Finite Groups in which Every Irreducible Character Vanishes on at Most Two Conjugacy Classes.

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Abstract

It is known that if G is a finite non-abelian group in which every irreducible character vanishes on at most one conjugacy class, then G is a Frobenius group with a Frobenius complement of order 2 and Frobenius kernel of odd order. In particular G is solvable. This paper studies the class of finite groups G in which every irreducible character vanishes on at most two conjugacy classes. There are such nonsolvable groups. We show that A_5 and $PSL(2,7)$ are the only nonsolvabe groups in this class. We also show that each solvable group in this class is either a certain type of a Frobenius group, or is very closed to being one.

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1 Introduction

It is shown in [1] and in dependently in [2] (Proposition 1.6) that if G is a finite non-abelian group in which every irreducible character vanishes on at most one conjugacy class, then G is a Frobenius group with a Frobenius complement of order 2 and Frobenius kernel of odd order. In particular G is solvable. This paper studies the class of finite groups G in which every irreducible character vanishes on at most two conjugacy classes. There are such nonsolvable groups, witness A_5 and $PSL(2,7)$. We show that A_5 and $PSL(2,7)$ are the only nonsolvable groups in this class, and give a description of the solvable groups in this class. Our result is:

Theorem 1 *Let G be a non-abelian finite group in which every irreducible ordinary character vanishes on at most two conjugacy classes. Then one of the following holds:*

a. $G \cong A_5$ or $PSL(2,7)$.

b. G is solvable and one of the following holds:

1. G has a subgroup Z with $|Z| \leq 2$ such that G/Z is a Frobenius group with a Frobenius complement of order 2 and an abelian Frobenius kernel of odd order.

2. $G/Z = F \cdot A$ a semi-direct product, where $|A| \leq 2$, $|Z| \leq 2$, and F is a Frobenius group with Frobenius complement of order 3 and a nilpotent Frobenius kernel of class at most 2.

The proof, which is in section 2, depends on the classification of the finite simple groups. Our notation is standard and is taken mainly from [4] and [6]. For a finite group G, we will denote by $n(\chi)$ the number of conjugacy classes on which the character χ of G vanishes, and by $cl_G(a)$ the G-conjugacy class of the element a of G. All groups considered are finite.

2 Proof

We prove Theorem 1.1 via a series of lemmas. We will use the fact that the property: "$n(\chi) \leq 2$ for every $\chi \in Irr(G)$" is inherited by factor groups.

Lemma 2 *Let G be a finite simple non-abelian group in which $n(\chi) \leq 2$ for all $\chi \in Irr(G)$. Then $G \cong A_5$ or $PSL(2,7)$.*

Proof. Assume first that G has a 2-block of defect zero. Then there exists $\chi \in Irr(G)$ vanishing on all elements of even order in G (see [6], p.133). By assumption, G has at most two conjugacy classes of elements of even order. Let S be a Sylow 2-subgroup of G. If the exponent of S is bigger than 4, G would have conjugacy classes of elements of order 2, 4 and 8, a contradiction. So the exponent of S is either 2 or 4. If the exponent is 2, S is abelian and by John Walter's theorem ([4], p. 485), $G \cong PSL(2,q)$ for some values of q, J_1, or a Ree type group. A Ree type group X has an involution t such that $C_G(t) = \langle t \rangle \times PSL(2,r)$, where r is a power ot 3. So X has at least three conjugacy classes of elements of even order. Each of the $PSL(2,q)$'s and J_1 , with the exception of $PSL(2,5) \cong A_5$ or $PSL(2,7)$ have a $\theta \in Irr(G)$ with $n(\theta) \geq 3$ (See [8]).

So we may assume that the exponent of S is 4. If an involution in S commutes with an element of odd order, then G would have conjugacy classes of elements of order 2, 4 and twice an odd number, a contradiction again. So the centralizer of an involution in G is 2-group. By ([10] and [11]), $G \cong PSL(2,q)$, $S_z(q)$, or $PSL(3,4)$. Again, each of these groups, with the exception of $PSL(2,5)$ or $PSL(2,7)$ have a $\theta \in Irr(G)$ with $n(\theta) \geq 3$ (See [11] and[8]).

To complete the proof we show that that G must have a 2-block of defect zero. Assume the contrary. By a theorem of Michler and Willems ([13]), $G \cong M_{12}$, M_{22}, M_{24}, J_2, HS, Suz, Ru, $C1$, $C3$, BM, A_n, for some values of n. Using [3], we see that each of these groups, as well as the groups A_n, $n \leq 13$ has a $\theta \in Irr(G)$ with $n(\theta) \geq 3$. Thus $G \cong A_n$, for some values of $n \geq 14$. Let $\eta \in Irr(G)$, $\eta(1) = n-1$ and $\eta(x)$ is the number of fixed points of x minus one. If n is odd, let

$$u = (1,2,3,\ldots,n-4)(n-3,n-2,n-1)(n),$$
$$v = (1,2,3,\ldots,n-6)(n-5,n-4,n-3,n-2,n-1)(n),$$
$$w = (1,2,3,\ldots,n-8)(n-7,n-6,n-5,n-4,n-3,n-2,n-1)(n).$$

Then u, v, w are non-conjugate elements of G with $\eta(u) = \eta(v) = \eta(w) = 0$, forcing $n(\eta) \geq 3$, a contradiction. So n is even. In this case let

$$u_1 = (1,2,3,\ldots,n-1)(n),$$
$$v_1 = (1,2,3,\ldots,n-5)(n-4,n-3)(n-2,n-1)(n),$$
$$w_1 = (1,2,3,\ldots,n-9)(n-8,n-7)(n-6,n-5)(n-4,n-3)(n-2,n-1)(n).$$

Then u_1, v_1, w_1 are non-conjugate elements of G with $\eta(u) = \eta(v) = \eta(w) = 0$, forcing $n(\eta) \geq 3$, a final contradiction. ■

Lemma 3 *Let G be a finite perfect group in which $n(\chi) \leq 2$ for all $\chi \in Irr(G)$. Then $G \cong A_5$ or $PSL(2,7)$.*

Proof. We use induction on $|G|$. Let G be a counter-example of minimal order. By Lemma 1.1, G is not simple. So let N be a minimal normal subgroup of G, $1 < N < G$. Clearly G/N is perfect, so induction implies that $G/N \cong A_5$ or $PSL(2,7)$. Let $p = 5$ if $G/N \cong A_5$ and $p = 7$ if $G/N \cong PSL(2,7)$.

Then G/N has exactly two conjugacy classes of elements of order p, each class consisting of $\frac{|G|}{p|N|}$ elements. So G/N has exactly $k = \frac{2|G|}{p|N|}$ elements of order p. We can chose p-elements of $a_i \in G$, $1 \leq i \leq k$ such that $\{a_iN \mid 1 \leq i \leq k\}$ is the set of all elements of order p in G/N. Set $T = \bigcup_{i=1}^{k} a_iN$. Moreover, there exists $\chi \in Irr(G)$, $N \subseteq \ker(\chi)$, $\chi(1) = p$ and $\chi(a_iN) = \chi(a_i) = 0$ for all i, $1 \leq i \leq k$. It follows that χ vanishes on T. As $n(\chi) \leq 2$, we get that $T \subseteq C_1 \cup C_2$ for some conjugacy classes C_1, C_2 of G. Since $|T| = \frac{2|G|}{p}$, there exists an element $b = a_iu \in T$, for some i and $u \in N$, such that a_iu has at least $\left|\frac{T}{2}\right| = \frac{|G|}{p}$ conjugates in G. It follows that $|C_G(b)| = \frac{|G|}{|cl_G(b)|} \leq \frac{|G|}{|G|/p} = p$. Since the order of b is clearly divisible by p, we conclude that $|C_G(b)| = p$. Thus b has order p and so $|G|_p = p$, and $(p, |N|) = 1$. As $b \notin N$, b acts without fixed points on N and consequently N is nilpotent ([4] p. 337). As N is minimal normal, N is an elementary abelian s-group for some prime $s \neq p$.

As G/N has exactly two conjugacy classes of elements of order p, G has at least two such. But χ vanishes on elements of order p ([6] p. 133), so G has exactly two conjugacy classes of elements of order p. For each $x \in G$ of order p we have that $|C_G(x)| = \left|C_{G/N}(xN)\right| = p$. Let $\theta \in Irr(G)$ be such that $N \nsubseteq \ker(\theta)$. Then the second orthogonality relation implies that θ vanishes on all elements of order p, implying that $n(\theta) \geq 2$. If G has an element y such that $|C_G(y)| = 3$, then y will have order 3, $|G|_3 = 3$, $(3, |N|) = 1$ and so $|C_G(y)| = \left|C_{G/N}(yN)\right| = 3$. As above, this will imply $\theta(y) = 0$, forcing $n(\theta) \geq 3$. This is impossible, so G has no element with centralizer of order 3.

Next, G/N has exactly one class of elements of order 3, whose size is $r = \frac{|G|}{3|N|}$. We can chose 3-elements $b_i \in G$, $1 \leq i \leq r$ such that $\{b_iN \mid 1 \leq i \leq r\}$ is the set of all elements of order 3 in G/N. Set $T_1 = \bigcup_{i=1}^{r} b_iN$. Moreover, there exists $\chi_1 \in Irr(G)$, $N \subseteq \ker(\chi_1)$, $\chi_1(1) = 3$ and $\chi_1(b_iN) = \chi_1(b_i) = 0$ for all i, $1 \leq i \leq r$. It follows that χ_1 vanishes on T_1. As $n(\chi_1) \leq 2$, we have that $T_1 \subseteq C_3 \cup C_4$ for some conjugacy classes C_3, C_4 of G. Since $|T_1| = \frac{|G|}{3}$, there is an $h \in T_1$ that has at least $\frac{|T_1|}{2} = \frac{|G|}{6}$ conjugates, so that $|C_G(h)| \leq 6$. As the orders of all elements of T_1 are divisible by 3 and as $|C_G(h)| \neq 3$, we conclude that $|C_G(h)| = 6$. Hence, $|cl_G(h)| = \frac{|G|}{6}$ and $cl_G(h)$ is contained in, say, C_3. Then $T_1 - cl_G(h) \subseteq C_4$, and as $|T_1| = \frac{|G|}{3}$, we get that if $g \in T_1 - cl_G(h)$ then g has at least $\frac{|G|}{6}$ conjugates in G. As above, we conclude $|C_G(g)| = 6$ and hence $|C_G(u)| = 6$ for all $u \in T_1$. In particular $|C_G(b_1)| = 6$ and as b_1 is a 3-element, b_1 must have order 3, and so $|G|_3 = 3$ and $(3, |N|) = 1$. Let t be the unique involution in $C_G(b_1)$. As $\left|C_{G/N}(b_1N)\right| = 3$, $t \in N$ and consequently N is an elementary abelian 2-group.

We now break the proof into two cases.

Case 1. $G/N \cong A_5$. Note that b_1 fixes exactly one nonidentity element of N. So if we set $|N| = 2^m$ then $2^m \equiv 2 (\text{mod } 3)$. As powers of 4 are congruent to 1 modulo 3, $m = 2l + 1$ is odd, for some integer l. Recall that G has an element of order 5 acting fixed point freely on N, so $2^m \equiv 1 (\text{mod } 5)$. On the other hand $2^m = 2 \cdot 4^l \equiv \pm 2 (\text{mod } 5)$, a contradiction.

Case 2. $G/N \cong PSL(2, 7)$. In this case there exists $\varphi \in Irr(G)$, $\varphi(1) = 6$, $N \subseteq \ker(\varphi)$ with $\varphi(b_1) = \varphi(w) = 0$ for some 2-element w of G, such that wN has order 4 in G/N (See [6], p.289). As $t \in N$ we have that $\varphi(b_1t) = 0$ as well, and b_1t has order 6. So b_1, b_1t and w belong to different conjugacy classes. It follows that $n(\varphi) \geq 3$, a final contradiction. ■

In the next two lemmas we show, among other things, that if G is non-perfect then G is solvable.

Lemma 4 *Let G be a finite non-perfect non-abelian group in which $n(\chi) \leq 2$ for all $\chi \in Irr(G)$. Then one of the following holds:*
1. G is a Frobenius group with a Frobenius complement of order 2 and Frobenius kernel of odd order.

2. $G = F \cdot A$ *a semi-direct product, where* $|A| \leq 2$*, and* F *is a Frobenius group with a Frobenius complement of order* 3 *and a Frobenius kernel which is nilpotent of class at most* 2*.*
3. G *has an element* y *with* $T = C_G(y)$ *of order* 4 *and* $C_G(T) = T$*.*

Proof. Suppose first that $|C_G(y)| > 4$ for all $y \in G$. Then by ([2], Theorem 1.4), we get that $G' = G''$, each $\varphi \in Irr(G')$ has the form $\theta_{G'}$ for some $\theta \in Irr(G)$, every G'-conjugacy class is a G-conjugacy class, and every non-linear $\chi \in Irr(G)$ has at least one zero outside G'. It follows that $n(\varphi) \leq 1$ for all $\varphi \in Irr(G')$. By ([2], Proposition 1.6) we have that G' is solvable, contradicting $G' = G''$. We conclude that G has an element y such that $|C_G(y)| \leq 4$. If there exists such a y with $|C_G(y)| = 2$, then claim 1. holds (e.g., see [2] Lemma 2.3).

Next assume that there exists $y \in G$ with $|C_G(y)| = 3$. Set $S =< y >$, then S is a Sylow 3-subgroup of G of order 3 and $C_G(S) = S$. As $G \neq G'$, G contains a normal subgroup M with $|G : M| = p$, a prime number. If $S \nsubseteq M$, then $G = SM$, $|G| = 3\,|M|$, and y acts fixed point freely on M. Thus G is a Frobenius group with Frobenius kernel M of nilpotency of class at most 2 ([5], p. 500). So claim 2. holds. Hence we may assume $S \subseteq M$. The Frattini argument implies that $G = N_G(S)M$. Now, $G \neq M$ so $N_G(S) \nsubseteq M$, and as $|N_G(S) : C_G(S)|$ divides 2, we get that $|N_G(S)| = 6$ and $|N_G(S) \cap M| = 3$. It follows that $|G : M| = 2$ and that $N_M(S) = C_M(S) = S$. By Burnside's Theorem M has a normal 3-complement on which S acts fixed point freely. Note that $N_G(S) = S \cdot A$, with $|A| = 2$. Thus $G = M \cdot A$ and so claim 2. holds.

So we may suppose that G has an element y with $T = C_G(y)$ has order 4. Clearly $T \subseteq C_G(T) \subseteq C_G(y) = T$, so claim 3. is true. ■

Lemma 5 *Let* G *be a finite non-perfect group in which* $n(\chi) \leq 2$ *for all* $\chi \in Irr(G)$*. Then* G *is solvable.*

Proof. Let G be a counter-example of minimal order. Then G is non-abelian and so satisfies the conclusion of the previous lemma. Since the groups in conclusions 1. and 2. of Lemma 2.3 are solvable, we get that G has an element y with $T = C_G(y)$ of order 4 and $C_G(T) = T$.

Suppose that G has a minimal normal solvable subgroup M, $1 < M < G$. Then M is an elementary abelian s-subgroup for some prime s. If G/M is non-perfect, we get by induction that G/M is solvable and hence so is G. This is a contradiction, so $G/M = (G/M)' = (G'M)/M$ and thus $G = G'M$. As $G \neq G'$, we have that $M \nsubseteq G'$ and therefore $G' \cap M = 1$, which implies that $G = G' \times M$. In particular $M \subseteq Z(G)$. Write $y = y'm$ with $y' \in G'$ and $m \in M$, then $C_G(y') \subseteq C_G(y)$. So $|C_G(y')| = 2$ or 4 and y' is a 2-element. Now, $G' \cong G/M$ is perfect, so $|G'|_2 \geq 4$. Consequently $|C_{G'}(y')| = |C_G(y')| = 4$. But then $C_{G'}(y') \times M \subseteq C_G(y')$ forcing $M = 1$, a contradiction again.

Thus G has no solvable normal subgroups. In particular $O(G) = 1$. We now use Theorems 1 and 2 of [15] (where $O(G)$ is denoted by K), to conclude that G has a normal subgroup N with $N \cong PSL(2, q)$, $G \subseteq Aut(N)$ and $|G : N| = 2$. All the other groups mentioned in these theorems are either solvable or perfect.

If q is even, Wong's results implies that $G \cong PGL(2,q)$ (the groups $H(q)$ defined in [15] exists only for odd q). But $PGL(2,q) \cong PSL(2,q)$ for q even, so G is perfect, a contradiction. So q is odd.

Suppose that $q \equiv 1 (\mathrm{mod}\ 4)$, and write $q = 4r+1$. Then N has a cyclic subgroup C, of order $2r$. If $r \geq 3$, then C contains elements a, b, c of orders 2, r, $2r$, so no two of them are conjugate in G. Furthermore, every irreducible character of N of degree $q-1$ vanishes on a, b, and c ([8]). Let $\theta \in Irr(N)$ with $\theta(1) = q-1$ and let $\chi \in Irr(G)$ be such that $[\theta, \chi_N] \neq 0$. By Clifford's Theorem χ_N is a sum of irreducible characters of N of degree $q-1$. It follows that $\chi(a) = \chi(b) = \chi(c) = 0$, implying that $n(\chi) \geq 3$, a contradiction. So $r = 1$ or 2 and so $q = 5$ or 9. Using [3], we see that each possible extension of index 2 of $PSL(2,5)$ and $PSL(2,9)$ has an irreducible character χ_1, with $n(\chi_1) \geq 3$, a contradiction.

So $q \equiv 3 (\mathrm{mod}\ 4)$ and we have a similar argument. Set $q = 4r+3$. Then N has a cyclic subgroup C, of order $2(r+1)$. If $r \geq 2$, then C contains elements a, b, c of orders 2, $r+1$, $2(r+1)$, so no two of them are conjugate in G. Let $\theta \in Irr(N)$ with $\theta(1) = q+1$ and let $\chi \in Irr(G)$ be such that $[\theta, \chi_N] \neq 0$. Then χ_N is a sum of irreducible characters of N of degree $q+1$, each of them is zero on a, b and c. So $\chi(a) = \chi(b) = \chi(c) = 0$, implying that $n(\chi) \geq 3$, a contradiction. It follows that $r = 1$ and $N \cong PSL(2,7)$ forcing $G = PGL(2,7)$. Then (by [3]) there exists $\chi_1 \in Irr(G)$ with $n(\chi_1) \geq 3$, a final contradiction. ■

Proof of Theorem 1.1. We may assume that $G \ncong A_5$ or $PSL(2,7)$. By Lemmas 2.2 and 2.4 we get that G is solvable. In particular $G \neq G'$ and we can apply Lemma 2.3. If one of the two first conclusions of Lemma 2.3 holds, then conclusion b. of the theorem holds, and we are done. Thus we may assume that G has an element y with $T = C_G(y)$ of order 4 and $C_G(T) = T$. Set $K = O(G)$, $\bar{G} = G/K$ and $\bar{a} = aK$ for $a \in G$. Looking on the solvable groups in the lists of Theorems 1 and 2 of [15] we get that $\bar{G}$ is isomorphic to one of the following groups: $GL(2,3)$, $PGL(2,2) \cong S_3$, $PGL(2,3) \cong S_4$, $PSL(2,3) \cong A_4$, a dihedral (including $Z_2 \times Z_2$) or semi-dihedral 2-group, $SL(2,3)$, the extension J of $SL(2,3)$ by an element γ such that γ^2 is the involution of $SL(2,3)$, a generalized quaternion group, a cyclic group of order 4. Note that the group $H(q)$ in [15] is defined only for q which is a square. Also see ([14], pp 104-105 for more details on J, details that will be used). Recall that $n(\eta) \leq 2$ for all $\eta \in Irr(\bar{G})$. We now break the proof into two cases.

Case 1: $|\bar{G}| \neq 4$.

First $\bar{G} \ncong S_3$, as $O(S_3) > 1$. Next, $\bar{G} \ncong SL(2,3)$ or $GL(2,3)$, as each of these two groups has an irreducible character η with $n(\eta) \geq 3$ (see [6], p. 288 and [9], p.226). Each of the 2-groups of order more than 4 in the above list, has a factor group isomorphic to a non-abelian group B of order 8 ([4], p 191). But there exists $\zeta \in Irr(B)$ with $n(\zeta) = 3$ ([7], p. 381). So if $\bar{G}$ is a 2-group, then $|\bar{G}| = 4$, this will be discussed in case 2.

We now show that $\bar{G} \ncong J$. Assume the contrary, that $\bar{G} \cong J$. Set $L = SL(2,3)$, we will use the character table of L ([6], p.288). The Sylow 2-subgroups of J and L are generalized quaternoin groups of order 16 and 8, respectively. So there exists an element of order 8 in $J - L$. Also, γ has order 4 and $\gamma \in J - L$.

So $J-L$ consists of at least two conjugacy classes of J. Let $\chi \in Irr(J)$, $\chi(1) \neq 1$. If χ_L is reducible, then $\chi_L = \theta + \theta_1$ where θ,$\theta_1 \in Irr(L)$ are distinct and J-conjugate. But then the inertia group of θ is L and so $(\theta)^J \in Irr(J)$ and as $\left[(\theta)^J, \chi\right] = [\theta, \chi_L] = 1$, we get that $(\theta)^J = \chi$. So χ vanishes outside L. Now, as θ,θ_1 have the same degree, they have a common zero $v \in L$, so $\chi(v) = \theta(v) + \theta_1(v) = 0$. Thus $n(\chi) \geq 3$. As this is impossible, we conclude that $\chi_L \in Irr(L)$ for all nonlinear $\chi \in Irr(J)$. This is clearly true for linear characters as well.

Now, $J' = L$ and so J has a unique linear $\lambda \in Irr(J) - \{1_G\}$. Clearly $\ker(\lambda) = L$. If $\chi\lambda = \chi$ for some $\chi \in Irr(J)$, then $\chi(1) \neq 1$ and χ has to vanish outside $\ker(\lambda) = L$. But $\chi_L \in Irr(L)$, so χ has a zero in L as well. Again $n(\chi) \geq 3$, a contradiction. So $\chi\lambda \neq \chi$ for all $\chi \in Irr(J)$. It follows each $\theta \in Irr(L)$ has exactly two extensions to J of the form χ and $\lambda\chi$. Let $\delta \in L$, then

$$|C_J(\delta)| = \sum_{\chi \in Irr(J)} |\chi_L(\delta)|^2 = 2 \sum_{\theta \in Irr(J)} |\theta(\delta)|^2 = 2\,|C_L(\delta)|\,.$$

This implies that

$$|cl_J(\delta)| = |J : C_J(\delta)| = \frac{2\,|L|}{2\,|C_L(\delta)|} = |cl_L(\delta)|\,.$$

Therefore any two non-conjugate elements of L are non-conjugate in J as well. Finally, there exists $\varphi \in Irr(L)$, vanishing on four conjugacy classes of L. An extension $\chi_1 \in Irr(J)$ of φ will then have $n(\chi_1) \geq 4$. A contradiction. Hence $\bar{G} \ncong J$.

Assume now that $\bar{G} \cong A_4$ or S_4. Then there exists a 3-element $b \in G$, such that $\bar{b}$ has order 3 and $r = \left|cl_{\bar{G}}(\bar{b})\right| = \frac{|\bar{G}|}{3}$. We can chose 3-elements of $b_i \in G$, $1 \leq i \leq r$ such that $\left\{\bar{b}_i \mid 1 \leq i \leq r\right\}$ is the set of all elements of order 3 in $\bar{G}$. Set $I = \bigcup_{i=1}^{r} b_i K$. Moreover, there exists $\chi_2 \in Irr(G)$, $K \subseteq \ker(\chi_2)$, $\chi_2(1) = 3$ and $\chi_2(b_i K) = \chi_2(b_i) = 0$ for all i, $1 \leq i \leq r$ (see [7], pp180-181). It follows that χ_2 vanishes on I. As $n(\chi_2) \leq 2$, $I \subseteq C_1 \cup C_2$ for some conjugacy classes C_1, C_2 of G. Since $|I| = |K|\frac{|\bar{G}|}{3} = \frac{|G|}{3}$, there is an $h \in I$ that has at least $\frac{|I|}{2} = \frac{|G|}{6}$ conjugates, so that $|C_G(h)| \leq 6$. The order of h is divisible by 3, so $|C_G(h)| = 3$ or 6. If $|C_G(h)| = 6$, let u be an involution in $C_G(h)$. Then $u \notin K$ (as $|K|$ is odd) and as $\bar{h}$ has order 3, the involution $\bar{u}$ commutes with $\bar{h}$. This is impossible, as $\bar{G}$ has no element of order 6. So $|C_G(h)| = 3$. Now read the second paragraph of the proof of Lemma 2.3, to conclude that claim b.2. of the theorem is true.

Case 2: $\left|\bar{G}\right| = 4$. We will use the notation $cd(G)$ for the set of all irreducible character degrees of G.

Here $T = C_G(y)$ is a Sylow 2-subgroup of order 4 of G, and $G = K \cdot T$, a semi-direct product. If every element of T acts on K with no fixed points, then G is a Frobenius group with T a Frobenius complement and K the Frobenius kernel. Then G has exactly three conjugacy classes of 2-elements and each irreducible character of G which is induced from an irreducible character of K, will vanish on these three classes ([6], p. 94). As this is impossible, there is some $\tau \in T - \{1\}$ such that τ commutes with some element of K, so $|C_G(\tau)|$ is not a power of 2. If τ has order 4, then $T = \langle\tau\rangle$, contradicting $T = C_G(T)$. Hence τ is an involution.

If 4 divides $\nu(1)$ for some $\nu \in Irr(G)$, then ν has 2-defect zero and so ν vanishes on elements of order 2, 4, and twice an odd number. As this is impossible, $4 \notin cd(G)$. Also, G is not nilpotent because otherwise $T \subseteq Z(G)$ and so $C_G(T) = T$ would imply that G is abelian.

If y is an involution, then y acts fixed point freely on K, so that $F = \langle y\rangle K$ is a Frobenius group with the abelian kernel K . By Ito's Theorem ([6], p.84) the degree of every irreducible character of G, divides $|G : K| = 4$ and as $4 \notin cd(G)$ we get that $cd(G) = \{1, 2\}$. As $|G|_2 = 4$, we use [6], p. 203 to conclude that G has an abelian normal subgroup L of index 2. Clearly $K \subseteq L$ and as $G = KT$, we have that $L = K(L \cap T)$. Let $\tau_1 \in (L \cap T) - \{1\}$. As L is abelian, $G = \langle L, T\rangle \subseteq C_G(\tau_1)$ so that $\tau_1 \in Z(G)$. Now $\frac{G}{\langle\tau_1\rangle}$ is a non-abelian (as G is not nilpotent) of order twice an odd number, and has an abelian Hall $2'$-subgroup. Thus Lemma 2.3 implies that $\frac{G}{\langle\tau_1\rangle}$ is a Frobenius group with a complement of order 2. So claim b.1. of the theorem holds.

So we may assume that y has order 4 and so T is cyclic. Clearly $\tau = y^2$.

We claim that $< \tau > \lhd G$. Assume the contrary that $< \tau > \ntriangleleft G$. Let $N = K \cdot < \tau >$. Then $|G : N| = 2$. If $\alpha(1)$ is odd for every $\alpha \in Irr(N)$, then N has a normal Sylow 2$-$ subgroup ([6], p. 216). So $N = K \times < \tau >$ and therefore $< \tau > \lhd G$, a contradiction. So there exists $\beta \in Irr(N)$ with $\beta(1)$ even. Let $\sigma \in Irr(G)$ with $[\beta, \sigma_N] \neq 0$. Since 4 does not divide $\sigma(1)$ we know that σ_N cannot be a sum of two conjugates of β. Hence $\sigma_N = \beta$. As β has 2-defect zero in N we get that β, and hence σ, vanishes on every element of even order of N ([6], p.133). In particular $\sigma(\tau) = 0$, every involution of N is G-conjugate to τ and any two non-involutions elements of N of even order are G-conjugate.

Write $C_N(\tau) = < \tau > \times U$, with $U > 1$ of odd order. Clearly $C_G(\tau) = TU$. If $|U|$ is not a p-group for some prime p, then U would have two elements of different orders and so $C_N(\tau)$ would have two non-involution elements of different even orders. These two elements cannot be conjugate in G, a contradiction. Thus U has prime power order. Let $u \in Z(U) - \{1\}$be fixed, and let g be an arbitrary element of $U - \{1\}$. Then τu and τg are two non-involution elements of N of even order, so they are conjugate in G. Let $h \in G$ be such that $\tau^h u^h = \tau g$. Now, $\left[\tau^h, g^h\right] = [\tau, h] = 1$ and the uniqueness of the decomposition of an element into its 2 and $2'$-parts implies that $\tau^h = \tau$ and $g = u^h$. So $h \in C_G(\tau) = UT = U\langle y\rangle$ and we can write $h = u_1 y^j$. Since $u \in Z(U)$, we get that $g = u^{y^j}$. But y has order 4 and $y^2 = \tau$ commutes with u. So $g = u$, or $u^y = u^{y^{-1}}$. Thus $|U| = 3$ and so $|C_N(\tau)| = 6$, and $|C_G(\tau)| = 12$. Furthermore, every element of even order on N is either an involution conjugate in G to τ, or an element of order 6 conjugate in G to τu.

The group G/K' has a normal abelian subgroup K/K', so by Itô's Theorem ([6], p.84), the degree of every irreducible character of G/K' divides $|G/K' : K/K'| = 4$. Since $4 \notin cd(G)$, we conclude that $cd(G/K') \subseteq \{1, 2\}$. Since $|G|_2 = 4$, we get by ([6], p. 203) that G/K' has an abelian normal subgroup M/K' with $|G/K' : M/K'| = 2$. Then $|G : M| = 2$. Clearly $K \subseteq M$ and as $G = KT$, we get that $M = K(M \cap T)$. In particular $\tau \in M$ so that $M = N$. Let $dK' \in K/K'$ be a nonidentity element. Then dK' and $\tau K'$ are in the abelian group N/K' and so $\tau dK'$ is of even order, but not a 2-element. Then τd is of even order, but not a 2-element. So there exists $vy^i \in G = NT$, $v \in N$ such that $(\tau d)^{vy^i} = \tau u$. Hence $\tau^{vy^i} d^{vy^i} K' = \tau u K'$. As τ,d,u,$v \in N$ and N/K' is abelian we get that $\tau^{y^i} d^{y^i} K' = \tau u K'$. But $[y, \tau] = 1$, so $\tau d^{y^i} K' = \tau u K'$ and so $d^{y^i} K' = uK'$. So every element of $K/K' - \{1\}$ has the form $u^{y^j} K'$. Since $y^2 \in N$, $u^{y2} K' = uK'$, we get that $K/K' = \{K', uK', u^y K'\}$ and that $u \notin K'$.

It follows that $|K/K'| = 3$ and that $C_G(\tau) \cap K' = 1$. So τ acts fixed point freely on K' and therefore K' is abelian of index 12 in G. Note that now $G = K' \cdot C_G(\tau)$, a semi-direct product. As $\langle y \rangle = T \subseteq C_G(\tau)$ and $C_G(T) = T$, we get that $G/K' \cong C_G(\tau)$ is non-abelian. In particular $|C_G(y)| = \left|C_{G/K'}(yK')\right|$ and also $\left|C_G(y^{-1})\right| = \left|C_{G/K'}(y^{-1}K')\right|$. Clearly $|C_G(\tau)| = \left|C_{G/K'}(\tau K')\right| = 12$. So if there exists $\chi \in Irr(G)$ such that $K' \nsubseteq \ker(\chi)$, then the second orthogonality relation implies that $\chi(\tau) = \chi(y) = \chi(y^{-1}) = 0$. Note that y and y^{-1} are not conjugate in G. Indeed, if $y^w = y^{-1}$ then $w \in N_G(T) = C_G(T) \times X = T \times X$ for some subgroup X of odd order. But $C_G(T) = T$ so $X = 1$ and $N_G(T) = T$ forcing w to centralize T. So y and y^{-1} are not conjugate in G. Hence $n(\chi) \geq 3$, a contradiction. Therefore $K' \subseteq \ker(\chi)$ for all $\chi \in Irr(G)$, implying that $K' = 1$ and $G = C_G(\tau)$. But now $\tau \in Z(G)$ so $< \tau > \lhd G$, which contradicts our assumption.

So our **claim** is true and $< \tau > \lhd G$. Then $\tau \in Z(G)$. As G is not nilpotent we see that $\frac{G}{<\tau>}$ is non-abelian and 4 does not divide $\left|\frac{G}{<\tau>}\right|$. Now Lemma 2.3 implies that claim b. holds. ■

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