On the Projective Schur Group of a Field

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If \( k \) is a field, the projective Schur group \( \text{PS}(k) \) of \( k \) is the subgroup of the Brauer group \( \text{Br}(k) \) consisting of those classes which contain a projective Schur algebra, i.e., a homomorphic image of a twisted group algebra \( k^nG \) with \( G \) finite, \( n \in H^2(G, k^*) \). It has been conjectured by Nels and Van Oystaeyen (J. Algebra 137 (1991), 501–518) that \( \text{PS}(k) = \text{Br}(k) \) for all fields \( k \). We disprove this conjecture by showing that \( \text{PS}(k) \neq \text{Br}(k) \) for rational function fields \( k_q(x) \) where \( k_0 \) is any infinite field which is finitely generated over its prime field.

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0. INTRODUCTION

Let \( k \) be any field. A \( k \)-central simple algebra \( B \) is called a Schur algebra (over \( k \)) if it is the homomorphic image of a group algebra \( kG \) for some finite group \( G \). Equivalently, a Schur algebra over \( k \) is a finite-dimensional \( k \)-central simple algebra \( B \) which is spanned over \( k \) by a finite subgroup of the units of \( B \). The Schur group \( S(k) \) of \( k \) is the subgroup of the Brauer group \( \text{Br}(k) \) of \( k \) generated by (in fact consisting of) classes in \( \text{Br}(k) \) that are represented by Schur algebras. The Schur group is trivial for fields of positive characteristic [CR81, p. 148, Proof of Cor. 7.11], whereas for fields of characteristic zero there is the fundamental

**Theorem 0.1 (Brauer–Witt) [Y70, Chap. 3].** Every Schur algebra \( B \) over a field \( k \) of characteristic zero is Brauer equivalent to a cyclotomic algebra.

(Recall that a crossed product algebra \( K_q \langle \zeta \rangle / k = (K/k, \alpha) \) is a cyclotomic algebra if \( K = k(\zeta) \) is a cyclotomic extension (\( \zeta \) a root of unity) and \( \alpha \in H^2(Q, K^*) \) has a representative 2-cocycle whose values lie in a finite

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subgroup of $K^*$; $t$ signifies the action of $Q = G(K/k)$ on $K$. In particular, Theorem 0.1 implies that every Schur algebra has a cyclotomic splitting field."

The notions of Schur algebra and Schur group were generalized by Lorenz and Opolka [LO78], by replacing an ordinary group algebra by a twisted group algebra $k^nG$, where $\alpha \in H^2(G, k^*)$ with $G$ acting trivially on $k^*$, and obtaining the analogous notions of projective Schur algebra and the projective Schur group of $k$. More precisely, let $B$ be a finite-dimensional $k$-central simple algebra. Then $B$ is a projective Schur algebra over $k$ if it is the homomorphic image of a twisted group algebra $k^nG$ for some finite group $G$ and some $\alpha \in H^2(G, k^*)$. In characteristic zero $k^nG$ is semisimple, so a projective Schur algebra is a direct summand of a twisted group algebra. In general, a projective Schur algebra may be characterized as a $k$-central simple algebra $B$ that contains a group $\Gamma$ in the group of units of $B$ that is finite modulo $k^*$ and spans $B$ over $k$.

The basic example of a projective Schur algebra is a symbol algebra. If $B = (a, b, \zeta)_n$, where $\zeta$ is a primitive $n$th root of unity in $k$; $a, b, \zeta$ are generators of $B$; and $a^n = a, b^n = b, \zeta = \zeta$, then $B$ is isomorphic to a twisted group algebra $k^n(\mathbb{Z}/n \oplus \mathbb{Z}/n)$, where $a$ is defined by the given relations.

The projective Schur group $PS(k)$ of $k$ is defined as the subgroup of $Br(k)$ generated by (and again, consisting of) classes containing projective Schur algebras.

The projective Schur group is much larger than the Schur group. For example, if $k$ is a number field then $PS(k) = Br(k)$ [LO78]. This is a consequence of the fact that every central simple algebra over a number field $k$ has a cyclic splitting field which is contained in a cyclotomic extension of $k$ and therefore is similar to a crossed product $L^nG$ where $L = k(\zeta)$ is a cyclotomic extension, $G = G(L/k)$, and $\alpha$ has a representing cocycle with values in $k^*$. The subgroup $E$ of $L^nG^*$ given by the extension

$$\alpha: 1 \to k^* \to E \to G \to 1$$

is finite mod $k^*$, hence so is the composite $\Gamma = E(\zeta)$, which spans $L^nG$ over $k$. It follows that $L^nG$ is a projective Schur algebra over $k$.

It has been conjectured [NVO91] that $PS(k) = Br(k)$ for all fields $k$. The main purpose of this paper is to disprove this conjecture. We will show for example that $PS(k) \neq Br(k)$ for rational function fields $k_0(x)$ with $k_0$ any infinite field which is finitely generated over its prime field. In a previous paper [AS93] the authors showed that every projective Schur algebra has an abelian splitting field $L$ (i.e., $L/k$ is an abelian extension). This result was not sufficient, however, to prove that $PS(k) \neq Br(k)$, since it is still an outstanding open question whether or not every central simple
algebra has an abelian splitting field. Using the results of [AS93] on the structure of projective Schur algebras, we prove in Section 1:

*Every projective Schur algebra over a field $k$ has an abelian splitting field which is contained in a (finite) radical extension of $k$."

By a radical extension of $k$ we mean an extension of the form $K = k(A)$, where $A$ is a subgroup of $K^*$ such that $Ak^*/k^*$ is a torsion group, i.e., $K = k(\sqrt[n]{a}, \sqrt[n]{b}, \ldots)$ with $a, b, \ldots \in k^*$. In Section 2 we prove that an abelian extension of $k$ which is contained in a radical extension of $k$ is contained in the composite of a cyclotomic extension and a Kummer extension of $k$. Finally, in Section 3 we give examples of (cyclic) algebras over various fields $k$ which do not have such splitting fields, which shows that, for such $k$, $PS(k) \neq Br(k)$.

1. SPLITTING FIELDS OF PROJECTIVE SCHUR ALGEBRAS

Let $B = k(\Gamma)$ be a projective Schur algebra over $k$, where the group $\Gamma$ spans $B$ over $k$ and is finite modulo $k^*$. Let $H$ be a (normal) subgroup of \(\Gamma\) that contains the commutator subgroup $\Gamma'$ of $\Gamma$. By [AS93] the subalgebra spanned by $H$ over $k$ is a semisimple algebra. Moreover, if it is not simple one shows that $B \cong M_n(S)$, where $S$ is a projective Schur algebra over $k$ and $n$ is the number of simple components in $k(H)$. Consequently we shall assume that $k(H)$ is a simple algebra for every $\Gamma' \subset H \subset \Gamma$. Next, since $\Gamma$ is center-by-finite, the group $\Gamma'$ is finite and so $k(\Gamma')$ is a Schur algebra over its center. Consider a maximal Schur algebra (over its center) of the form $k(\Gamma_0)$, where $\Gamma' \subset \Gamma_0 \subset \Gamma$. Denote by $L$ the center of $k(\Gamma_0)$ and set $G = \text{(the abelian group)} \Gamma/\Gamma_0$.

**Theorem 1.1 [AS93].**

(a) Conjugation of $k(\Gamma_0)$ by $\Gamma$ induces an isomorphism $G \cong G(L/k)$, where $L$ is the center of $k(\Gamma_0)$. In particular, $L/k$ is an abelian extension.

(b) $k(\Gamma)$ is isomorphic to a ring-theoretic crossed product $k(\Gamma_0)^*G$.

(c) If $F/L$ is a finite extension that splits $k(\Gamma_0) \cong M_4(D)$, then it also splits $k(\Gamma)$.

Now since $k(\Gamma_0)$ is a Schur algebra over $L$, it is split by a cyclotomic extension $L(\xi)$ of $L$. (In characteristic $p > 0$, $k(\Gamma_0)$ is already split.) Furthermore, since $L/k$ is an abelian extension, $L(\xi)$ is abelian over $k$ and we obtain

**Theorem 1.2 [AS93].** Every projective Schur algebra $k(\Gamma)$ is split by an abelian extension of $k$. 
As explained in the Introduction, our objective is to show that the above splitting field of $k(\Gamma)(L(\xi))$ in characteristic zero, $L$ in positive characteristic, can be embedded in a radical extension $k(A)$ of $k$.

**Theorem 1.3.** Let $k$ be any field, and let $B = k(T)$ be a simple $k$-algebra which is spanned over $k$ by a group $T$ such that $T$ is finite modulo $k^*$. If $L = Z(k(T)) \supseteq k$ is Galois over $k$ then it is contained in a radical extension $K = k(A)$ of $k$.

**Proof.** Write $S = T/k^*$ and let $\alpha \in H^2(S, k^*)$ correspond to the extension

$$\alpha : 1 \rightarrow k^* \rightarrow T \rightarrow S \rightarrow 1.$$  

Then the algebra $B$ is a homomorphic image of the twisted group algebra $k^*S$. We consider first the case $\text{char}(k) = 0$. Then $k(T)$ is a direct summand of the semisimple algebra $k^*S$. If $K/k$ is an extension, then $K \otimes_k k(T) \cong K(T)$ is a direct summand (not necessarily simple) of $K \otimes_k k^*S$, where $\alpha'$ is the image of $\alpha$ under the map $H^2(S, k^*) \rightarrow H^2(S, K^*)$. There is a radical (Galois) extension $K_1/k$ such that the image $\alpha'$ of $\alpha$ in $H^2(S, K_1^*)$ is represented by a 2-cocycle with values in the group of $m$th roots of unity $\mu_m$, where $m$ is a positive integer such that $\alpha^m = 1$ (e.g., $m = |S|$). (The following argument goes back to [B32] (see [NVO91]): let $f(\sigma, \tau)$ be a cocycle representing $\alpha$. Then $f(\sigma, \tau)^m = g(\sigma)g(\tau)g(\sigma\tau)^{-1}$, where $g : S \rightarrow k^*$ is a map (1-cocochain). Let $K_1 = k(\sqrt[m]{g(\sigma)} : \sigma \in S, \mu_m)$. Set $f'(\sigma, \tau) = f(\sigma, \tau)g(\sigma)^{-1/m}g(\tau)^{-1/m}g(\sigma\tau)^{1/m}$. Then $f'(\sigma, \tau)$ represents $\alpha' \in H^2(S, K_1^*)$ and $f'(\sigma, \tau)^m = 1$, i.e., $f'(\sigma, \tau) \in \mu_m$, for all $\sigma, \tau \in S$. Then $K_1(T) = K_1(S_1)$, where $S_1$ is the finite group in the exact sequence

$$1 \rightarrow \mu_m \rightarrow S_1 \rightarrow S \rightarrow 1$$

defined by $\alpha'. K_1(S_1)$ is a direct summand of the group algebra $K_1S_1$. By the Brauer splitting theorem [CR81] there exists a cyclic extension $K = K_1(\xi)$ which splits $K_1S_1$, i.e., $K_1S_1 = K \otimes_{K_1} K_1S_1 \cong \otimes K_1(K)$. Now $K$ is a radical (Galois) extension of $k$ and since $k(T)$ is a direct summand of $k^*S$, $K(T) = K_1(S_1)$ is a direct summand of $K_1S_1$, so $K(T) = K(S_1)$ is a direct summand of $K_1S_1$. Now $K(T) = K \otimes_{K_1} K_1(T) \cong K \otimes_{K_1} (K_1 \otimes_{k^*} k(T)) \cong K \otimes_{k^*} k(T) \equiv K \otimes_{K_1} M_1(D)$ is a direct summand of $K_1S_1 \equiv \otimes M_1(K)$. It follows from the uniqueness part of the Wedderburn Theorem that $K \otimes_{k^*} M_1(D)$ is $k$-isomorphic to a direct sum of some of the $M_1(K)$. The center of $\otimes_{K_1} M_1(K)$ is $k$-isomorphic to $\otimes_{K_1} K$. On the other hand, the center of $K \otimes_{K_1} M_1(D)$ is $k$-isomorphic to $K \otimes_{K_1} L \equiv \oplus KL$, where $KL$ is the composite of $K$ and $L$ in an algebraic closure of $k$. We therefore have a $k$-embedding of $KL$ into $\otimes_{K_1} K$, which is possible only if $KL = K$, i.e., $L \subseteq K$. 


If \( \text{char} (k) = p \neq 0 \), the twisted group algebra \( k^\text{S} \) is not necessarily semisimple. However, since \( L = Z(k(T)) \) is Galois over \( k \), the algebra \( K \otimes_k k(T) \) is semisimple for every finite extension \( K/k \). Furthermore, it is a homomorphic image of \( K \otimes_k k^\text{S} \cong K^\text{S} \). Now an argument similar to the above shows that there exists a radical extension \( K_1 \) of \( k \), possibly nonseparable, such that \( K_1 \otimes_k k(T) \) is a homomorphic image of a group algebra \( K_1 S_1 \) with \( S_1 \) finite. Since \( K_1 \otimes_k k(T) \) is semisimple, it is a homomorphic image of \( K_1 S_1 / \text{rad}(K_1 S_1) \) and again by the Brauer splitting theorem there exists a cyclotomic extension \( K = K(\xi) \) which splits \( K_1 S_1 \), i.e., \( KS_1 / \text{rad}(KS_1) \cong M_2(K) \oplus \cdots \oplus M_r(K) \). Now \( K \otimes_k K_1 \otimes_k k(T) \cong K \otimes_k K_1 \otimes_k k(T) \) is semisimple, so it is a homomorphic image of \( KS_1 / \text{rad}(KS_1) \).

Since \( L/k \) is Galois, the center of \( K \otimes_k k(T) \) is isomorphic to \( \oplus KL \) (direct sum of copies of the composite \( KL \)). The last part of the argument is the same as in characteristic zero.

**Corollary 1.4.** Let \( k(\Gamma) \) be a projective Schur algebra over \( k \). Then \( k(\Gamma) \) has an abelian splitting field \( F \) contained in a radical extension \( K \) of \( k \).

**Proof.** Let \( I_0 \) be as in Theorem 1.1 above, and take \( T = I_0 \) in Theorem 1.3. Then the center \( L \) of \( k(\Gamma_0) \) is embedded in a radical extension of \( K \) of \( k \). It follows that the splitting field \( L(\xi) \) of \( k(\Gamma) \) in Theorem 1.2 is embedded in the radical extension \( K(\xi) \) of \( k \).

2. SCHINZEL EXTENSIONS

In Section 1 we proved that every projective Schur algebra over a field \( k \) has a splitting field \( L \) which is abelian over \( k \) and is embedded in a radical extension of \( k \). We prove next a purely field-theoretic result about such field extensions. If \( k \) is a field containing the \( m \)th roots of unity, an extension of the form \( k(\sqrt[n]{U})/k \), where \( U \) is a subgroup of \( k^* \), is called a Kummer extension. Let \( b \in k^* \), \( n \) a positive integer prime to \( \text{char} (k) \), and \( m \) the number of \( n \)th roots of unity in \( k \). A theorem of Schinzel [Sc77, Theorem 2; K88, p. 235] states that if the Galois group of \( x^n - b \) over \( k \) is abelian, then \( b^m = c^n \) for some \( c \in k^* \). Extracting \( m \)th roots of both sides of this equation yields \( \sqrt[m]{b^n} = \xi \sqrt[n]{c} \), where \( \xi \) is an \( m \)th root of unity. Thus Schinzel's theorem implies that \( k(\sqrt[m]{b^n}) \subseteq k(\xi, \sqrt[n]{c}) \), which is the composite of a cyclotomic extension of \( k \) with a Kummer extension of \( k \).

Let us call a finite abelian extension \( L \) of \( k \) a Schinzel extension of \( k \) if \( L \) is contained in the composite of a cyclotomic extension of \( k \) with a Kummer extension of \( k \).
PROPOSITION 2.1. Let $k$ be a field, $L$ a finite abelian extension of $k$ which is contained in a finite radical extension $E = k^{(m)}$, and $U$ a subgroup of $k^*$. Then $L$ is a Schinzel extension of $k$.

Proof. First we factor $m = p_0^{n_0}s$, where $p_0 = \text{char}(k)$ is prime to $s$. Then $L$ is contained in the maximal separable subextension of $E$, which is $k^{(\sqrt{U})}$. Hence we may assume $m$ is prime to $p_0$.

Without loss of generality $L$ is a cyclic extension of prime power degree $p'$ of $k$. Let $k_1 = k(\mu_m)$, with $\mu_m$ the group of $m$th roots of unity, $L_1 = Lk_1$, and $E_1 = Ek_1$. Then $E_1 = k(\sqrt[m]{U})$ is a finite Kummer extension of $k_1$. By Kummer theory, the cyclic subextension $L_1$ of $E_1$ is of the form $k((\sqrt{m}b))$, where $b \in U k_1^{*m}$ and hence without loss of generality $b \in U$. Now $L_1 = Lk_1$ is an abelian extension of $k$, hence its subextension $k((\sqrt{m}b))$ is also abelian over $k$. By Schinzel's theorem above, $k((\sqrt{m}b))$ is a Schinzel extension of $k$, hence so is the composite $L_1 = k((\sqrt{m}b))$ and hence its subfield $L$. 

COROLLARY 2.2. Let $k \subseteq L \subseteq k^{(m)}$ be as in Proposition 2.1, and assume that for every prime $p | [L : k]$, $k$ does not contain the $p$th roots of unity. Then $L$ is contained in a cyclotomic extension of $k$.

Proof. By Proposition 2.1, $L \subseteq CK$, with $C/k$ cyclotomic, $K/k$ Kummer. $K$ is a direct composite of Kummer extensions $K_r$ of prime power degree $q^r$, its "$q^r$-primary components". If $q \nmid [L : k]$, then we can erase $K_r$ from the composite $CK$. On the other hand, for every $p | [L : k]$, the component $K_p$ is trivial, since $k$ does not contain $\mu_p$. Thus $K = k$ and $L \subseteq C$.

COROLLARY 2.3. Every projective Schur algebra over $k$ is split by a Schinzel extension of $k$. In other words, $\text{PS}(k) \subseteq \text{Br}(\Omega/k)$, where $\Omega$ is the maximal Schinzel extension of $k$.

COROLLARY 2.4. Let $p$ be a prime such that $k$ does not contain the $p$th roots of unity. Then $\text{PS}(k)_p \subseteq \text{Br}(k(\mu)/k)_p$, where $\mu$ is the group of all roots of unity in the algebraic closure of $k$, and the subscript $p$ denotes the $p$-primary component.

Proof. Let $B$ be a projective Schur algebra over $k$ of order $p'$ in $\text{Br}(k)$. By Corollary 2.3, $B$ is split by a Schinzel extension $L/k$, and we may assume $[L : k]$ is a power of $p$. By Corollary 2.2 and the assumption that $k$ does not contain the $p$th roots of unity, $L$ is contained in a cyclotomic extension of $k$. 

3. \( \text{PS}(K) \neq \text{Br}(K) \)

We will show that \( \text{PS}(K) \neq \text{Br}(K) \) if \( K = k(t) \) is a rational function field in one variable over any field \( k \) satisfying a mild hypothesis, which holds for example if \( k \) is any infinite field which is finitely generated over its prime field.

**Lemma 3.1.** Let \( k \) be a field, \( p \) a prime \( \neq \text{char}(k) \). Suppose there exists a finite extension \( F \) of \( k \) and a cyclic extension \( L/F \) of degree \( p' > 1 \) such that \( L \cap F(\mu) = F \), where \( \mu \) is the group of all roots of unity in the algebraic closure of \( k \). Then \( \text{Br}(k(t)) \) has an element of order \( p' \) which is of order \( p' \) also modulo \( \text{Br}(k(\mu, t)/k(t)) \).

**Proof.** By the theorem of Faddeev-Auslander-Brumer [FS81], we have an isomorphism

\[
\text{Br}(k(t))_p \cong \text{Br}(k)_p \oplus \{ \oplus_{\sigma} \text{Hom}(G_k(\mathcal{P}), \mathbb{Q}/\mathbb{Z})_p \}
\]

where \( \mathcal{P} = \mathcal{P}(t) \) runs through the set \( k[t]_n \) of monic irreducible polynomials in \( k[t] \), where \( k \) is the separable closure of \( k \), \( G_k = G(k/k) \) is the absolute Galois group of \( k \) and \( G_k(\mathcal{P}) \) denotes the subgroup of \( G_k \) fixing \( \mathcal{P} \). Note that \( \mathcal{P} \) is of the form \( \sigma^n - \alpha \) for some \( \alpha \in k \), so \( G_k(\mathcal{P}) = G_k(\sigma^n) \). Write \( k' = k(\mu) \). There is a commutative diagram

\[
\begin{array}{ccc}
\text{Br}(k(t))_p & \xrightarrow{\sim} & \text{Br}(k)_p \oplus \{ \oplus_{\sigma} \text{Hom}(G_k(\mathcal{P}), \mathbb{Q}/\mathbb{Z})_p \} \\
\downarrow \text{res} & & \downarrow \text{res} \\
\text{Br}(k'(t))_p & \xrightarrow{\sim} & \text{Br}(k')_p \oplus \{ \oplus_{\sigma} \text{Hom}(G_k(\mathcal{P}), \mathbb{Q}/\mathbb{Z})_p \}
\end{array}
\]

where on the right side the summand \( \text{Hom}(G_k(\mathcal{P}), \mathbb{Q}/\mathbb{Z})_p \) is mapped to \( \oplus_{\sigma} \text{Hom}(G_k(\sigma(\mathcal{P})), \mathbb{Q}/\mathbb{Z})_p \), where \( \sigma \) runs through a set of representatives of the double cosets of \( (G_k, G_k(\mathcal{P})) \). (See [So90, Lemma 1].) We note here that the statement of the Faddeev-Auslander-Brumer theorem and Lemma 1 and its proof in [So90] applies only in characteristic zero. They must be modified as above to apply to finite characteristic. Choose \( \alpha \in k \), so that the extension \( F \) given in the statement of Lemma 3.1 is \( k(\alpha) \), and set \( \mathcal{P} = t - \alpha \). Let \( \chi \in \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \) such that the fixed field of \( \ker(\chi) \) is \( L \). Then the condition \( L \cap k'(\alpha) = L \cap F(\mu) = F = k(\alpha) \) implies that \( \text{res} \) maps \( \chi \) faithfully to \( \oplus_{\sigma} \text{Hom}(G_k(\sigma(\mathcal{P})), \mathbb{Q}/\mathbb{Z})_p \). Thus the element of \( \text{Br}(k(t))_p \) corresponding to \( \chi \) is of order \( p' \) modulo \( \ker(\text{res}) = \text{Br}(k'(t)/k(t))_p \).

We therefore have

**Theorem 3.2.** Let \( p \) be a prime, \( k \) a field of characteristic \( \neq p \) not containing the \( p \)th roots of unity. Suppose there exists a finite extension \( F \) of \( k \) and a cyclic extension \( L/F \) of degree \( p' > 1 \) such that \( L \cap F(\mu) = F \). Then
\( \operatorname{Br}(k(t))_p \) has an element of order \( p' \) which is of order \( p' \) also modulo \( \operatorname{PS}(k(t))_p \). In particular, \( \operatorname{PS}(k(t))_p \neq \operatorname{Br}(k(t))_p \).

**Proof.** By Corollary 2.4 above, \( \operatorname{PS}(k(t))_p \subseteq \operatorname{Br}(k'(t)/k(t))_p \), so the assertion follows from Lemma 3.1.

Recall that a field \( F \) is Hilbertian iff given any irreducible polynomial \( f(x, y) \in F[x, y] \), there exists \( c \in F \) such that \( f(c, y) \in F[y] \) is irreducible. It is known [FJ86, Chap. 12] that any infinite field which is finitely generated over its prime field is Hilbertian.

**Corollary 3.3.** Let \( k \) be a Hilbertian field, \( p \) a prime \( \neq \text{char}(k) \), and assume that \( k \) does not contain the \( p \)-th roots of unity. Then \( \operatorname{PS}(k(t))_p \neq \operatorname{Br}(k(t))_p \); in fact, \( \operatorname{Br}(k(t))_p/\operatorname{PS}(k(t))_p \) is infinite.

**Proof.** By Theorem 3.2, it suffices to show that for any positive integer \( r \), there exists a finite extension \( F \) of \( k \) and a cyclic extension \( L/F \) of degree \( p' \) such that \( L \cap F(\mu_r) = F \). Since \( k \) is Hilbertian, the symmetric group \( S_{p'} \) is realizable as a Galois group over \( k \), say \( G(L/k) \cong S_{p'} \). Let \( F \) be the fixed field of an element of order \( p' \). Then \( L/F \) is cyclic of order \( p' \). Since \( [L \cap k(\mu_r):k] = 1 \) or 2 and \( p \) is odd by hypothesis, it follows that \( L \cap F(\mu_r) = F \).

**Corollary 3.4.** Let \( k \) be an infinite field which is finitely generated over its prime field. Then \( \operatorname{Br}(k(t))_p/\operatorname{PS}(k(t))_p \) is infinite for all but finitely many primes \( p \).

**Proof.** For almost all \( p \), \( k \) does not contain the \( p \)-th roots of unity. Since \( k \) is Hilbertian, the result follows from Theorem 3.3.

It is interesting to consider also formal power series fields \( k((t)) \). Here we have Witt’s theorem [Se79, p. 186]

\[
\operatorname{Br}(k((t)))_p \cong \operatorname{Br}(k)_p \oplus \operatorname{Hom}(G_1, \mathbb{Q}/\mathbb{Z})_p
\]

where \( p \) is a prime \( \neq \text{char}(k) \). In this direct sum, \( \operatorname{Br}(k) \) represents the algebras \( D \otimes_k k((t)) \), where \( D \) is a \( k \)-central division algebra. In the second summand \( \operatorname{Hom}(G_1, \mathbb{Q}/\mathbb{Z}) \), the elements are represented by cyclic algebras of the form \( (L((t))/k((t)), \sigma, t) \), where \( L \) is the fixed field of \( \ker(\chi) \), \( \chi \in \operatorname{Hom}(G_1, \mathbb{Q}/\mathbb{Z}) \), and \( \sigma \) is a generator of the cyclic group \( G(L/k) \) such that \( \chi(\sigma) = 1/n (\text{mod} \mathbb{Z}) \), where \( n \) is the order of \( \chi \).

**Proposition 3.5.** Let \( k \) be a field, \( p \neq \text{char}(k) \), and assume that \( k \) does not contain all \( p \)-power roots of unity. Let \( s \) be maximal such that \( k \) contains the \( p^s \)th roots of unity. Let \( r > 2s \) and assume that there exists a cyclic extension \( L/k \) of degree \( p' \) such that \( L \cap k(\mu_r) = k \). Then \( \operatorname{Br}(k((t))) \) contains an element of order \( p' \) which is of order \( \geq p'^{r-2s} \text{mod} \operatorname{Br}(\Omega/k((t))) \),
where $\Omega$ is the maximal Schinzel extension of $k((t))$. This element lies in the subgroup of $\text{Br}(k((t)))$ corresponding to $\text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ in Witt's Theorem. In particular, $\text{Br}(k((t)))_p/\text{PS}(k((t)))_p$ is of order $\geq p^{-2s} > 1$.

**Remark.** If $k$ contains all $p$-power roots of unity, then by the Merkur’ev–Suslin theorem $\text{Br}(k((t)))_p = \text{PS}(k((t)))_p$.

**Proof of Proposition 3.5.** Set $k' = k(\mu)$. Let $K$ be the maximal Kummer $p$-extension of $k((t))$. Then $G(K/k((t)))$ has exponent $p^s$ and $\Omega = K(\mu)$ by Section 2 above. Write $K = k((t))(k((t))^{1/p})$. Decomposing $k((t))^s = k^s \times \langle t \rangle \times U_1$, where $U_1 = 1$–units = $\{1 + \sum_{a \geq 1} a_n t^n : a_n \in k\}$, and observing that $UP = U_1$ for $p \not\mid \text{char}(k)$ (by Hensel's Lemma), we have $k((t))^s/k((t))^{p^s} = k^s/k^{*p} \times \langle t \rangle / \langle t^p \rangle$. Set $K_1 = k((t))^{(1/p^s)}$, so that $K = K_1(t^{1/p^s})$. We may write $K_1 = k_i k((t))$, where $k_i = k(k^{*p})$. Let $D = (L((t))/k((t)), \sigma, t)$. Tensoring up to $K_1' = k_i k' k((t))$, we get $D_i = D \otimes_{k((t))} K_1' = (k_i k' k((t)) / k_i k' k((t)), \sigma, t)$ since $[L \cap k_i : k] = p^s$. We claim $D_i$ is of order $\geq p^{-2s}$ in $\text{Br}(K_i')$. Indeed, it is enough to consider finite extensions $F$ of $k$ which are contained in $k_i k'$. There is a commutative diagram [Sc79, p. 187] (Witt's Theorem)

$$
\begin{align*}
\text{Br}(k((t)))_p & \quad \longrightarrow \quad \text{Br}(k)_p \oplus \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})_p \\
\downarrow & \quad \downarrow \\
\text{Br}(F((t)))_p & \quad \longrightarrow \quad \text{Br}(F)_p \oplus \text{Hom}(G_F, \mathbb{Q}/\mathbb{Z})_p
\end{align*}
$$

where the component $\text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})_p$ is mapped into $\text{Hom}(G_F, \mathbb{Q}/\mathbb{Z})_p$ (restriction to $G_F$). Then $D$ corresponds to a character $\chi: G_i \rightarrow \mathbb{Q}/\mathbb{Z}$ with kernel $G_i$. Since $[L \cap F : k] \leq p^s$, $\chi|_{G_i}$ is of order $\geq p^{-s}$, i.e., $D \otimes_{k((t))} F((t))$ is of order $\geq p^{-2s}$ in $\text{Br}(F((t)))$. Now the fields $F((t)), k \subseteq F \subseteq k_i k'$, $[F : k] < \infty$, form a direct system and $K_i' = k_i k' k((t))$ is the direct limit. Consequently, $D_i = D \otimes_{k((t))} K_1'$ is of order $\geq p^{-2s}$ in $\text{Br}(K_i')$. Since we already know that this order is $\leq p^{-2s}$, we have equality, proving the claim. Finally, since $D = K_1'(t^{1/p})$ is of degree $= p^s$ over $K_i'$, we have $D \otimes_{k((t))} \Omega = D_i \otimes_{k_i} \Omega$ is of order $\geq (\text{in fact, } =) p^{-2s}$ in $\text{Br}(\Omega)$.  

**Remark 3.6.** For many fields $k$, the hypothesis in Proposition 3.5 holds for all $r$, e.g., $k$ a proper finite extension of $\mathbb{Q}_p$ or $k$ a number field which is not totally real or $k = \mathbb{Q}(t)$, etc., so in such cases $\text{Br}(k((t)))_p/\text{PS}(k((t)))_p$ is infinite, even for all $p$ in the latter two examples.

In contrast:

**Example 3.7.** $\text{PS}(\mathbb{Q}(t)) = \text{Br}(\mathbb{Q}(t)), \text{PS}(\mathbb{Q}_p(t)) = \text{Br}(\mathbb{Q}_p(t))$.

**Proof.** By Witt's Theorem [Sc79, p. 186], $\text{Br}(\mathbb{Q}(t)) = \text{Br}(\mathbb{Q}) \oplus \text{Hom}(G_{\mathbb{Q}}, \mathbb{Q}/\mathbb{Z})$. In this direct sum, $\text{Br}(\mathbb{Q})$ represents the algebras $D \otimes_{\mathbb{Q}}$
\( \mathcal{Q}(t) \), where \( D \) is a \( \mathcal{Q} \)-central division algebra. Hence \( \text{Br}(\mathcal{Q}) = \text{PS}(\mathcal{Q}) \subseteq \text{PS}(\mathcal{Q}(t)) \). In the second summand \( \text{Hom}(G, \mathcal{Q} / \mathbb{Z}) \), the elements are represented by cyclic algebras of the form \((L(t))/\mathcal{Q}(t), \sigma, \tau)\), where \( L/\mathcal{Q} \) is cyclic. By the Kronecker–Weber Theorem, \( L \) is contained in a cyclotomic extension \( K \) of \( \mathcal{Q} \), so this cyclic algebra is similar to a crossed product \((K(t))/\mathcal{Q}(t), G, \alpha)\) where \( \alpha \) is representable by a cocycle with values in \( \mathcal{Q}(t)^* \), which is a projective Schur algebra. Hence this second summand is also contained in \( \text{PS}(\mathcal{Q}(t)) \). The proof for \( \mathcal{Q}_p \) is identical.

From the results in this paper it seems that the projective Schur group of a field \( k \) is in general a proper subgroup of \( \text{Br}(k) \). The natural question arises: what (proper) subgroup is \( \text{PS}(k) \)? For example, is it a relative Brauer group, i.e., is there an algebraic extension \( K/k \) such that \( \text{PS}(k) = \text{Br}(K/k) \)? Here is an example where this happens.

**Example 3.8.** Let \( k = \mathbb{F}_2 \), \( K = k(x, y) \), a rational function field in two variables over \( k \). By Corollary 3.4, \( \text{Br}(K(\mu)/K) \neq \text{Br}(K) \). By Corollary 2.4, \( \text{PS}(K) \subseteq \text{Br}(K(\mu)/K) \), since \( K \) does not contain any roots of unity other than 1. On the other hand, \( \text{Br}(K(\mu)/K) \subseteq \text{PS}(K) \) since (see Introduction) every element of \( \text{Br}(K) \) which is split by a cyclic cyclotomic extension of \( K \) is represented by a projective Schur algebra. (Since \( \text{char}(K) \) is finite, every cyclotomic extension of \( K \) is cyclic.) Therefore \( \text{PS}(K) = \text{Br}(K(\mu)/K) \).

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We are grateful to the referee for pointing out that if one does not restrict oneself to algebraic extensions, every subgroup of \( \text{Br}(k) \) is a relative Brauer group \( \text{Br}(K/k) \) for some extension \( K \) of \( k \) (e.g., one can take \( K \) to be a composite of generic splitting fields).

**REFERENCES**


