

The Cahn-Hilliard Equation:  
From Backwards Diffusion to Surface Diffusion

Amy Novick-Cohen



# Contents

<b>1</b>	<b>An overview</b>	<i>page</i> 1
1.1	The Cahn-Hilliard equation	1
1.2	Backwards diffusion and regularization	3
1.3	The Cahn-Hilliard equation and phase separation	6
1.4	Two prototype formulations	13
1.5	Long time behavior and limiting motions	20
<b>2</b>	<b>The derivation</b>	27
2.1	A conservation law	29
2.2	Onsager relations and chemical potentials	31
2.3	Diffusion and heat equation analogues	33
2.4	Constitutive assumptions for the mobilities	35
2.5	Constitutive assumptions for the chemical potentials	36
2.6	Gradient energy regularization	43
2.7	The first variation and the boundary conditions	45
2.8	The problem formulations	54
<b>3</b>	<b>Energetic properties</b>	60
3.1	Mass conservation	60
3.2	Energy dissipation	60
3.3	Gradient flow	60
3.4	Energy minimization	60
<b>4</b>	<b>Linear theory and the spinodal</b>	70
4.1	The spinodal and the fastest growing mode	71
4.2	Regularization and well-posedness	84
4.3	Towards a nonlinear theory	86
4.4	The functional setting	87

<b>5</b>	<b>Basic theory for the constant mobility case</b>	89
5.1	Existence and uniqueness	89
5.2	Additional regularity	89
5.3	Steady states and their stability in one-dimension	89
5.4	Higher dimensional steady states	97
<b>6</b>	<b>Early behavior</b>	98
6.1	Spinodal decomposition	98
6.2	The prolonged linear regime	98
6.3	Scaling predictions	98
<b>7</b>	<b>The Cahn-Hilliard equation as a dynamical system</b>	99
7.1	Existence of a global attractor	99
7.2	The structure of the global attractor in one dimension	99
7.3	Dynamics in the neighborhood of the global attractor	99
<b>8</b>	<b>Qualitative long time dynamics in higher dimensions</b>	100
8.1	The Mullins-Sekerka problem: a formal derivation	100
8.2	The Mullins-Sekerka Problem: rigorous justification	100
<b>9</b>	<b>Coarsening</b>	101
9.1	Some distinguished limits...	102
9.2	Some auxiliary and technical results	105
9.3	Three lemmas	107
9.4	Proofs of the three lemmas	108
9.5	Predictions of the Lemmas	115
<b>10</b>	<b>The deep quench limit</b>	117
10.1	The physical model	117
10.2	Existence and regularity	117
10.3	Steady states	117
10.4	A sharp interface limit	117
10.5	Coarsening	117
<b>11</b>	<b>The degenerate Cahn-Hilliard equation</b>	118
11.1	Existence, regularity, and non-uniqueness	118
11.2	Steady states and spreading	118
11.3	Surface diffusion: a formal derivation	118
<b>12</b>	<b>A survey of numerical and computational results</b>	119
12.1	A finite element formulation	119
12.2	Practical implementation	119
12.3	A panoply of results	119
12.4	Alternative approaches	119

<b>13</b>	<b>Various generalizations of the Cahn-Hilliard model</b>	120
13.1	Non-isothermal systems and phase field models	120
13.2	The viscous Cahn-Hilliard equation	120
13.3	Gurtin type models	120
13.4	Models with memory and other non-local effects	120
13.5	Reaction terms and control problems	120
13.6	Coupling conserved and nonconserved dynamics	120
<b>14</b>	<b>Additional applications of the Cahn-Hilliard model</b>	121
14.1	Cahn-Hilliard equations: a characterization	121
14.2	Biofilms	124
14.3	An augmented thin film equation	129
14.4	The rings of Saturn	136
14.5	Image processing	142
14.6	River beds	142
14.7	Ecology	142
<b>15</b>	<b>Further directions and open questions</b>	143
	<i>Appendix 1</i> Some results from functional analysis	144
	<i>References</i>	148