Frobenius group of odd order $\frac{p^n-1}{2}p^n$ for some odd prime p, with an abelian kernel G' of order p^n and exactly two nonlinear irreducible characters of degree $\frac{p^n-1}{2}$.*

Finally, consider the case

(8) $|Chd(G)| = \frac{k+1}{2}$.

By the Feit-Thompson theorem, groups of odd order are solvable and consequently G must satisfy condition (10) and $g/g' = 3$. Hence Theorem 7 implies that $g = \frac{p^n-1}{2}p^n$ for some odd prime p, $g' = p^n$ and $(p^n - 1)/2 = 3$. Thus we get

Theorem 8 *Let G be a nonabelian group of odd order. Then G has exactly two nonprincipal irreducible characters of each degree if and only if G is the nonabelian group of order* 21.

The irreducible character degrees of such G are $\{1, 1, 1, 3, 3\}$.

References

[1] Y. Barnea and I.M. Isaacs,*Lie algebras with few centralizer dimensions*, J. Algebra **259** (2003), 284–299.

[2] Y. Berkovich, D. Chillag and M. Herzog, *Finite groups in which the degrees of the nonlinear irreducible characters are distinct*, Proc. Amer. Math. Soc. **115** (1992), 955–959.

[3] M. Bianchi, A. Gillio, M. Herzog, G. Qian and W. Shi, *Characterization of non-nilpotent groups with two irreducible character degrees*, J. Algebra (to appear).

[4] D. Chillag and M. Herzog, *Finite groups with almost distinct character degrees*, J. Algebra (to appear).

[5] S. Dolfi, *Prime factors of conjugacy class lengths and irreducible character-degrees in finite solvable groups*, J. Algebra **174** (1995), 749-752.

[6] S. Dolfi, *Arithmetical conditions on the lengths of the conjugacy classes of a finite group*, J. Algebra **174** (1995), 753-771.

[7] M. Herzog and J. Schönheim, *Groups of odd order with exactly two non-central conjugacy classes of the same size*, Arch. Math. (to appear).

[8] B. Huppert, *Character Theory of Finite Groups*, Walter de Gruyter, Berlin (1998).

[9] I.M. Isaacs, *Character Theory of Finite Groups*, Dover Publications, Inc., New-York (1994).

[10] K. Ishikawa, *On finite p-groups which have only two conjugacy lengths*, Israel J. Math., **129** (2002), 119–123.

[11] N. Ito *On finite groups with given conjugate types I*, Nagoya Math. J.**6** (1953), 17–28.

[12] R. Knörr, W. Lempken and B. Thielcke, *The* S_3 *conjecture for solvable groups*, Israel J. Math. **91** (1995), 61–76.

[13] A. Mann, *Minimal elements in finite p-groups*, Personal communication.

[14] L. Verardi, *On groups whose noncentral elements have the same finite number of conjugates*, Bull. Unione Mat. Italia (7) **2A** (1988), 391–400.

[15] J.P. Zhang, *Finite groups with many conjugate elements*, J. Algebra **170** (1994), 608–624.