4
Linear theory and the spinodal

In §1.2, the spatially uniform state $u(x, t) = \frac{1}{2}$ was shown to be unstable within the framework of a very naive stability analysis in which both nonlinear and gradient energy terms were neglected. The analysis there was not particularly satisfactory since the perturbations about the uniform state were governed by an equation, the backwards diffusion equation, which is ill-posed. In the present chapter we shall revisit the stability analysis about the uniform state $u(x, t) = \frac{1}{2}$, this time taking into account the gradient energy terms but neglecting the nonlinear terms as before. In this sense, it can be said that a *linear stability* analysis about the uniform state $u(x, t) = \frac{1}{2}$ is undertaken. Within this framework, it shall again be possible to obtain a formal solution to the linearized problem (the *linear stability problem*) and to demonstrate that instability of the uniform state $u(x, t) = \frac{1}{2}$ is predicted as before. The resultant solution will no longer be obviously ill-posed, as was the case for the backwards diffusion equation. With this positive result in hand, we then generalize the discussion of linear stability about $u(x, t) = \frac{1}{2}$ and consider the evolution of perturbations about the uniform state $u(x, t) = \bar{u}$, where $\bar{u} \in R$ is arbitrary. In this manner, we shall see that it is possible to distinguish between compositions $\bar{u} \in (\bar{u}_-, \bar{u}_+)$ which are exponentially unstable and compositions $\bar{u} \in R \setminus (\bar{u}_-, \bar{u}_+)$ which are stable, at least in the sense *uniform stability* in which that perturbations which begin small stay small and only possibly get smaller.

The stability bounds mentioned above turn out to depend on both ϵ/L, where L is the size of the domain. While ϵ reflects a material property of the system, the parameter L can be varied with relative ease. Since in most systems, the size of the system is very large relative to the size of the (micro-)structures under consideration, the behavior of the stability limits as $\epsilon/L \to 0$ is of physical relevance. The limiting compositions, $\lim_{\epsilon/L\to 0} \bar{u}_{\pm}(\epsilon/L)$, are known as the *spinodal* compositions.

In §4.2, it is demonstrated that the equation governing the evolution of perturbations about an arbitrary uniform state is indeed well-posed, and hence our conclusions with regard to linear stability have a firm mathematical basis. Clearly it would be desirable to consider a fuller *nonlinear stability problem* in which nonlinear terms are retained. Some remarks are made with regard to certain aspects of the analysis here which may be carried over and used in considering well-posedness of the nonlinear stability problem, which should accompany any reasonable study of the nonlinear stability. A more complete discussion is postponed until later, since it can be done more easily and rapidly after a discussion of the existence for the original equation has been given. Basically the analysis of the nonlinear stability problem is similar though simpler than that for the original (nonlinear) Cahn-Hilliard equation. To ease our approach into the more complete nonlinear formulation, in the last two sections of this chapter we present the main existence results which can be obtained for the full nonlinear theory for both Cases I and II, and the definitions of various spaces which are utilized.

Readers who are already well versed in linear stability theory in a Hilbert space setting might like to skip directly to the main results of the linear stability theory which are given ... and to the statement of the existence results which are given in §4.3.

4.1 The spinodal and the fastest growing mode

In §1.2, we considered the linear stability of the one-dimensional Cahn-Hilliard equation in the context of *Case I*. There the growth of perturbations about the initial state $u_0(x) \equiv \frac{1}{2}$ was analysed, and it was demonstrated that if the gradient energy term was neglected, then the evolution of perturbations was governed by the backwards diffusion equation. By constructing formal solutions, we found that the backwards diffusion equation was ill posed; i.e., that arbitrarily small perturbations could become arbitrarily large, arbitrarily fast. Such a result is clearly problematic for any model which claims to be physically viable. We shall now return to reconsider the linear stability analysis as before. This time, however, the effects of the gradient energy term shall be included. As a result the linear problem, which should be approximately satisfied by sufficiently small perturbations, will no longer be obviously ill-posedness, as it was in the case of the backwards diffusion equation. There afterwards, we shall generalize the analysis to treat perturbations about an arbitrary spatially uniform state, $u_0(x) \equiv \bar{u}$, where $\bar{u} \in R$ is constant. This shall allow us to formalize our definition of the *spinodal*, which corresponds to the *limit of linear stability* as $\epsilon/L \to 0$, or in other words, as the domain becomes arbitrarily large and ϵ is held fixed. In the section which follows, Section 4.2, the issue of well-posedness is treated with greater care.

Let us now consider perturbations about the spatially uniform state, $u_0(x) \equiv \frac{1}{2}$, and in accordance with (1.26) we shall assume that $M(u) = M_0$ and $f(u) = -u + u^3$. Setting

$$u(x,\, t) = \frac{1}{2} + \tilde{u}(x,\, t), \tag{4.1}$$

and substituting (4.1) into the one-dimensional Cahn-Hilliard equation defined on $(0,\, L) \times (0,\, T)$ for $L > 0$, $T > 0$,

$$\tilde{u}_t = M_0[-\tilde{u} + \frac{3}{4}\tilde{u} + \frac{3}{2}\tilde{u}^2 + \tilde{u}^3 - \epsilon^2 \tilde{u}_{xx}]_{xx}, \quad (x,\, t) \in (0,\, L) \times (0,\, T), \tag{4.2}$$

$$\tilde{u}_x(0,\, t) = \tilde{u}_x(L,\, t) = \tilde{u}_{xxx}(0,\, t) = \tilde{u}_{xxx}(L,\, t) = 0, \quad t \in (0,\, T), \tag{4.3}$$

$$\tilde{u}(x,\, 0) = \tilde{u}_0(x), \quad x \in [0,\, L]. \tag{4.4}$$

The perturbation $\tilde{u}_0(x)$ shall be assumed to be prescribed and to satisfy $|\tilde{u}| \ll 1$ for all $x \in [0,\, L]$. Moreover, let us assume that $\tilde{u}_0$ is smooth, or at least that $\tilde{u}_0$ is in $L^2([0,\, L])$, the space of square integral functions.† As in §1.2, we shall neglect the nonlinear terms in $\tilde{u}$ in (4.2), however this time we shall maintain the (linear) term $\epsilon^2 u_{xxxx}$.

Thus, we must solve the linear initial-boundary value problem:

$$\tilde{u}_t = M_0(-\frac{1}{4}\tilde{u} - \epsilon^2 \tilde{u}_{xx})_{xx}, \quad (x,\, t) \in (0,\, L) \times (0,\, T), \tag{4.5}$$

$$\tilde{u}_x(0,\, t) = \tilde{u}_x(L,\, t) = \tilde{u}_{xxx}(0,\, t) = \tilde{u}_{xxx}(L,\, t) = 0, \quad t \in (0,\, T), \tag{4.6}$$

$$\tilde{u}(x,\, 0) = \tilde{u}_0(x), \quad x \in [0,\, L]. \tag{4.7}$$

The question now arises as to how we might best proceed to solve (4.5)-(4.7). In fact, there are various ways in which one might proceed at this point. A possible approach is to attempt to solve the above system by the method of separation of variables [106, 104, 88], which was seen in §1.2 to be useful in our stability analysis when gradient effects were neglected. That approach would imply here to look for solutions of the

† See Appendix 1 for a more precise definition of $L^2([0,\, L])$.

form

$$u(x,\, t) = X(x)T(t). \tag{4.8}$$

By substituting (4.8) into (4.5), $X(x)$ and $T(t)$ can be seen to satisfy

$$\frac{T'}{M_0 T} = \frac{(-\frac{1}{4}X - \epsilon^2 X'')''}{X} = \lambda, \tag{4.9}$$

where λ is independent of both x and t. Since we wish to solve (4.9) and to guarantee that the solution given by (4.8) also satisfies the boundary condition (4.6), we must look for the eigenvalues and eigenvectors of the fourth order eigenvalue problem

$$(-\frac{1}{4}X - \epsilon^2 X'')'' = \lambda X, \quad x \in (0,\, L), \tag{4.10}$$

$$X'(0) = X'(L) = X'''(0) = X'''(L) = 0. \tag{4.11}$$

This approach seems plausible. Moreover, it can demonstrated that there exists a countable set of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ and associated eigenvectors $\{X_n(x)\}_{n=0}^{\infty}$ for (4.10)-(4.11), with fairly reasonable properties, as we shall explain shortly. However, there is a far easier approach based on using the well studied set of eigenvalues and eigenvectors which we made use of earlier in §1.2, namely the eigenvalues and the eigenvectors for the problem

$$X'' = \lambda X, \quad x \in (0,\, L), \tag{4.12}$$

$$X'(0) = X'(L) = 0. \tag{4.13}$$

For this second eigenvalue problem, we can recall (or easily calculate) that the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ and eigenvectors $\{X_n(x)\}_{n=0}^{\infty}$ are given by

$$\lambda_k = \frac{k\pi}{L}, \quad X_n(x) = \cos(k\pi x/L), \quad k = 0,\, 1,\, 2,\, \ldots.$$

It is well known that the set of functions

$$\cos(k\pi x/L), \quad k = 0,\, 1,\, 2,\, \ldots,$$

spans $L^2([0,\, L])$. This follows both from standard results in Fourier analysis [89, 104] and from standard results in Sturm-Liouville theory. We shall explain Sturm-Liouville theory [106, 104, 88] briefly, since its results can be readily generalized to a much wider setting.

Definition 4.1.1 *An ordinary differential equation which can be written* *in the form:*

$$(p(x)u_x)_x + [q(x) + \lambda r(x)]u = 0, \tag{4.14}$$

$$\alpha u(a) + \beta u_x(a) = 0, \quad \gamma u(b) + \delta u_x(b) = 0,$$

where λ *is a parameter,* p', p, q, r *are continuous for* $x \in [a, b]$ *and* p, $r > 0$, *and* α, β, γ, δ *are constants such that* $\alpha^2+\beta^2 \neq 0$ *and* $\gamma^2+\delta^2 \neq 0$, *is called a regular Sturm-Liouville problem.*

Theorem 4.1.1 *For a regular Sturm-Liouville problem defined on the interval* $[a, b]$*, the eigenvalues* $\{\lambda_n\}_{n=1}^{\infty}$ *are real and can be arranged in ascending order,*

$$\lambda_0 \leq \lambda_1 \leq \ldots.$$

Moreover, $\lim_{n\to\infty} \lambda_n = \infty$*, and the eigenfunctions* $\{X_n\}_{n=0}^{\infty}$ *which correspond to* λ_n *can be prescribed as an orthonormal sequence which constitutes a* <u>*complete basis*</u> *for* $L^2([a, b])$*; i.e., for any* $w \in L^2([a, b])$

$$\sum_{n=0}^{\infty} | < w, \phi_n > |^2 = ||w||_{L^2([a, b])},$$

where $||w||^2_{L^2([a, b])} = |< w, w >|$, *and* $< w, X_n >:= \int_a^b w(x)X_n(x)r(x)\,dx.$

Proof See, for example, Courant-Hilbert [25] or Nagy-Riesz [72]. □

We comment that since (4.14) is a second order boundary value problem, for each $\lambda \in R$ there are always either 0, 1, or 2 linearly independent solutions to (4.14), and by the definition of an eigenvalue, λ constitutes an eigenvalue if there is one or more nontrivial solutions. To reduce the eigenvectors to an orthonormal sequence, by Theorem 4.1.1 one can first order the λ_n as an increasing sequence. Then, if for a given eigenvalue there is only one associated eigenfunction, it is only necessary to scale the eigenfunction so that its norm will be unity. If there are two eigenfunctions, they can be reduced to orthonormal form via the Gram-Schmidt procedure. The notions of *spanning* $L^2([a,b])$ and comprising a *complete basis for* $L^2([a,b])$ are equivalent.

Turning now to (4.12)-(4.13), we see that in this context $a = 0$, $b = L$, $\alpha = \gamma = 0$, $\beta = \delta = 1$, $p(x) = r(x) = 1$. Hence the conclusions of the theorem above hold, and thus the set of eigenfunctions given by the cosine series, $\{\cos(k\pi x/L)\}_{k=0}^{\infty}$ comprises a complete basis in $L^2([0, L])$, as claimed earlier. We remark that that the functions in the series

$\{\cos(k\pi x/L)\}_{k=0}^{\infty}$ are orthogonal; i.e., $< \cos(i\pi x/L), \cos(j\pi x/L) >= 0$ if $i \neq j$, and to obtain an *orthonormal basis* in the sense of Theorem 4.1.1, the series $\{\cos k\pi x/L\}_{k=0}^{\infty}$ must be further normalized. However it is convenient to continue to work here with the Fourier analysis normalization which is commonly adopted in seeking formal solutions to PDEs, [89]. Let us now suppose that for all $t \geq 0$ our solution lies in $L^2([0, L])$. If this is the case then our solution can be represented in the form

$$\tilde{u}(x, t) = \frac{A_0(t)}{2} + \sum_{1}^{\infty} A_k(t) \cos(k\pi x/L), \quad t \geq 0. \tag{4.15}$$

Note that for each k, $k = 0, \ldots, \infty$, $X_k(x) = \cos(k\pi x/L)$ in fact satisfies all of the boundary conditions in (4.5)-(4.7), even though in considering (4.12)-(4.13), the functions $X_k(x)$ were only required to satisfy the Neumann boundary conditions which also appear in (4.5)-(4.7). Hence the boundary conditions in (4.6) will be satisfied by (4.15), if (4.15) can be differentiated term by term. Therefore, it seems reasonable to look for a formal solution to (4.5)-(4.7) of the form given in (4.15). We may now proceed, as in the method of separation of variables, to obtain a formal solution, with the hope of being able to justify the formal solution afterwards.

In order to satisfy the initial conditions, we should require that

$$\tilde{u}_0(x) = \frac{A_0(0)}{2} + \sum_{1}^{\infty} A_k(0) \cos(k\pi x/L). \tag{4.16}$$

Since it has been assumed that $u_0(x) \in L^2([0, L])$, the coefficients in (4.16) may [89] be determined by

$$A_k(0) = \frac{2}{L} \int_0^L \tilde{u}_0(x) \cos(k\pi x/L)\, dx, \quad k = 0, 1, 2, \ldots. \tag{4.17}$$

Given now (4.15) and (4.17), it remains to determine the functions $A_k(t)$, $k = 0, 1, \ldots,$ for $t > 0$. Substituting (4.15) into (4.5) and differentiating the series term by term as necessary, we obtain that

$$\frac{A_0'}{2} + \sum_{1}^{\infty} A_k' \cos(k\pi x/L) = M_0 \sum_{1}^{\infty} \Big(\frac{k^2\pi^2}{4L^2} - \frac{\epsilon^2 k^4 \pi^4}{L^4} \Big) A_k(0) \cos(k\pi x/L). \tag{4.18}$$

Since the sequence $\{\cos(n\pi x/L)\}_{n=0}^{\infty}$ is orthogonal, (4.18) yields that

$$\begin{cases} A_0' = 0, \\ A_k' = M_0 \Big(\frac{k^2\pi^2}{4L^2} - \frac{\epsilon^2 k^4 \pi^4}{L^4} \Big) A_k, \quad k = 1, 2, \ldots. \end{cases} \tag{4.19}$$

Solving (4.19), we obtain that

$$\begin{cases} A_0(t) = A_0(0), \\ A_k(t) = A_k(0)e^{\frac{k^2\pi^2}{L^2}[\frac{1}{4}-\frac{\epsilon^2 k^2\pi^2}{L^2}]t}, \quad k = 1, 2, \ldots . \end{cases} \tag{4.20}$$

Thus, in summary, we have obtained the formal solution

$$u(x, t) = \frac{A_0(0)}{2} + \sum_1^\infty A_k(0)e^{\frac{k^2\pi^2}{L^2}[\frac{1}{4}-\frac{\epsilon^2 k^2\pi^2}{L^2}]t} \cos(k\pi x/L). \tag{4.21}$$

Before discussing the behavior and physical implications of (4.21), let us see what would have happened had we continued to work within the framework of (4.10)-(4.11) and (4.9). For this purpose, let us consider the generalization of Sturm-Liouville problems which we referred to earlier, namely eigenvalue problems for certain *self-adjoint operators* [67].

Definition 4.1.2 *Suppose that $A : \mathcal{D}(A) \to X$ is a linear operator, where X is a Hilbert spaces and $\overline{\mathcal{D}(A)} = X$. We say A is a self-adjoint operator if $\mathcal{D}(A) = \mathcal{D}(A)$ and $A = A^*$.*

In Definition 4.1.2, A^* denotes the operator which is *adjoint* to A. The definition of an adjoint operator in a Hilbert space setting may be given as follows:

Definition 4.1.3 *Let $A : \mathcal{D}(A) \to X$ be a linear operator, where $\overline{\mathcal{D}(A)} = X$ and X is a Hilbert space. Let A^* denote the Hilbert space adjoint to A. Then $y \in \mathcal{D}(A^*)$ iff there exists an $z \in X$ such that*

$$(Ax, y) = (x, z), \qquad \textit{for all } x \in \mathcal{D}(A),$$

and for $y \in \mathcal{D}(A^*)$, we may define $A^* : \mathcal{D}(A^*) \to X$ by $A^*y = z$.

To aid in identifying self-adjoint operators, we remark that a linear operator $A : \mathcal{A} \to X$ where X is a Hilbert space, is said to be *symmetric* if

$$(Ax, y) = (x, Ay) \qquad \text{for all}\, x, y \in \mathcal{D}(A).$$

Hence it follows from Definition 4.1.2 that $A : \mathcal{D}(A) \to X$ is self-adjoint iff it is symmetric, $\overline{\mathcal{D}(A)} = X$, and $\mathcal{D}(A) = \mathcal{D}(A^*)$.

The eigenvalues and eigenvectors for many self-adjoint operators behave very much like the eigenvalues and eigenvectors of Sturm-Liouville problems. To elucidate this point, we first recall the definitions of the *resolvent set*, the *resolvent operator*, and *compact operators*.

Definition 4.1.4 *Let A be a linear operator in a normed space X. The resolvent set of A, denoted by $\rho(A)$, is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists a bounded linear operator $R(\lambda) : X \to X$ such that*

(1) for every $y \in X$, $R(\lambda)y \in \mathcal{D}(A)$ and $(A-\lambda)R(\lambda)y = y$,

(2) $R(\lambda)(A-\lambda)x = x$ for all $x \in \mathcal{D}(A)$.

When $\lambda \in \rho(A)$, $R(\lambda)$ is called the resolvent of A at x and is usually denoted by $(A-\lambda)^{-1}$.

Definition 4.1.5 *Let X be a Banach space and let $A : X \to X$ be a bounded linear operator. Then A is said to be compact if for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, where n_k are integers such that $n_1 < n_2 < \dots$, such that $\{Ax_{n_k}\}_{k=1}^{\infty}$ converges to an element of X.*

Theorem 4.1.2 *Let X be a Hilbert space, and let $A : \mathcal{D}(A) \to X$ be a self-adjoint operator whose resolvent $(A-\lambda)^{-1}$ is compact. Then the spectrum of the eigenvalue problem*

$$Ax + \lambda x = 0, \quad x \in X,$$

is real and consists of a countable set of eigenvalues. To each eigenvalue there corresponds a finite set of linearly independent eigenfunctions. The eigenvalues may be arranged in ascending order $\lambda_0 \le \lambda_1 \le \dots$, where eigenvalues are repeated in accordance with their multiplicity (the dimension of the span of their eigenfunctions), and $\lim_{n\to\infty} = \infty$. Moreover, the corresponding eigenfunctions, $\{\Phi_k(x)\}_{k=0}^{\infty}$, can be prescribed as an orthonormal sequence spanning X.

Proof The results stated here follow from Theorems 1.7.16 and 2.6.8 in [67]. □

We remark that if X is a Hilbert space and $A : \mathcal{D}(A) \to X$ is self-adjoint, and suppose that

(1) there exists $\theta \in R$ such that

$$(Ax, x) \ge \theta(x, x) \text{ for all } x \in \mathcal{D}(A),$$

and

(2) $(A-\lambda_0)^{-1}$ is compact for some $\lambda_0 < \theta$,

then (see Theorem 2.6.6 in [67]) $\lambda_0 \in \rho(A)$ and by Theorem 1.7.16 in [67], $(A-\lambda)^{-1}$ is compact for all $\lambda \in \rho(A)$. Therefore the conclusions of Theorem 4.1.2 again follow.

In reference to the operator appearing in problem (4.10)-(4.11), we may state the following

Lemma 4.1.1 *The operator $A : \mathcal{D}(A) \to X$ defined by*

$$Au := M_0(-(1/4)u_{xx} - \epsilon^2 u_{xxxx}),$$

where $X = \{u \in H^4([0, L]) \,|\, u_x(0) = u_{xxx}(0) = u_x(L) = u_{xxx}(L) = 0\}$, *and* $Y = L^2([0, L])$, *is self-adjoint.*

Proof We define A, X, Y by .... See ... for definition. □

From (4.1.1), we may conclude that if for all $t \geq 0$ the solution to (4.5)-(4.7) lies in $L^2([0, L])$, then it may be represented in the form

$$\tilde{u}(x, t) = \sum_{k=0}^{\infty} B_k(t) Y_k(x), \tag{4.22}$$

where $\{Y_n(x)\}_{n=0}^{\infty}$ denotes the complete set of eigenfunctions for (4.10)-(4.11). It is possible to calculate the eigenvectors (eigenfunctions) explicitly, see Exercise 1. However, as anyone who bothers to solve the exercise can verify, the eigenfunctions $\{Y_n(x)\}_{n=0}^{\infty}$ are considerably more clumsy and uninviting than the cosine series. Thus working with the cosine series seems definitely preferable.

Let us now rewrite the formal solution obtained in (4.21) in the more convenient form

$$u(x, t) = \frac{A_0(0)}{2} + \sum_{1}^{\infty} A_k(0) e^{\sigma(k)t} \cos(k\pi x/L), \tag{4.23}$$

where the coefficients $\{A_k(0)\}_{k=0}^{\infty}$ are defined in (4.17) and where $\sigma(k)$ is given by

$$\sigma(k) = \frac{k^2\pi^2}{L^2}\Big[\frac{1}{4} - \frac{\epsilon^2 k^2 \pi^2}{L^2}\Big], \tag{4.24}$$

for $k \in \mathcal{Z}^+$, where $\mathcal{Z}^+$ denotes the set of non-negative integers. From (4.24), we see that we may distinguish between the *growing modes* which we define by $\{k \in \mathcal{Z}^+ \,|\, \sigma(k) > 0\}$, the *neutral modes* given by $\{k \in \mathcal{Z}^+ \,|\, \sigma(k) = 0\}$, and the *decaying modes* given by $\{k \in \mathcal{Z}^+ \,|\, \sigma(k) < 0\}$. Even though from the point of view of (4.23) it is only necessary to consider $k \in \mathcal{Z}^+$, it is constructive to consider $\sigma(k)$ as a function defined for all $k \in R$. Moreover, $\sigma(k) = -\sigma(-k)$, it is only necessary to consider $\sigma(k)$ for $k \geq 0$, see Fig. 4.1. The function $\sigma = \sigma(k)$ is known as the *growth rate* or *dispersion relation*. Examining $\sigma(k)$ it is

Fig. 4.1. The dispersion relation $\sigma = \sigma(k)$ as a function of k.

easily seen that $\sigma(k)$ vanishes at $k_1 = 0$ and at $k_2 = L/(2\epsilon\pi)$, it is positive for $k \in (k_1, k_2)$, and has a unique critical point (a maximum) at $k_3 = L/(2\sqrt{2}\epsilon\pi)$, and it is negative elsewhere. In particular, we may conclude that

$$\sigma(k) \leq \sigma(k_3) = \frac{1}{64\epsilon^2}, \quad k \geq 0. \tag{4.25}$$

Even if $k_3 \notin \mathcal{Z}^+$, the mode k_3 is known as the *fastest growing mode.*

From (4.24) and from (4.20), we may conclude that for $k > k_2$, $A_k(t)$ decreases (in absolute value) as a function of time. Moreover the number of growing modes is finite and depends on the parameters of the problem, ϵ and L. Let us amplify this last remark. The number of growing modes is determined by the number of integers $k \in \mathcal{Z}^+$ such that $k_1 < \sigma(k) < k_2$. Thus considering (4.24), we see that

$$\# \text{ growing modes } = \left[\frac{L}{2\epsilon\pi}\right], \tag{4.26}$$

where the $[s]$ refers to the *integer value* of s. It follows from (4.26) that as L increases or ϵ decreases, the number of growing modes increases. Note that if L is sufficiently small or ϵ is sufficiently large, then $[\frac{L}{2\epsilon\pi}] = 0$ and there are no growing modes at all.

Up to now in this section we have assumed that $\bar{u} = 1/2$. However, this assumption has not strongly influenced our analysis. Indeed it is readily verified that if the assumption (4.1) is replace by the more general assumption that

$$u(x, t) = \bar{u} + \tilde{u}(x, t), \tag{4.27}$$

then (4.5)-(4.7) should be replaced by the more general initial value problem

$$\tilde{u}_t = M_0((-1 + 3\bar{u}^2)\tilde{u} - \epsilon^2\tilde{u}_{xx})_{xx}, \quad (x, t) \in (0, L) \times (0, T), \tag{4.28}$$

$$\tilde{u}_x(0,\,t) = \tilde{u}_x(L,\,t) = \tilde{u}_{xxx}(0,\,t) = \tilde{u}_{xxx}(L,\,t) = 0, \quad t \in (0,\,T), \quad (4.29)$$

$$\tilde{u}(x,\,0) = \tilde{u}_0(x), \quad x \in [0,\,L]. \quad (4.30)$$

The formal solution to (4.5)-(4.7) given in (4.20) and (4.21) will also constitute a formal solution to (4.28)-(4.29) if $\sigma(k)$ is defined by

$$\sigma(k) = \frac{M_0 k^2 \pi^2}{L^2}\Big[(1 - 3\bar{u}^2) - \epsilon^2 k^2 \pi^2 / L^2\Big], \quad (4.31)$$

which generalizes (4.24), and the number of growing modes can now be seen to be given by

$$\#\text{ growing modes } = \begin{cases} \left[\frac{L\sqrt{1-3\bar{u}^2}}{\epsilon\pi}\right], & -\frac{1}{\sqrt{3}} < \bar{u} < \frac{1}{\sqrt{3}} \\ 0 & \text{otherwise} \end{cases}. \quad (4.32)$$

Note in particular that for $\bar{u} \in (-\infty,\,-1/\sqrt{3}) \cup (1/\sqrt{3},\,\infty)$ for all $L > 0$ and $\epsilon > 0$, there are no perturbations $\bar{u}_0(x) \in L^2(\Omega)$ which are predicted to grow. See Exercise 2.

Let us pause to summarize what we have done so far. In Chapter 1 the evolution of perturbations about the uniform, steady state $u(x,\,t) = 1/2$ were considered within the context of the Cahn-Hilliard equation as prescribed in *Case I.* Nonlinear terms in the perturbation were neglected, the coefficient ϵ was set to zero, and as a result, and backwards diffusion was predicted. Were we to replace the assumption $u(x,\,t) = 1/2 + \tilde{u}(x,\,t)$ in the analysis there by the more general assumption (4.27) as we did here, we would readily find that we could get a formal solution to the problem considered there, which in fact would correspond to the formal solution obtained in (4.20), (4.21), (4.31) with ϵ set to zero. Note however that setting $\epsilon = 0$ in (4.28)-(4.30) yields the forward (regular) diffusion equation for $\bar{u} \in (-\infty,\,-1/\sqrt{3}) \cup (1/\sqrt{3},\,\infty)$ and it yields the backwards diffusion equation for $\bar{u} \in (-1/\sqrt{3},\,1/\sqrt{3})$. In other words, there are values of $\bar{u}$ for which the uniform state $u(x,\,t) = \bar{u}$ is stable in the sense that

$$u(x,\,t) \to \bar{u} + \frac{A_0}{2} \quad \text{as} \quad t \to \infty. \quad (4.33)$$

Since $\tilde{u}_0(x)$ has been assumed to be small as a function in $L^2([0,\,L])$, it follows from Parseval's equality that $A_0(0)$ must also be small. Hence from (4.33) we see that if $A_0(0) = 0$, then $u(x,\,t)$ converges to the uniform steady state $u(x,\,t) = \bar{u}$, and if $A_0(0)$ is small but non-zero, then the solution $u(x,\,t)$ converges to a "nearby" uniform (steady) state. On the other hand, there is also an interval of values of $\bar{u}$ for which perturbations grow. In fact, it is easy to check that unless $A_k(0) = 0$ for all other than $k = 0$, then the predicted growth is exponential. However, the conclusion that $(-1/\sqrt{3},\, 1/\sqrt{3})$ corresponds to an "unstable regime" is somewhat dubious in this context, since the backwards diffusion equation has been seen to be ill-posed and therefore of questionable validity in modelling a physical process.

Turning our attention to the analysis which was undertaken in the present section, we can see that as in the discussion above, if $\bar{u}$ lies outside the interval $(-1/\sqrt{3},\, 1/\sqrt{3})$, then $\sigma(k) \le 0$ for all $k \in \mathcal{Z}^+$. Moreover by considering (4.21),(4.31), we see that

$$u(x,\, t) \to \bar{u} + \frac{A_0}{2} + \xi A_{k_2}(0)\cos(k_2\pi x/2) \qquad \text{as} \qquad t \to \infty, \tag{4.34}$$

where $\xi = 1$ if $k_2 \in \mathcal{Z}^+$ and $\xi = 0$ otherwise. Again, we may argue that the limit is close, if not identical, to the spatially homogeneous steady state, $u(x,\, t) \equiv \bar{u}$. If $\bar{u}$ lies outside the interval $(-1/\sqrt{3},\, 1/\sqrt{3})$ where "stable behavior" is predicted, it is not difficult to demonstrate well-posedness of the linear problem satisfied by the perturbations, (4.28)-(4.30), both when $\epsilon > 0$ and when $\epsilon = 0$. See Exercise 3.

By considering (4.32), we can see that if $\bar{u} \in (-1/\sqrt{3},\, 1/\sqrt{3})$, then there exist growing modes if $L\sqrt{1 - 3\bar{u}^2}/\epsilon\pi > 1$, in other words if

$$\frac{-1}{\sqrt{3}}\left[1 - \left(\frac{\epsilon\pi}{L}\right)^2\right)^{1/2} < \bar{u} < \frac{1}{\sqrt{3}}\left[1 - \left(\frac{\epsilon\pi}{L}\right)^2\right]^{1/2}. \tag{4.35}$$

From the solution (4.23), it can be seen that exponential growth is predicted, so long as at least some of the amplitudes of the growing modes are non-vanishing in the initial perturbation, $\tilde{u}_0(x)$. The interval designated in (4.35) may be considered as defining the range *linear instability*, so long as the linear stability problem given in (4.28)-(4.30) can be seen to be well-posed. This indeed is not very difficult to demonstrate, as we shall see in the section which follows. Note that it follows from (4.35) that the regime of linear instability depends on ϵ and on L, as well as on $\bar{u}$. It is physically reasonable to assume (outside of the nano context in which systems on the order of *nanometers*, i.e. 10^{-9} meters, are considered) that $\epsilon L^{-1} \ll 1$. Hence we may approximate the regime of instability by

$$\frac{-1}{\sqrt{3}} < \bar{u} < \frac{1}{\sqrt{3}}. \tag{4.36}$$

Note that this corresponds precisely to our earlier ($\epsilon = 0$) predictions.

Thus our earlier, very simplistic calculation, had indeed set us thinking in the right direction. The interval (4.36) can also be considered as the unstable regime in the limit as $L \to \infty$; in other words it corresponds to the unstable regime on an infinite domain. Since the approximation which lead us to (4.36) seems well justified on physical grounds, it should roughly correspond to the range of instability as measured in experiment. As such, the bounding concentrations for this interval, $\bar{u} = \pm 1/\sqrt{3}$, have achieved a name of their own and are known as the *spinodal* compositions. More generally, as discussed in Chapter 1, the spinodal concentrations can be expected to be temperature dependent, and should thus correspond to the locus sketched in Figure 1.2. In the following section, we shall discuss in what sense the formal solutions obtained here do indeed constitute solutions to (4.28)-(4.30).

Arguably, to obtain these limiting concentrations, one should directly consider the Cahn-Hilliard equation defined on the whole real line, $(-\infty, \infty)$. This approach is possible and is undertaken in Exercise 4, using Fourier transforms and a continuously defined dispersion relation.

Certainly it would make good physical sense to extend the discussions above to the more physically realistic context of domains $\Omega \subset R^n$, with $n = 2$ or 3. This is indeed possible, and the conclusions are quite similar. However, the analysis is somewhat less explicit except in some very geometries, since in general one must work with eigenfunctions which are no longer explicitly prescribed.

In the multi-dimensional case, one may easily verify that the linearization of the Cahn-Hilliard equation about a spatially homogeneous steady state, $u(x, t) = \bar{u}$, in other words the *linear stability problem*, is given by

$$(LP) \begin{cases} \tilde{u}_t = M_0((1 - 3\bar{u}^2)\triangle\tilde{u} - \epsilon\triangle^2\tilde{u}), & (x, t) \in \Omega_T, \\ \mathbf{n} \cdot \nabla\tilde{u} = \mathbf{n} \cdot \nabla\triangle\tilde{u} = 0, & (x, t) \in \partial\Omega_T, \\ \tilde{u}_0(x, 0) = \tilde{u}_0(x), & x \in \Omega. \end{cases} \tag{4.37}$$

To solve (4.37), we rely again on the results mentioned earlier on self-adjoint operators, this time in regard to the Laplacian with Neumann boundary conditions

$$\begin{aligned} \mathcal{L}u := \triangle u = 0, \quad & x \in \Omega, \\ \mathbf{n} \cdot \nabla u = 0, \quad & x \in \partial\Omega, \end{aligned} \tag{4.38}$$

where $\mathcal{L}$ is now considered as an operator from $H^4(\Omega)$† to $L^2(\Omega)$. Within this setting, one may again conclude based on Theorem 4.1.2 that there is a countable sequence $\{\lambda\}_{n=0}^{\infty}$ of real eigenvalues, each with finite multiplicity, which approaches infinity as $n \to \infty$, and that the corresponding eigenfunctions can be prescribed as an orthonormal sequence $\{\Phi_k\}_{k=0}^{\infty}$ where $\Phi_k = \Phi_k(x)$ for $x \in \Omega$, which spans $L^2(\Omega)$.

In analogy with (4.15)-(4.17), we may assume that

$$\tilde{u}(x,\, t) = \frac{A_0(t)}{2} + \sum_{k=1}^{\infty} A_k(t)\Phi_k(x), \tag{4.39}$$

where

$$A_k(0) = \int_{\Omega} \tilde{u}_0(x),\; \Phi_k(x)\, dx.$$

Substituting (4.39) into (4.37) and making use of the orthonormality, we obtain a system of equations analogous to the system (4.19) obtained earlier, which may be solved to yield that

$$\tilde{u}(x,\, t) = \frac{A_0(0)}{2} + \sum_{k=1}^{\infty} A_k(0)e^{\sigma(k)t}\Phi_k(x,\, t),$$

where

$$\sigma(k) = ((1 - 3\bar{u}^2) - \epsilon^2\lambda_k^2)\lambda_k^2,$$

where λ_k are the eigenvalues of the Laplacian, discussed above.

Again, we remind the reader that the discussion above only makes sense if well-posedness can be demonstrated for (LP) in some appropriate sense. This is undertaken in the section which follows.

So far we have focused on the Cahn-Hilliard equation in the context of Case I. It is natural to inquire at this point with regard to linear stability and the spinodal compositions (concentrations) for the degenerate Cahn-Hilliard equation, *Case II*. It turns out that the analysis is quite similar, and that a very similar discussion can be given with regard to linear stability and the spinodal, see Exercise 5.

Exercises

4.1 Find the eigenvalues and eigenfunctions for (4.10)-(4.11).

4.2 Verify (4.28)-(4.30), (4.31), and (4.32).

† A definition of $H^4(\Omega)$ is given in §4.4.