Finite groups with restrictions on the zero sets of their irreducible characters.

David Chillag
Department of mathematics, Technion,
Israel Institute of Technology.
Haifa 32000. ISRAEL.
chillag@techunix.technion.ac.il

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Abstract

It is known that certain restrictions on the the character degrees, or the sizes of the conjugacy classes of a group, imply certain structural properties of that group.

In this paper we study the effect on the structure of a group, of similar restrictions on the zero sets of its irreducible characters.

1 Introduction

It is known that certain conditions concerning the degrees of the characters, or the sizes of the conjugacy classes of a finite group have very restrictive implications for the structure of the group. Examples of results of this kind can be found, e.g., in Chapters 27, 31, 32, 33 of [8]. In this article we obtain analogous results concerning the implications of some similar conditions on the zero sets of the irreducible characters of a group on its structure.

Let G be a finite group and θ an irreducible character of G. The set $O_G(\theta) = \{g \in G \mid \theta(g) = 0\}$ is called the zero set of θ . A theorem of Burnside states that if θ is nonlinear then $O_G(\theta)$ is not empty.

Our first result determines all finite groups which have the property that any two distinct nonlinear irreducible characters have distinct zero sets (See Theorem 1.1). It turns out that these groups can also be characterized by the property that distinct nonlinear irreducible characters have distinct degrees. This is explained further in the remarks following Theorem 1.1.

Our second result (in some sense "dual" to the first) states that S_3 is the only nonabelian finite group for which the number of irreducible characters vanishing on a noncentral conjugacy class is different for each such class (See Theorem 1.2).

Finally, we study groups G which satisfy a conditions which is at the opposite extreme to the one considered in our first result, namely finite groups in which all nonlinear characters have the same zero sets. We show that such groups are solvable and have a normal p - complement for some prime dividing $|G : G'|$.

Theorem 1.1 *Let G be a finite nonabelian group such that $O_G(\theta) \neq O_G(\eta)$ for every two distinct nonlinear irreducible characters θ and η of G. Then one of the following holds:*

1. *G is an extra special 2 - group.*
2. *G is a Frobenius group of order $p^n(p^n - 1)$ for some prime power p^n, with a cyclic Frobenius complement of order $(p^n - 1)$.*
3. *G is a Frobenius group of order 72 with Frobenius complement isomorphic to the quaternion group of order 8.*

REMARKS.

1. In cases 1 and 2 G has exactly one nonlinear irreducible character.
2. The condition "$O_G(\theta) \neq O_G(\eta)$ for every two distinct nonlinear irreducible characters θ and η of G" is equivalent to the condition "$\theta(1) \neq \eta(1)$ for every two distinct nonlinear irreducible characters θ and η of G". This is because the latter condition yields the same list of groups as the former, as shown in [8] Theorem 32.9, p. 437.
3. The classification of the finite simple groups is used (via the Feit-Seitz characterization of simple rational groups) for the case $G = G'$. It is not used in case $G \neq G'$.

The proof of Theorem 1.1 can be found in Section 2.

Let $a \in G$ (G a finite group). Set $V_G(a) = \{\chi \in Irr(G) | \chi(a) = 0\}$. If C is conjugacy class of G and $b \in C$ we use the notation $V_G(C) = V_G(b)$.

Theorem 1.2 *Let G be a finite group such that $|V_G(C)| \neq |V_G(D)|$ for every two distinct noncentral conjugacy classes C and D of G. Then either G is abelian or $G \simeq S_3$.*

REMARK. It is conjectured that S_3 is the only finite group for which the condition $|V_G(C)| \neq |V_G(D)|$ (in Theorem 1.2) is replaced by $|C| \neq |D|$. The conjecture is proved for solvable groups in [16] and independently in [12]

The proof of Theorem 1.2 is in Section 3. In Section 4 we consider finite groups in which all nonlinear characters have the same zero sets. We show that they are solvable and have a normal p - complement for some prime dividing $|G : G'|$.

In addition to standard notation taken mainly from [10] we will use the following notation: Lin(G) will stand for the set of all linear characters of G.

2 Groups in which distinct nonlinear characters have different zero sets.

For the proof of Theorem 1.1 we need few lemmas.

Lemma 2.1 *Let $k = p^n$ be a prime power and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \varepsilon_m$ be (not necessarily distinct) kth roots of unity. Assume that $\sum_{i=1}^{m} \varepsilon_i = 0$. Then p divides m.*

Proof. Let α be a primitive kth root of unity, then each ε_i is a power of α and the sum $\sum_{i=1}^{m} \varepsilon_i = 0$ can be written as

$$a_0 \cdot 1 + a_1\alpha + a_2\alpha^2 + \ldots + a_{p^n-p^{n-1}-1}\alpha^{p^n-p^{n-1}-1} + a_{p^n-p^{n-1}}\alpha^{p^n-p^{n-1}} + \\ + a_{p^n-p^{n-1}+1}\alpha^{p^n-p^{n-1}+1} + \ldots + a_{p^n-2}\alpha^{p^n-2} + a_{p^n-1}\alpha^{p^n-1} = 0 \qquad (1)$$

where the a_i' s are nonnegative integers and $m = \sum_{i=0}^{k-1} a_i$. Next $\alpha^{p^{n-1}}$ is a primitive pth root of unity and so

$$\alpha^{p^n-p^{n-1}} = -1 - \alpha^{p^{n-1}} - \alpha^{2p^{n-1}} - \alpha^{3p^{n-1}} - \ldots - \alpha^{p^n-2p^{n-1}} \qquad (2)$$

We now multiply equation (2) by $1, \alpha,\ \alpha^2,\ \alpha^3, \ldots, \alpha^{p^{n-1}-2},\ \alpha^{p^{n-1}-1}$ respectively, and subtitute the rusulting equalities into (1). We get

$$
\begin{aligned}
&a_0 \cdot 1 + a_1\alpha + a_2\alpha^2 + \ldots + a_{p^n-p^{n-1}-1}\alpha^{p^n-p^{n-1}-1} + \\
&a_{p^n-p^{n-1}}\left(-1 - \alpha^{p^{n-1}} - \alpha^{2p^{n-1}} - \alpha^{3p^{n-1}} - \ldots - \alpha^{p^n-2p^{n-1}}\right) + \\
&a_{p^n-p^{n-1}+1}\left(-\alpha - \alpha^{p^{n-1}+1} - \alpha^{2p^{n-1}+1} - \alpha^{3p^{n-1}+1} - \ldots - \alpha^{p^n-2p^{n-1}+1}\right) + \\
&+ \ldots + \qquad (3)\\
&a_{p^n-2}\left(-\alpha^{p^{n-1}-2} - \alpha^{2p^{n-1}-2} - \alpha^{3p^{n-1}-2} - \alpha^{4p^{n-1}-2} - \ldots - \alpha^{p^n-p^{n-1}-2}\right) + \\
&a_{p^n-1}\left(-\alpha^{p^{n-1}-1} - \alpha^{2p^{n-1}-1} - \alpha^{3p^{n-1}-1} - \alpha^{4p^{n-1}-1} - \ldots - \alpha^{p^n-p^{n-1}-1}\right) = 0.
\end{aligned}
$$

Equation (3) is a zero linear combination with rational coefficients of $1, \alpha, \alpha^2, \ldots, \alpha^{p^n-p^{n-1}-1}$. However the minimal polynomial of α over the rational has degree $\phi(p) = p^n - p^{n-1}$ (see [11] p. 273). It follows that the coefficient of every power of α in (3) is equal to zero. In addition (3) shows which powers of α have the same coefficient. It follows (noting that $p^n - 2p^{n-1} = (p-2)p^{n-1}$ and $p^n - p^{n-1} = (p-1)p^{n-1}$) that the "set" $\{a_i \mid i = 0, 1, 2, \ldots, p^{n-1}\}$ is a disjoint union of "subsets" containing p identical $a_i' \, s$ each. Hence $m = \sum_{i=0}^{k-1} a_i$ is divisible by p. ■

Lemma 2.2 *Let G be a finite nonabelian group and M a normal subgroup of G. Let θ a be nonlinear irreducible character of G/M. Then*

1. $O_{G/M}(\theta) = \{xM \mid x \in O_G(\theta)\}$.
2. $O_G(\theta) = \bigcup \{xM \mid x \in O_G(\theta)\}$.

Proof. Let $x \in G$. Then $xM \in O_{G/M}(\theta) \iff \theta(xM) = 0 \iff \theta(xm) = \theta(x) = 0$ for all $m \in M \iff x \in O_G(\theta)$. This proves 1. Next, if $y \in \bigcup \{xM \mid x \in O_G(\theta)\}$ then $y = xm$ with $\theta(x) = 0$ and $m \in M \subseteq \ker(\theta)$. So $\theta(y) = 0$ and $y \in O_G(\theta)$. Thus the union is contained in $O_G(\theta)$. The other containment of 2. is trivial as $x \in xM$ for all $x \in O_G(\theta)$. ■

Corollary 2.3 *Let G be a finite nonabelian group such that $O_G(\theta) \neq O_G(\eta)$ for every two distinct nonlinear irreducible characters θ and η of G. Let M be a normal subgroup of G. Then $O_{G/M}(\varphi) \neq O_{G/M}(\psi)$ for every two distinct nonlinear irreducible characters φ and ψ of G/M.*

Proof. Suppose that $O_{G/M}(\varphi) = O_{G/M}(\psi)$ for some nonlinear φ and ψ in $Irr(G/M)$. Then $\{xM \mid x \in O_G(\varphi)\} = \{xM \mid x \in O_G(\psi)\}$ by 1. of the previous lemma, and by the 2. of the previous lemma we get $O_G(\varphi) = O_G(\psi)$, contradicting the assumption on G. ■

Lemma 2.4 *Let G be a Frobenius group with an abelian Frobenius kernel of order m. Assume that $O_G(\theta) \neq O_G(\eta)$ for any two distinct nonlinear irreducible characters θ and η of G. Then $m = p^n$ and $|G| = p^n(p^n - 1)$ for some prime p and a positive integer n.*

Proof. Let K be the Frobenius kernel and H a Frobenius complement. Let p be a prime divisor of $m = |K|$ and P a Sylow p - subgroup of K. We write $K = P \times M$ where M is the p - complement of K. Then M is normal in G and G/M is a Frobenius group with Frobenius kernel isomorphic to P and Frobenius complement isomorphic to H ([14]12.6.6 p. 351). Denote images modulo M by bars, then $\bar{G} = \bar{K}\bar{H}$ where $\bar{K} \simeq P$ is the Frobenius kernel and $\bar{H} \simeq H$ is a Frobenius complement. Set $|P| = \left|\bar{K}\right| = p^n$ and let $\lambda \in Irr(\bar{K}) - \{1_{\bar{K}}\}$. As $\bar{K}$ is abelian, λ is linear and hence $\lambda(\bar{a})$ is a p^n th root of unity for every $\bar{a} \in \bar{K}$. Next, the action of $\bar{H}$ on $Irr(\bar{K}) - \{1_{\overline{K}}\}$ is fixed-point-free, so λ has exactly $|H|$ $\bar{G}$-conjugates. Now $\lambda^{\bar{G}} \in Irr(\bar{G})$ (see [8], Theorem 18.7, p.239) and for every $\bar{g} \in \bar{K}$ we have (by Clifford's theorem) that $\lambda^{\bar{G}}(\bar{g}) = e\sum_{\bar{h}\in\bar{H}} \lambda^{\bar{h}}(\bar{g})$ for some positive integer e. Now $|H|$ divides $p^n - 1$ and so $(|H|, p) = 1$. Therefore $\lambda^{\bar{G}}(\bar{g})$ is e times a sum of p^nth roots of unity and the number of these roots of unity is not divisible p. Lemma 2.1 now implies that $\lambda^{\bar{G}}(\bar{g}) \neq 0$ for every $\bar{g} \in \bar{K}$. Hence $O_{\bar{G}}(\lambda^{\bar{G}}) = \bar{G} - \bar{K}$ for all $\lambda \in Irr(\bar{K}) - \{1_{\bar{K}}\}$. However Lemma 2.3 implies that $O_{\bar{G}}(\alpha^{\bar{G}}) \neq O_{\bar{G}}(\beta^{\bar{G}})$ for distinct $\alpha, \beta \in Irr(\bar{K}) - \{1_{\bar{K}}\}$. It follows (again [8], Theorem 18.7, p.239) that all the elements of $Irr(\bar{K}) - \{1_{\bar{K}}\}$ are $\bar{H}-$ conjugates and that $\left|Irr(\bar{K}) - \{1_{\bar{K}}\}\right| = \left|\bar{H}\right| = |H|$. As $\bar{K}$ is abelian $\bar{K} \simeq Irr(\bar{K})$ and consequently $|H| = |P| - 1$. Since p was arbitrary, $|H| + 1$ is p - power for every prime divisor p of $|K|$. So $|K| = p^n$ a prime power, $|H| = p^n - 1$, $M = 1$ and $|G| = p^n(p^n - 1)$ as claimed. ■

Lemma 2.5 *Let G be a Frobenius group with a cyclic Frobenius complement. Assume that $O_G(\theta) \neq O_G(\eta)$ for any two distinct nonlinear irreducible characters θ and η of G. Then the Frobenius kernel must be abelian.*

Proof. Let K be the Frobenius kernel and H a Frobenius complement. We will show that K has to be abelian. To do that we assume the contrary and reach a contradiction. First, let $\bar{G} = G/K'$ and denote images modulo K' by bars. Then $\bar{G}$ is a Frobenius group with abelian Frobenius kernel $\bar{K} = K/K'$ and a Frobenius complement $\bar{H} = HK'/K' \simeq H$. By Lemma 2.4 (and using Lemma 2.3) $\left|\bar{K}\right| = p^n$ and $|H| = p^n - 1$. Since K is nilpotent (as a Frobenius kernel) $\left|\bar{K}\right|$ is divisible by every prime divisor of $|K|$, thus K is a p - group. In fact, since K is not abelian, $|H| = p^n - 1$ is odd (see [8] Theorem 16.7, p. 203) so that $p = 2$.

Furthermore, all nonidentity elements of $\bar{K}$ are conjugate under the action of $\bar{H}$ so that $\bar{K}$ is an elementary abelian p - group. It follows that the elements of $Irr(\bar{K}) - \{1_{\bar{K}}\}$ are conjugate under the action of $\bar{H}$. Hence all elements of $Lin(K) - \{1_K\}$ are conjugate under the action of H.

Next let $X = G/[K,K,K]$. Then X is Frobenius group with Frobenius kernel $Y = K/[K,K,K]$ and a Frobenius complement $W \simeq H$. Moreover $Y' = K'/[K,K,K]$ and $Y/Y' \simeq \bar{K}$ is an elementary abelian 2 - group. Clearly the nilpotency class of Y is equal to 2 and hence $Z(Y) \supseteq Y'$. Recall that $|W| = 2^n - 1$ and $|Y/Y'| = 2^n$. Since $X/Z(Y)$ is a Frobenius group with Frobenius kernel $Y/Z(Y)$ and Frobenius complement of order $2^n - 1$ we have that $2^n - 1$ divides $|Y/Z(Y)| - 1$. Hence $|Y/Z(Y)| \geq 2^n = |Y/Y'|$ and so $|Z(Y)| \leq |Y'|$. But $Z(Y) \supseteq Y'$ so $Z(Y) = Y'$. As $Lin(Y) \simeq Y/Y'$ is of order 2^n, all elements of $Lin(Y) - \{1_Y\}$, as well as all nonidentity elements of $Y/Y' = Y/Z(Y)$ are conjugate under the action of W. Set $W = \{w_1 = 1, w_2, w_{3,\dots,}w_{2^n-1}\}$. Let $y \in Y - Z(Y)$, then $Y/Z(Y) - \{1\} = \left\{ w_i y w_i^{-1} Z(Y) \mid i = 1,2,3,\dots,2^n - 1\right\}$.

Let $\lambda \in Lin(Y) - \{1_Y\}$, then $\lambda^X \in Irr(X)$ with $O_X(\lambda^X) \supseteq X - Y$. Let $a \in Y$, then $\lambda^X(a) = e\sum_{w\in W}\lambda^w(a)$. (the fixed point free action of W on $Lin(Y)$ implies that there are $2^n - 1$ summands in this sum). On the other hand each $\lambda^w(a) = \lambda(waw^{-1})$ is a $|Y|$th root of unity and $|Y|$ is a 2 - power. Now Lemma 2.1 implies that $\lambda^X(a) \neq 0$. Thus $O_X(\lambda^X) = X - Y$. Note also $\lambda^X = \mu^X$ for all λ, $\mu \in Lin(Y) - \{1_Y\}$.

Let θ an arbitrary nonlinear element of $Irr(Y)$. Again $\theta^X \in Irr(X)$. Clearly $Z(\theta) \supseteq Z(Y)$. Let $b \in Z(Y) \subseteq Z(\theta)$. As $Z(Y) \lhd X$, $wbw^{-1} \in Z(Y) \subseteq Z(\theta)$ for all $w \in W$. Now $\theta(b) = \theta(1)\beta(b)$ for some $\beta \in Lin\,(Z(\theta))$ (see [10] p. 27), so $\beta(b)$ is a $|Y|$th root of unity. Next

$$\theta^X(b) = f \sum_{w\in W}\theta(wbw^{-1}) = f\ \theta(1) \sum_{w\in W}\beta(wbw^{-1}),$$

where f is a positive integer. The fixed point free action of W on $Z(Y)$ (and hence on $Lin(Z(Y))$) implies that $\sum_{w\in W}\beta(wbw^{-1})$ is a sum of $2^n - 1$ $|Y|$th roots of unity, $|Y|$ being a 2 - power. Lemma 2.1 implies that $\theta^X(b) \neq 0$. This means that $X - Y \subseteq O_X(\theta^X) \subseteq X - Z(Y)$.

Clearly $Z(Y) = Y' \subseteq Z(\theta)$ so $Y/Z(\theta)$ is abelian. By ([10] p.27,28) there exists a linear character α of $Z(\theta)$ such that for every $g \in Y$ we have

$$\theta(g) = \begin{cases} 0 \ \text{ if } g \notin Z(\theta) \\ \theta(1)\alpha(g) \ \text{ if } g \in Z(\theta) \end{cases} . \tag{4}$$

Here $\alpha \in Lin((Z(\theta))$.

We claim that $Z(\theta) = Z(Y)$. Assume the contrary that $Z(\theta) \supsetneq Z(Y)$. For every $y \in Y - Z(Y)$ we let

$$A(y) = \left\{ w_i y w_i^{-1} \mid w_i y w_i^{-1} \in Z(\theta) \right\},$$
$$\bar{A}(y) = \left\{ w_i y w_i^{-1} Z(Y) \mid w_i y w_i^{-1} Z(Y) \in Z(\theta)/Z(Y) \right\}.$$

We have already seen that $W = \{w_1 = 1, w_2, w_{3,\ldots,} w_{2^n-1}\}$ acts fixed point freely on Y and $WZ(Y)/Z(Y) \cong W$ acts with no fixed points on nonidentity elements of $Y/Z(Y)$. Hence the map $w_i y w_i^{-1} \ \rightarrow w_i y w_i^{-1} Z(Y)$ from $A(y)$ to $\bar{A}(y)$ is one-to-one and onto. It follows that $|A(y)| = \left|\bar{A}(y)\right|$. Furthermore, since $Y/Z(Y) - \{1\} = \left\{ \ w_i y w_i^{-1} Z(Y) \mid i = 1, 2, 3, \ldots, 2^n - 1 \right\}$ we have that $\bar{A}(y) = Z(\theta)/Z(Y) - \{1\}$. As $|Z(\theta)/Z(Y)|$ is a 2 - power, we conclude that $\left|\bar{A}(y)\right| = |A(y)|$ is odd.

Next, for any $y \in Y - Z(Y)$ and $w \in W$ equality (5) implies that $\theta(wyw^{-1}) = 0$ for $wyw^{-1} \notin A(y)$. It follows that

$$\theta^X(y) = f \sum_{wyw^{-1} \in A(y)} \theta(wyw^{-1}) = f\theta(1) \sum_{wyw^{-1} \in A(y)} \alpha(wyw^{-1}) \tag{5}$$

where $\alpha \in Lin(Z(\theta))$. Again, the fixed-point-free action implies that the number of summands is exactly $|A(y)|$ which is odd. Now, each $\alpha(wyw^{-1})$ is a $|Y|$th root of unity. As $|Y|$ is a 2 - power we get by Lemma 2.1 that $\theta^X(y) \neq 0$.

We have already shown that $\theta^X(b) \neq 0$ for $b \in Z(Y)$ so $O_X(\theta^X) = X - Y$. As $O_X(\lambda^X) = X - Y$ for $\lambda \in Lin(Y) - \{1_Y\}$ our assumption (and Lemma 2.3) imply that $\theta^X = \lambda^X$ for $\lambda \in Lin(Y) - \{1_Y\}$. This is a contradiction since $\lambda^X(1) = |W| < \theta(1)\,|W| = \theta^X(1)$.

We conclude that $Z(\theta) = Z(Y) = Y'$ for all nonlinear $\theta \in Irr(Y)$. By equality (5) every nonlinear character of Y vanishes on $Y - Z(Y) = Y - Y'$. Take $y \in Y - Y'$, by the second orthogonality relations we know that

$$2^n = \left|C_{Y/Y'}(yY')\right| = |C_Y(y)|.$$

However, $Z(Y) = Y' < \langle Y', y \rangle \subseteq C_Y(y)$. This implies that $|Z(Y)| \leq 2^n - 1$. This is impossible as each nonidentity element of $Z(Y)$ must have exactly $|W| = 2^n - 1$ W - conjugates, all of them must lie in $Z(Y)$. This final contradiction finishes the proof. ■

DEFINITION. Let G be a finite group and N a proper nonidentity normal subgroup of G. If $|C_G(x)| = \left|C_{G/N}(xN)\right|$ for all $x \in G - N$ then we call the pair (G, N) a Camina pair and N the Camina kernel. We will use properties of camina pairs from [2], [4], [5] and [13]. In the literature Camina pairs are also referred to as F2-pairs. Clearly the Camina pair condition is equivalent to the condition that all irreducible characters of G not containing N in their kernel vanish outside N (second orthogonality relations).

Proof. OF THEOREM 1.1 Let $\chi \in Irr(G) - Lin(G)$. Let ξ be a primitive $|G|$th root of unity and $\sigma \in Gal\left(\mathbb{Q}(\xi)/\mathbb{Q}\right)$. Then clearly $O_G(\chi) = O_G(\chi^\sigma)$ so that $\chi = \chi^\sigma$, and χ must be rational.

Suppose first that $G = G'$. Then G has a normal subgroup N such that G/N is a nonabelian simple group. By the above, G/N is a rational group and by [6] $G/N \cong S_2(6)$ or $O_8^+(2)$. Looking in [1] we see that each of this groups has two characters θ and η with $O_{G/N}(\theta) = O_{G/N}(\eta)$ (in the atlas notation $\{\theta, \eta\} = \{\chi_{17}, \chi_{18}\}$ for $S_2(6)$ and $\{\theta, \eta\} = \{\chi_{35}, \chi_{36}\}$ for $O_8^+(2)$). This contradicts Lemma 2.3.

Therefore $G > G'$, so that $Lin(G) \neq \{1_G\}$. Let $\lambda \in Lin(G) - \{1_G\}$ and $\chi \in Irr(G) - Lin(G)$. Since $O_G(\lambda\chi) = O_G(\chi)$ we have that $\lambda\chi = \chi$ so that χ vanishes on $G - \ker(\lambda)$ for all $\lambda \in Lin(G)$. Therefore every element of $Irr(G) - Lin(G)$ vanishes on $G - \bigcap\{\ker(\lambda) \mid \lambda \in Lin(G)\} = G - G'$. Consequently (G, G') is a Camina pair in which G' is the kernel. It follows from[2] and [5](Corollary p. 788) that there are 4 possibilities: i. G' is a p - group for some prime p, ii. G is a p - group for some prime p. iii. G is a Frobenius group with G' the Frobenius kernel, iv. G is a Frobenius group with a Frobenius complement isomorphic to Q_8 (the quaternion group of order 8) and $|G : G'| = 4$. We now deal with each of these cases separately.

If G' is a p - group for some prime p, and G is not a Frobenius group with G' the Frobenius kernel, then Lemma 4.4 of [4] implies that $O_{p'}(G/G') = 1$. But G/G' is abelian and consequently G/G' must be a p - group. So G is a p - group and case i. is identical to case ii. We now show that in this case conclusion 1 of Theorem 1.1 holds. First, as G is not abelian, G has a nonlinear characters which, by the above, must be rational. Hence G has even order so that G is a 2 - group. A theorem of Macdonald [13](Theorem 3.1) implies that the nilpotency class of G is equal to 2. Therefore $G' \subset Z(G)$. Since no irreducible character is zero on $Z(G)$ and since elements of $Irr(G) - Lin(G)$ are zero outside G', we must have that $G' = Z(G)$. It follows that every element of $Irr(G) - Lin(G)$ vanishes on $G - G'$ and is nonzero on G'. So $O_G(\theta) = O_G(\eta)$ for all θ and η in $Irr(G) - Lin(G)$. Our assumption now implies that G has exactly one nonlinear character. A result of Seitz ([15]) now implies that G is an extra special 2 - group.

In case iii the Frobenius complement is abelian and hence cyclic ([9] 8.7 p.499). By Lemma 2.5 the Frobenius kernel is also abelian, so Lemma 2.4 implies that conclusion 2 of the theorem holds.

In case iv, a Frobenius complement is of even order and so the Frobenius kernel is abelian ([9] 8.8 p. 500). Lemma 2.4 implies that $|G| = 72$ and conclusion 3 of the theorem holds. ■

3 Groups in which distinct noncentral conjugacy classes have distinct number of irreducible characters vanishing of them

Proposition 3.1 *Let G be a finite group such that $V_G(C) \neq V_G(D)$ for every two distinct noncentral conjugacy classes C and D of G. Then either G is abelian or $Z(G) = 1$.*

Proof. Let G be a counterexample of minimal order. Then G is nonabelian and $Z = Z(G) \neq 1$.

Let $a \in G - Z$.

Let $z \in Z - \{1\}$, $\chi \in Irr(G)$ and T a representation affording χ. Thus $T(za) = T(z)T(a)$ and $T(z) = \lambda_\chi(z)I$ for some complex number $\lambda_\chi(z)$. So $\chi(za) = \lambda_\chi(z)\chi(a)$. So $\chi_Z = \chi(1)\lambda_\chi$, where $\lambda_\chi \in Irr(Z)$. From this we see that $\chi(za) = 0$ if and only if $\chi(a) = 0$ and hence $V_G(a) = V_G(az)$. Our assumption now implies that a is conjugate to az for for all $z \in Z - \{1\}$. Note that $\ker(\chi) \supseteq \ker(\lambda_\chi)$.

Let $\eta \in Irr(G)$ be such that $Z \not\subseteq \ker(\eta)$ and let $u \in Z - \ker(\eta)$, so $\lambda_\eta(u) \neq 1$. Then $\eta(ua) = \lambda_\eta(u)\eta(a) = \eta(a)$ so that $\eta(a) = 0$. We conclude that every irreducible character of G whose kernel does not contain Z, vanishes outside Z. Hence $(G,\, Z)$ is a Camina pair. Since Frobenius groups have trivial center, G is not a Frobenius group. So [2] implies that either Z or G/Z is a p - group for some prime p. If Z is a p - group, Lemma 4.3 of [4] implies that every p'- subgroup of G acts on Z with no fixed points. However, $Z = Z(G)$, so G itself is a p - group. Thus, in any case G/Z is a p - group.

Denote by $\mathcal{U}$ the set of all irreducible characters of G with kernels not containing Z. Then for all $g \in G - Z$ we have that $V_G(g) = V_{G/Z}(gZ) \cup \mathcal{U}$.

We claim that G/Z satisfies the assumption of the proposition. Indeed, let bZ and cZ be two non conjugate elements of G/Z. Then clearly b and c are not conjugate in G. Suppose that $V_{G/Z}(bZ) = V_{G/Z}(cZ)$. Then $V_G(b) = V_{G/Z}(bZ) \cup \mathcal{U} = V_{G/Z}(cZ) \cup \mathcal{U} = V_G(c)$, a contradiction. Hence $V_{G/Z}(bZ) \neq V_{G/Z}(cZ)$, as claimed. Since G/Z is a p - group, $Z(G/Z) \neq 1$ and by induction we get that G/Z is abelian. It follows that the elements of $Irr(G/Z)$ never vanish. Consequently $\mathcal{U} = V_G(g)$ for every $g \in G - Z(G)$. Our assumption implies that $G - Z$ is a conjugacy class, in particular $|G - Z|$ divides $|G|$. Since $|Z|$ divides $|G|$ we get that $|G/Z(G)| = 2$ which forces G to be abelian. This contradiction finishes the proof. ■

Proof. OF THEOREM 1.2. Suppose that G is not abelian, then $Z(G) = 1$ by the previous Proposition. Let $C_1 = \{1\}$, C_2, $C_3, \ldots, C_k$ be the conjugacy classes of G. The second orthogonality relation forbid the existence of an element on which all irreducible characters but 1_G vanish. Hence $A = \{|V_G(C_i)| \quad i = 2 \ldots k\} \subseteq \{0, 1, 2, 3, \ldots, k-2\}$. As all elements of A are distinct, there is a conjugacy class C with $|V_G(C)| = k - 2$. Let η be the unique nonprincipal irreducible character of G not vanishing on C. Then $1 \cdot 1 + \eta(1)\eta(C) = 0$ (here $\eta(C)$ is the value of η on any element of C). Then $\eta(C) = \frac{-1}{\eta(1)}$ a rational algebraic integer. Hence $\eta(1) = -\eta(C) = 1$ and so $|C_G(c)| = 2$ for $c \in C$. Therefore G is a Frobenius group with Frobenius complement of order 2 and an abelian Frobenius kernel K of odd order n (see for example [3] Lemma 2.3). Note that if $x \in K$ and $\chi \in Irr(G)$ with $x \notin \ker(\chi)$, then $\chi(x)$ is a sum of two nth roots of unity. Lemma 2.1 implies that $\chi(x) \neq 0$. Hence, K is a union of $\frac{n-1}{2}$ noncentral G-conjugacy classes on which no irreducible character vanishes. Since the assumption allows at most one such conjugacy class we get that $n = 3$ so $G \simeq S_3$. ■

4 Groups in which all nonlinear characters have the same zero sets.

We will show:

Theorem 4.1 *Let G be a finite nonabelian group in which all nonlinear irreducible characters have equal zero sets. Then G has a normal p - complement for some prime divisor p of $|G:G'|$. Moreover, G is solvable.*

We use the classification of the finite simple groups to prove solvability. The proof of the other statement (including the fact that $|G:G'|>1$) does not use the classification. The Theorem is a consequence of the next Theorem (the proof of all but part 1, do not use the classification).

We denote the conjugacy class in G of the element g by $cl_G(g)$.

Theorem 4.2 *Let G be a finite nonabelian group in which all nonlinear irreducible characters have equal zero sets. Then:*

1. *G is solvable.*
2. *If $g \in G$ is a zero of an irreducible character then $cl_G(g) = gG'$. In particular $g \notin G'$ and $G > G'$.*
3. *$Lin(G) = A \cup B$, $A \cap B = \Phi$ where*

   $$A = \{\lambda \in Lin(G) | \lambda\chi = \chi \text{ for all nonlinear } \chi \in Irr(G)\}$$

   and

   $$B = \{\lambda \in Lin(G) | \lambda\chi \neq \chi \text{ for all nonlinear } \chi \in Irr(G)\}.$$

4. *$|A| > 1$ and $|A|$ divides $|G:G'|$.*
5. *For every nonlinear $\chi \in Irr(G)$ define $e(\chi)$ and $t(\chi)$ by*

   $$\chi_{G'} = e(\chi)\left(\sigma_1 + \sigma_2 + \ldots + \sigma_{t(\chi)}\right) \text{ where } \sigma_i \in Irr(G') \text{ are } G\text{-conjugates}.$$

   Then for all nonlinear $\chi \in Irr(G)$ we have that $e(\chi)^2 \cdot t(\chi) = |A|$.
6. *G has a normal p - complement for every prime divisor p of $|A|$.*

Proof. First we show that the assumption is inherited by factor groups. Let X be a normal subgroup of G and $\alpha, \beta \in Irr(G/X)$. By assumption $O_G(\alpha) = O_G(\beta)$ and Lemma 2.2 implies that $O_{G/X}(\alpha) = O_{G/X}(\beta)$ as claimed.

Now let $g \in G$ be zero of some irreducible character, then $\chi(g) = 0$ for all $\chi \in Irr(G) - Lin(G)$. Hence

$$|C_G(g)| = \sum_{\chi \in Irr(G)} |\chi(g)|^2 = \sum_{\chi \in Lin(G)} |\chi(g)|^2 = |G : G'| .$$

It follows that $|cl_G(g)| = |G'|$. If $g \in G'$ then $cl_G(g)$ is properly contained in G' which is impossible. Thus $g \notin G'$. In particular $G \neq G'$.

Let $g' \in G'$. Then $\chi(g') = 1$ for all $\chi \in Lin(G)$. We show that gg' is conjugate to g. Indeed

$$\sum_{\chi \in Irr(G)} \chi(g)\overline{\chi(gg')} = \sum_{\chi \in Lin(G)} \chi(g)\overline{\chi(gg')} = \sum_{\chi \in Lin(G)} \chi(g)\overline{\chi(g)\chi(g')} = \sum_{\chi \in Lin(G)} \chi(g)\overline{\chi(g)} \neq 0.$$

So gg' is conjugate to g and so $cl_G(g) \supset gG'$. Since $|cl_G(g)| = |G'|$ part 2 follows.

Next suppose that G is nonsolvable. Let $i \geq 1$ be the first index such that $G^{(i)} = G^{(i+1)}$ and let M be largest among proper subgroups of $G^{(i)}$ which are normal in G (possible $M = 1$). Then $V = G^{(i)}/M$ is a minimal normal subgroup of $\overline{G} = G/M$. As $G^{(i)}$ has no abelian quotients, $G^{(i)}/M = S_1 \times S_2 \times \ldots \times S_w$ where the $S_j's$ are isomorphic nonabelian simple groups. We have shown that $\overline{G}$ satisfies the assumption of the theorem so no irreducible character of $\overline{G}$ vanishes inside V as $V \subseteq \overline{G}' = (G/M)' = G'/M$. Let $p \geq 5$ be a prime divisor of $|S_1|$. By [17], and [7]) there exists for every i an $\vartheta_i \in Irr(S_i)$ with p - defect equal to zero. So $\vartheta = \vartheta_1 \times \vartheta_2 \times \ldots \vartheta_w \in Irr(V)$ with p - defect equal to zero. Hence ϑ vanishes on p - elements of V ([10], 8.17, p.133). Since every $\overline{G}$ - conjugate of ϑ has degree equal to $\vartheta(1)$, it also has p - defect equal to zero, thus it also vanishes on p - elements of V. Pick $\xi \in Irr(\overline{G})$ with $(\xi_V, \vartheta) \neq 0$. Then ξ_V is a multiple of a sum of $\overline{G}$ - conjugates of ϑ, each vanishing on p - elements of V. We conclude that ξ vanishes on p - elements of V, which is a contradiction. So part 1 is proved.

Let $\lambda \in Lin(G)$. Suppose that $\lambda \notin B$. Then there exists $\theta \in Irr(G) - Lin(G)$ such that $\lambda\theta = \theta$. So θ vanishes on $G - \ker(\lambda)$. Therefore every element of $Irr(G) - Lin(G)$ vanishes on $G - \ker(\lambda)$. Take $\phi \in Irr(G) - Lin(G)$, then

$$\lambda(x)\phi(x) = \begin{cases} 0 & \text{if } x \in G - \ker(\lambda) \\ \phi(x) & \text{if } x \in \ker(\lambda) \end{cases} = \phi(x).$$

Hence $\lambda\phi = \phi$ for all $\phi \in Irr(G) - Lin(G)$, which proves part 3.

Next we prove parts 4 and 5. Clearly A is a subgroup of $Lin(G)$, in particular $|A|$ divides $|G : G'|$. Let $\{\mu_1 = 1_G, \mu_2, \ldots, \mu_s\}$ be a set of representatives for the cosets of A in $Lin(G)$. Then $\{\mu_2, \ldots, \mu_s\} \subseteq B$ and $s = |Lin(G)/A|$. Also, $Lin(G) = \bigcup_{i=1}^{s} A\mu_i$. Let $K = \bigcap \{\ker(\lambda) |\ \lambda \in A\}$.

Pick an arbitrary $\chi \in Irr(G) - Lin(G)$ and consider the character

$$\Gamma = \chi + \mu_2\chi + \mu_3\chi + \ldots + \mu_s\chi = \chi\left(1_G + \mu_2 + \mu_3 + \ldots + \mu_s\right),$$

which is a sum of s distinct irreducible characters. In particular $(\Gamma, \Gamma) = s$.

Let $a \in G - G'$. We will show that $\Gamma(a) = 0$. If $a \notin K$ then $a \notin Ker(\lambda)$ for some $\lambda \in A$. As $\lambda\chi = \chi$ and $\lambda(a) \neq 1$ we get that $\chi(a) = 0$ so that $\Gamma(a) = 0$. Next assume that $a \in K - G'$. We show that $\Gamma(a) = 0$ by showing that $\sum_{i=1}^{s} \mu_i(a) = 0$. Using the second orthogonality relation and the fact $\lambda(a) = 1$ for all $\lambda \in A$, we get

$$0 = \sum_{\lambda \in Lin(G/G')} \lambda(aG') = \sum_{\lambda \in Lin(G)} \lambda(a) = \sum_{i=1}^{s} \sum_{\lambda \in A} (\lambda\mu_i)(a) =$$
$$\sum_{i=1}^{s} \sum_{\lambda \in A} \mu_i(a) = |A| \sum_{i=1}^{s} \mu_i(a),$$

as claimed. So Γ is zero outside G'.

Note that $\Gamma_{G'} = s\chi_{G'} = s \cdot e(\chi)\left(\sigma_1 + \sigma_2 + \ldots + \sigma_{t(\chi)}\right)$. Therefore

$$|G|\, s = |G|\,(\Gamma, \Gamma) = \sum_{x \in G} |\Gamma(x)|^2 = \sum_{x \in G'} |\Gamma_{G'}(x)|^2 =$$
$$s^2\,(e(\chi))^2\,|G'|\left(\sigma_1 + \sigma_2 + \ldots + \sigma_{t(\chi)}, \sigma_1 + \sigma_2 + \ldots + \sigma_{t(\chi)}\right) = s^2\,(e(\chi))^2\,|G'|\,t(\chi).$$

It follows that $|G|\, s = s^2\,(e(\chi))^2\,|G'|\,t(\chi)$ which is the same as $e(\chi)^2 \cdot t(\chi) = \frac{|G:G'|}{s} = |A|$. This proves part 5.

Let $\chi \in Irr(G) - Lin(G)$. If $|A| = 1$ then by part 5 we have that $e(\chi) = t(\chi) = 1$. Hence $\chi_{G'} \in Irr(G)$ and since χ cannot vanish on G', $\chi_{G'}$ must be linear. But $\chi(1) = \chi_{G'}(1) = 1$, a contradiction. So part 4 is proved.

For part 6 note that if p is a prime divisor of $|A| = e(\chi)^2 \cdot t(\chi)$ then p divides $e(\chi)t(\chi)$ for all $\chi \in Irr(G) - Lin(G)$. But $\chi(1) = e(\chi)t(\chi)\sigma_1(1)$ so p divides the degree of every irreducible nonlinear character of G. Now Part 6 follows from a theorem of Thompson ([10], 12.2, p.199). ■

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