Induction from Elementary Abelian Subgroups

E. Aljadeff* and Y. Ginosar

Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel

Communicated by Kent R. Fuller

Received October 1, 1994

INTRODUCTION

Let $R$ be a ring with identity element 1 and $G$ a finite group. $RG$ denotes the group ring. Chouinard [C] showed that an $RG$-module $M$ is weakly projective (projective) iff it is $RH$ weakly projective (projective) for every elementary abelian subgroup $H$ of $G$ (see the definition of weak projective in Section 1). His proof is based on Serre’s theorem on products of Bockstein operators. Chouinard’s theorem can be generalized (using his original result) to arbitrary crossed products $R * G$. Recall that a crossed product $R * G$ is an associative ring which contains $R$ and has an $R$ basis $\{u_{\sigma}\}_{\sigma \in G}$. The multiplicative structure is given by the rules:

1. (“Twisting”) $u_{\sigma}u_{\tau} = \alpha(\sigma, \tau)u_{\sigma\tau}$, where $\alpha: G \times G \to U(R)$ (units of $R$).
2. (“Action”) $u_{\sigma}x = t_{\sigma}(x)u_{\sigma}$ where $t_{\sigma} \in \text{Aut}(R)$.

(See also [P, Chap. 1, Sect. 1].)

One of the main points of this generalization is that results on group actions on rings $R$ may be obtained from the structure of $R$ as a module over the skew group ring $R, G$.

For a given action of $G$ on $R$ (i.e., a homomorphism $t: G \to \text{Aut}(R)$) we construct the trace map for $H \leq G$:

$$\text{tr}_{H}: R \to R^H \quad (H\text{-invariants})$$
$$r \to \sum_{\sigma \in H} \sigma(r).$$

* This research was supported by the Fund for the Promotion of Research at the Technion.
It is easy to show (Section 2) that if $\text{tr}_G$ is surjective onto $R^G$ then $\text{tr}_H$ is surjective onto $R^H$ for every subgroup $H$ of $G$. In this paper we prove

**Theorem 1.** Let $R$ and $G$ be as above. If $\text{tr}_H$ is surjective onto $R^H$ for every elementary abelian subgroup $H$ of $G$, then $\text{tr}_G$ is surjective onto $R^G$.

Let us point out that in case that $R$ is a commutative ring, $\text{tr}_G$ is surjective onto $R^G$ iff $\text{tr}_P$ is surjective onto $R^P$ for every prime order subgroup $P$ of $G$ (see [A, Corollary 02]). This stronger result is known to fail for arbitrary non-commutative rings. Theorem 1 is a consequence of our main result.

**Theorem 2.** Let $R * G$ be a crossed product of $G$ over $R$ and let $M$ be an $R * G$-module. Then $M$ is weakly projective (projective) iff $M$ is $R * H$ weakly projective (projective) for every elementary abelian subgroup $H$ of $G$.

To see that it implies Theorem 1 above, let $R * G = R_G$ be the skew group algebra of $G$ over $R$ defined by the given action $t: G \to \text{Aut}(R)$ (trivial twisting) and note that $R$, as a principal $R_G$ module, is projective iff the trace map $\text{tr}_G$ is surjective onto $R^G$ (see [ARS, Proposition 1.7] or Section 2 below). From Theorem 2 we conclude also

**Theorem 3.** Let $R * G$ be a crossed product algebra and $M$ a left $R * G$ module. Then

$$\text{proj.dim}_{R * G} M = \sup_{H \leq G \text{ elementary abelian}} \{\text{proj.dim}_{R * H} M\}$$

and so

$$\text{gl.dim} R * G = \sup_{H \leq G \text{ elementary abelian}} \{\text{gl.dim} R * H\}.$$ 

This generalizes a theorem of Yi [Y, Corollary 5.4].

### 1. WEAK PROJECTIVE MODULES OVER CROSSED PRODUCTS

**Definition.** A module $M$ over $R * G$ is said to be weakly projective if for every diagram of $R * G$ modules (and $R * G$ morphisms)

$$\begin{array}{c}
\begin{array}{c}
M \\
\mu
\end{array} \\
A \xrightarrow{\phi} B \xrightarrow{} 0
\end{array}$$
in which \( \varphi \) may be lifted to an \( R \)-map, it may be lifted also to an \( R \ast G \) map.

For a left \( R \ast G \)-module \( M \), a diagonal action of \( G \) on \( \text{Hom}_R(M, M) \) is defined \( (\sigma(f) = u_\sigma f u_\sigma^{-1}, u_\sigma \in R \ast G) \) and hence we have the (trace) map

\[
\text{tr}_G: \text{Hom}_R(M, M) \to \text{Hom}_{R \ast G}(M, M)
\]

\[
f \mapsto \sum_{\sigma \in G} u_\sigma f u_\sigma^{-1}.
\]

**Lemma 4.** Let \( M \) be an \( R \ast G \) module. Then \( M \) is \( R \ast G \) weakly projective iff there exists an \( f \in \text{Hom}_R(M, M) \) with \( \text{tr}_G(f) = 1_M \).

**Proof.** Construct the (left) \( R \ast G \) module \( \mathbb{Z}G \otimes_\mathbb{Z} M \) with the diagonal action

\[
xu_\sigma(\sigma \otimes m) = \tau \sigma \otimes xu_\tau, \quad m \in M, \tau, \sigma \in G, \ x \in R.
\]

Consider the surjective \( R \ast G \) map

\[
\eta: \mathbb{Z}G \otimes_\mathbb{Z} M \to M
\]

\[
\sigma \otimes m \mapsto m.
\]

Clearly, \( \eta \) splits as an \( R \) map (say, by \( m \mapsto e \otimes m, \ e \) the identity element in \( G \)). Suppose \( M \) is \( R \ast G \) weakly projective, so \( \eta \) splits also as an \( R \ast G \) map, i.e., there is an \( R \ast G \) map \( \psi: M \to \mathbb{Z}G \otimes_\mathbb{Z} M \) such that \( \eta \circ \psi = \text{id}_M \).

Since \( G \) is a \( \mathbb{Z} \)-basis, \( \psi(m) = \sum_{\sigma \in G} \sigma \otimes f_\sigma(m) \) where \( f_\sigma: M \to M, \ \sigma \in G \) are uniquely defined. Furthermore, for every \( x \in R \)

\[
\sum_{\sigma \in G} \sigma \otimes f_\sigma(xm) = \sum_{\sigma \in G} \sigma \otimes xf_\sigma(m) \quad (\psi \text{ is } R\text{-linear})
\]

and so \( f_\sigma \in \text{Hom}_R(M, M) \) for every \( \sigma \in G \). Since \( \psi \) is \( R \ast G \) linear

\[
\sum_{\sigma \in G} \tau \sigma \otimes u_\tau f_\sigma(m) = \sum_{\sigma \in G} \sigma \otimes f_\sigma(u_\tau m).
\]

Comparing base elements, we obtain

\[
\tau e \otimes u_\tau f_c(m) = \tau \otimes f_c(u_\tau m)
\]

or

\[
u_\tau f_c(m) = f_c(u_\tau m),
\]

\[
u_\tau f_c(u_\tau m) = f_c(u_\tau m)
\]
and denoting \( u_\tau m = m' \) we get

\[
u_\tau f_\tau u_\tau^{-1}(m') = f_\tau(m')
\]

for every \( \tau \in G \) and \( m' \in M \). Since \( \psi \) splits \( \eta \),

\[
m' = \sum_{\tau \in G} f_\tau(m') = \sum_{\tau \in G} u_\tau f_\tau u_\tau^{-1}(m').
\]

Hence \( f_\tau \in \text{Hom}_R(M, M) \) satisfies \( \text{tr}_G(f_\tau) = 1_M \).

Conversely, let \( f \in \text{Hom}_R(M, M) \) with \( \text{tr}_G(f) = 1_M \) and let

\[
\begin{array}{c}
M \\
\mu
\end{array}
\]

\[
A \xrightarrow{\mu} B \xrightarrow{} 0
\]

be a diagram of \( R \ast G \) modules and \( R \ast G \) maps. Suppose that \( \mu \) can be lifted to an \( R \)-map \( s: M \to A \), i.e., \( \varepsilon \circ s = \mu \). Define the \( R \ast G \) map

\[
\tilde{s}: M \to A,
\]

\[
m \mapsto \sum_{\tau \in G} u_\tau s \circ f u_\tau^{-1}(m).
\]

Clearly, \( \tilde{s} \) is an \( R \ast G \) map. Furthermore, \( \varepsilon \circ \tilde{s}(m) = \varepsilon \sum_{\tau \in G} u_\tau s \circ f u_\tau^{-1}(m) = \mu(m) \), so \( \tilde{s} \) is a lifting of \( \mu \), and \( M \) is weakly projective. \( \square \)

We apply this to obtain Chouinard’s theorem for crossed products.

**Proof of Theorem 2.** By Lemma 4, for every elementary abelian subgroup \( H \) of \( G \) there exists \( f^{(H)} \in \text{Hom}_R(M, M) \) with

\[
\text{tr}_H(f^{(H)}) = \sum_{\sigma \in H} u_\sigma f^{(H)} u_\sigma^{-1} = 1_M.
\]

Every \( f^{(H)} \) induces a map

\[
f^{(H)}_*: \text{Hom}_R(M, M) \to \text{Hom}_R(M, M)
\]

\[
g \mapsto f^{(H)}g.
\]

We claim that \( \text{tr}_H(f^{(H)}_*) = \text{id}_{\text{Hom}_R(M, M)} \). Indeed, acting on \( g \in \text{Hom}_R(M, M) \)

\[
\text{tr}_H(f^{(H)}_*)(g) = \sum_{\sigma \in H} \sigma (f^{(H)}_* \sigma^{-1}(g))
\]
and evaluating at \( m \in M \)

\[
\text{tr}_H(f^*_g)(g)(m) = \sum_{\sigma \in H} \sigma(f^*_g\sigma^{-1}(g))(m)
\]

\[
= \sum_{\sigma \in H} u_{\sigma}(f^*_g\sigma^{-1}(g))(u_{\sigma}^{-1}m)
\]

\[
= \sum_{\sigma \in H} u_{\sigma}(f^*_g\sigma^{-1}g(u_{\sigma}u_{\sigma}^{-1}m)) = g(m).
\]

Thus, \( \text{Hom}_R(M, M) \) is \( \mathbb{Z}H \) weakly projective for every elementary abelian subgroup \( H \) of \( G \). By Chouinard’s theorem [C, Theorem 1], it follows that \( \text{Hom}_R(M, M) \) is \( \mathbb{Z}G \) weakly projective and hence \( H^n(G, \text{Hom}_R(M, M)) = 0 \) (the \( n \)th Tate cohomology). The vanishing of \( H^0(G, \text{Hom}_R(M, M)) \) says that the map \( \text{tr}_G: \text{Hom}_R(M, M) \rightarrow \text{Hom}_R(M, M)^G \) is surjective. In particular, there exists \( f \in \text{Hom}_R(M, M) \) with \( \text{tr}_G(f) = 1_M \). By Lemma 4 we conclude that \( M \) is \( R \ast G \) weakly projective. Finally, the statement about projectivity (rather than weak projectivity) follows from the fact that projectivity of \( R \ast G \) modules is the same as weak projectivity together with projectivity over \( R \).

**Corollary 5** (compare with [P, 18.10]). \( R \ast G \) is semisimple artinian iff \( R \ast H \) is semisimple artinian for every elementary abelian subgroup \( H \) of \( G \).

**Proof.** This is clear.

**Proof of Theorem 3.** A projective resolution of an \( R \ast G \) module \( M \) remains projective over \( R \ast H \), and recall that \( \text{proj.dim}_{R \ast H} M \leq n \) is equivalent to \( R \ast H \) projectivity of the \( n \)th syzygy module \( K_n \) of any resolution of \( M \). The statement on the global dimension follows at once.

**Remark.** The results of this paper hold for \( G \)-strongly graded rings (generalizing crossed products \( R \ast G \)) (see [P, Chap. 1, Sect. 2]). The proofs are basically the same. Let us write the diagonal action in this case. Let \( S \) be a \( G \)-strongly graded ring, that is, \( S = \sum_{g \in G} S_g \) (the sum being direct) such that for every \( \sigma, \tau \in G \), \( S_{\sigma\tau} = S_{\sigma\tau} \) (rather than just \( S_{\sigma\tau} \subseteq S_{\sigma\tau} \) as required for \( G \)-graded rings). Let \( M, N \) be left \( S \) modules. Then the action of \( G \) on \( \text{Hom}_S(M, N) \) (e the identity element in \( G \)) is given by

\[
(\sigma h)(m) = \sum_{i=1}^{n_{\sigma}} x_i h(y_i m),
\]

where \( x_i \in S_{\sigma^i}, y_i \in S_{\sigma^{-i}} \), and \( \sum_{i=1}^{n_{\sigma}} x_i y_i = 1 \). One verifies that this action of \( G \) is well defined.
2. INDUCTION ON TRACE MAPS

We consider now a special case of crossed products—the skew group ring \( R^G \). \( R \) is an \( R^G \) module (in particular a \( G \)-module) so the trace map

\[
\text{tr}_G: R \to R^G \\
x \mapsto \sum_{\sigma \in G} \sigma(x)
\]

is defined.

It is obvious that \( \text{tr}_G: R \to R^G \) is surjective if and only if \( 1 \) is in the image of this map, and so surjectivity of \( \text{tr}_G: R \to R^G \) implies surjectivity of \( \text{tr}_H: R \to R^H \), \( H \) any subgroup of \( G \), for if \( g_1, \ldots, g_r \) are representatives of (left) cosets of \( G:H \) and \( x \in R \) is such that \( \text{tr}_G(x) = 1 \) then the element \( y = \sum_{i=1}^r g_i(x) \) satisfies \( \text{tr}_H y = 1 \). Since \( \text{Hom}_R(R, R) \simeq R \) as \( G \)-modules, Lemma 4 implies that \( R \) is an \( \mathcal{R} \mathcal{G} \) projective module if and only if there exists an element \( x \in R \) with \( \text{tr}_G(x) = 1 \). Together with Theorem 2 we obtain again Theorem 1.

It might be interesting to obtain a formula for \( x_G \) with \( \text{tr}_G(x_G) = 1 \) depending on all \( y_H \) with \( \text{tr}_H(y_H) = 1 \) for every \( H \leq G \) elementary abelian. The existence of such a formula was proved by S. Shelah.

**Proposition 6 (S. Shelah).** An expression for \( x_G \), of the required form, does exist. Moreover, fixing the finite group \( G \), the formula does not depend on the ring \( R \). Such a formula for \( x_G \) has the form

\[
\sum_{i=1}^n a_i \sigma_{i_1}(y_{H_1}) \sigma_{i_2}(y_{H_{i_2}}) \ldots \sigma_{i_s}(y_{H_{i_s}}),
\]

where \( s = s(i) \), \( a_i \in \mathbb{Z} \), \( \sigma_i \in G \), \( H_i \leq G \) elementary abelian, \( y_{H_i} \in R \) with \( \text{tr}_H(y_{H_i}) = 1 \).

**Proof.** Let \( K_G \) be the variety (i.e., defined by a set of equations)

\[
\left\{ \left( R, \varphi, y_H \right)_{\sigma \in G}, \varphi: G \to \text{Aut}(R), y_H \in R, \text{tr}_H(y_H) = 1 \right\}
\]

and let

\[
\mathcal{F}_G = \left\{ \left( R^F, \varphi_{G}^{F}, y_H \right)_{\sigma \in G}, \varphi_{G}^{F}: G \to \text{Aut}(R), y_H \in R, \text{tr}_H(y_H) = 1 \right\}
\]
be the free algebra in $K$ (i.e., it is generated by $0, 1, \ldots, y, \ldots$) and by the operations $\{+, -, \times, \varphi_\sigma\}_{\sigma \in G}$. By Theorem 1, there exists $x_G \in R^G$ with

$$\sum_{\sigma \in G} \varphi_\sigma(x_G) = 1. \quad (\ast)$$

By algebraic manipulations $x_G$ is equal to some term as described above. Finally, by the freeness, $(\ast)$ holds for every $(R, \varphi, y_H) \in K$ as required.

If $R$ is commutative, an explicit formula for $x_G$ is given in [A, Theorem 2.1] depending on all $y, P \leq G$ (cyclic) of prime order. Here we show how to compute $x_G$ in the special case that $R$ is an algebra over $\mathbb{F}_2$ and $G$ is an abelian 2-group (or has such 2-Sylow subgroup).

First, let $G$ be a cyclic group of order $2^l$, $l > 1$ generated by $\sigma$, and $H = \langle 1, \sigma^{2^l-1} \rangle$ be its unique elementary abelian subgroup.

**Claim.** If $x \in R$ satisfies $tr_H(x) = x + \sigma(x)^{2^{l-1}} = 1_R$, then $y = x\sigma(x)$ satisfies $tr_G(y) = 1$.

**Proof.**

$$tr_G(x\sigma(x)) = \sum_{i=0}^{2^l-1-1} \sigma^i(x\sigma(x))$$

$$= \sum_{i=0}^{2^l-1-1} \sigma^i[x\sigma(x) + \sigma^{2^l-1}(x\sigma(x))]$$

$$= \sum_{i=0}^{2^l-1-1} \sigma^i[x\sigma(x) + \sigma^{2^l-1}(x)\sigma^{2^l-1+1}(x)]$$

$$= \sum_{i=0}^{2^l-1-1} \sigma^i[x\sigma(x) + (1-x)(1-\sigma(x))]$$

$$= \sum_{i=0}^{2^l-1-1} \sigma^i[1-x - \sigma(x)] = -x - \sigma^{2^l-1}(x) = 1_R$$

as desired.

Next, let $G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \oplus \cdots \oplus \langle \sigma_r \rangle$ be an abelian 2-group. By induction it is enough to compute $x_G$ with $tr_G(x_G) = 1$ in case that $G = \langle \sigma \rangle \oplus N$, $|\sigma| = 2^{l+1}$, $H = \langle \sigma^{2^l} \rangle \oplus N$, and $x_H \in R$ such that $tr_H(x_H) = 1$. Indeed, let $z = tr_N(x_H)$. Then $tr_{\langle \sigma^{2^l} \rangle}(z) = 1$ and by the cyclic case, $tr_{\langle \sigma^{2^l} \rangle}(z\sigma(z)) = 1$. Now, $z$ and therefore $z\sigma(z)$ are $N$ invariant, so there exists $y \in R$ with $tr_N(y) = z\sigma(z)$. Consequently $tr_G(y) = 1$. Following the steps above we have

$$y = tr_N(x_H)\sigma(tr_N(x_H))tr_{\langle \sigma^{2^l} \rangle}(x_H).$$
ACKNOWLEDGMENT

We thank the referee very much for his useful report.

REFERENCES


