Abstract

One of the open questions that has emerged in the study of the projective Schur group $PS(F)$ of a field $F$ is whether or not $PS(F)$ is an algebraic relative Brauer group over $F$, i.e. does there exist an algebraic extension $L/F$ such that $PS(F) = Br(L/F)$? We show that the same question for the Schur group of a number field has a negative answer. For the projective Schur group, no counterexample is known. In this paper we prove that $PS(F)$ is an algebraic relative Brauer group for all Henselian valued fields $F$ of equal characteristic whose residue field is a local or global field. For this, we first show how $PS(F)$ is determined by $PS(k)$ for an equicharacteristic Henselian field with arbitrary residue field $k$.

© 2006 Elsevier B.V. All rights reserved.

MSC: 11R52; 11S25; 12F05; 12G05; 13A20

1. Introduction

Let $F$ be a field, and $Br(F)$ its Brauer group. The Schur group $S(F)$ of $F$ is the subgroup of $Br(F)$ consisting of classes represented by Schur algebras over $F$. A finite dimensional central simple $F$-algebra $A$ is called Schur over $F$ if it is a homomorphic image of a group algebra $FG$ with $G$ finite. Equivalently, $A$ is Schur over $F$ if it is spanned as a vector space over $F$ by a finite subgroup $G$ of the group $A^*$ of invertible elements of $A$. In 1978 [23], Lorenz and Opolka introduced projective analogues to these notions. They defined the projective Schur group $PS(F)$ of $F$ to be the subgroup of $Br(F)$ consisting of classes represented by projective Schur algebras over $F$. A finite dimensional central simple $F$-algebra $A$ is projective Schur over $F$ if it is spanned as a vector space over $F$ by a subgroup $G$ of the group $A^*$ of invertible elements of $A$ which is finite modulo $F^*$, i.e. $GF^*/F^*$ is finite. In either case, when a subgroup $G$ of $A^*$ spans $A$ over $F$, we write $A = F(G)$.

In view of the fact that these two subgroups of $Br(F)$ are defined in the language of algebras, we can ask for a natural characterization of them in the language of Galois cohomology, just as $Br(F)$ is a Galois cohomology group $H^2(G_F, F^*_S)$, where $F_S$ denotes the separable closure of $F$. In the case of the Schur group, the Brauer–Witt theorem can be viewed as a positive answer to this question: let $F_{cyc}$ be the maximal cyclotomic extension of $F$ and let $\mu$
denote the group of all roots of unity in $F_s$. Then $S(F)$ is the image of the canonical map $H^2(G(F_{\text{cyc}}/F), \mu) \to H^2(G_F, F_s^\times) \cong Br(F)$ (cf. [34, Corollary 3.11]).

In the case of $PS(F)$, all known examples of projective Schur algebras are Brauer equivalent to radical Abelian algebras, defined as follows. Let $A = (L/F, G, f)$ be a crossed product algebra, where $L$ is a finite Galois extension field of $F$, $G$ is the Galois group $G(L/F)$, and $f \in H^2(G, L^\times)$. Then $A$ is said to be a radical algebra if $L = F(T)$ for some subgroup $T$ of $L^\times$ containing $F^\times$ such that $T/F^\times$ is finite and $f$ is represented by a 2-cocycle with values in $T$. This $A$ is called a radical Abelian algebra if in addition $G(L/F)$ is Abelian. It is easy to see that every radical algebra over $F$ lies in $PS(F)$. The first two authors have conjectured that all projective Schur algebras are Brauer equivalent to radical algebras, and even to radical Abelian algebras. The radical algebra conjecture is equivalent to the conjecture that $PS(F)$ is the image in $Br(F) = H^2(G(F_{\text{cyc}}/F), F_s^\times)$ of $H^2(G(F(T))/F, T)$, where $T$ is the subgroup of $F_s^\times$ consisting of elements of finite order modulo $F^\times$. This would provide an analogue for $PS(F)$ of the Brauer–Witt theorem for $S(F)$. The radical Abelian algebra conjecture has an analogous homological interpretation. The radical Abelian algebra conjecture has been proved for all fields of nonzero characteristic [5, Corollary 1.5]. In characteristic 0 only partial results are known [2,4,5,7].

Another way of describing some subgroups of $Br(F)$ is as algebraic relative Brauer groups. Let $M/F$ be a field extension. The relative Brauer group $Br(M/F)$ is the kernel of the restriction map $\text{res}_{M/F} : Br(F) \to Br(M)$. A subgroup $H$ of $Br(F)$ is called an algebraic relative Brauer group if there exists an algebraic extension $M/F$ such that $Br(M/F) = H$. It is known that every subgroup of $Br(F)$ is a relative Brauer group, by taking $M$ to be an iterated generic splitting field of the division algebras in $H$, cf. [17, Theorem 1]; but it is not true in general that every subgroup is an algebraic relative Brauer group (e.g., let $H$ be any nontrivial finite subgroup of $Br(F)$, if $F$ is a global field [18, Corollary 4]). Of course $Br(F)$ itself is an algebraic relative Brauer group by definition. We ask if $S(F)$ and $PS(F)$ are algebraic relative Brauer groups. The answer is negative for $S(F)$ even for $F$ a number field, as we will show in Section 6 below. For $PS(F)$ this question has an obvious affirmative answer for local and global fields $F$ since in that case $PS(F) = Br(F)$ by [23, Satz 3], or see [3, p. 531]. There is no good reason to believe that $PS(F)$ is an algebraic relative Brauer group for every field $F$, but so far no counterexample has been found.

This paper is concerned with the radical (Abelian) algebra conjecture and the algebraic relative Brauer group question for $PS(F)$ for fields $F$ with Henselian valuation such that the residue field $k$ has the same characteristic as $F$. We show in Corollary 4.6 that the radical (resp. radical Abelian) conjecture holds for $F$ if it holds for $k$. We prove in Section 5 that if $k$ is a local or global field, then $PS(F)$ is an algebraic relative Brauer group.

The proofs of our main results require detailed information about the Brauer group of a Henselian valued field $F$, which we give in Section 3. Beyond the known results, which we recall, we construct explicit splitting maps for the inertially split part of $Br(F)$ and for the tame part of $Br(F)$. The splitting maps are used in Section 4 to show exactly how $PS(F)$ is built from $PS(k)$, where $k$ is the residue field of the Henselian valuation on $F$ (assuming $\text{char}(k) = \text{char}(F)$). This generalizes to arbitrary equicharacteristic Henselian fields results in [7] for iterated power series fields.

Whenever a projective Schur algebra $F(G)$ has Abelian finite group $GF^*/F^*$, there is an associated symplectic pairing on $GF^*/F^*$ given by commutators. We will show in Section 2 how such pairings and their associated Lagrangians can elucidate the structure of an arbitrary reduced projective Schur algebra. This provides a unified approach to a number of previous results on projective Schur algebras, as well as being needed for the analysis of the Henselian situation in Section 4.

We point out in passing that the algebraic relative Brauer group question has been studied for the $m$-torsion subgroups $mBr(F)$ of $Br(F)$. In general $mBr(F)$ is not an algebraic relative Brauer group. Counterexamples exist for $F$ a power series field $k((t))$, with $k$ a local field [6, Sec. 4]. On the other hand, for $F$ a global field $mBr(F)$ is an algebraic relative Brauer group for every $m$, see [6,21,27,22].

We will use the following notation throughout the paper. If $C$ is a torsion Abelian group, we write $\exp(C)$ for the exponent of $C$, $nC$ for the $n$-torsion subgroup of $C$; and $(p)$ for the $p$-primary component of $C$. For $c \in C$, $\alpha(c)$ denotes the order of $c$. If $C$ is associated to a field $F$, $C'$ denotes the prime-to-$p$ part of $C$ if $\text{char}(F) = p \neq 0$, while $C' = C$ if $\text{char}(F) = 0$. We write $\mu(F)$ for the group of roots of unity in a field $F$; we write $\mu_n$ for the group of $n$ $n$-th roots of unity. If $S$ is a central simple algebra over $F$, $\deg(S) = \sqrt{\dim_F(S)}$ is the degree of $S$, and $\exp(S)$ is the exponent of $S$, which is the order of the class $[S]$ of $S$ in the Brauer group $Br(F)$. If $\mu_n \subseteq F$ and $a, b \in F^*$ we write $(a, b; F)_n$ for the symbol algebra of dimension $n^2$ over $F$ with generators $i, j$ and relations $i^2 = a$, $j^2 = b$, and $ij = \omega ji$, where $\omega$ is some primitive $n$-th root of unity in $F$. 

2. Projective Schur algebras of Abelian type and Lagrangians

A projective Schur algebra $A = F(A)$ is said to be of Abelian type if the finite group $A/F^*$ is Abelian. Associated to such an $A$ is a nonsingular symplectic pairing on $A/F^*$. We will first recall some properties of such algebras, which can be seen easily by using this pairing. The data about the Abelian case are relevant for more general projective Schur algebras because we will see in Proposition 2.2 below that every reduced projective Schur algebra admits after scalar extension a decomposition into a Schur algebra and a projective Schur algebra of Abelian type. Furthermore, we will show that Lagrangian subgroups of $A/F^*$ with respect to the symplectic pairing yield useful refinements of such tensor decompositions. The results in this section provide a unified approach to arguments in several papers by the first two authors.

Proposition 2.1. Let $A = F(A)$ be a projective Schur algebra of Abelian type, where $F^* \subseteq A \subseteq A^*$ (so $A/F^*$ is a finite Abelian group). Then,

(a) There is a well-defined pairing $B_A : A/F^* \times A/F^* \to F^*$ given by $(aF^*, bF^*) \mapsto aba^{-1}b^{-1}$.
(b) The pairing $B_A$ is nondegenerate, bimultiplicative and symplectic, and $\exp(B_A)$ is a finite (cyclic) subgroup of $\mu(F)$.
(c) $|A/F^*| = \dim_F(A)$ and $\exp(A/F^*) = |\exp(B_A)|$.
(d) $A \cong S_1 \otimes F \cdots \otimes F_{S_m}$, where each $S_i$ is a symbol algebra, with $\exp(A/F^*) = \text{lcm}_{1 \leq i \leq m} \deg(S_i)$.

Proof. This is mostly known, cf. [23, Hilfsatz 1] and [4, Theorem 1.1], but we will show how the use of the pairing $B_A$ facilitates the proof. For $a \in A$, let $\tilde{a} = aF^* \in A/F^*$. The pairing $B_A$ is well-defined because $F^*$ is central in $A^*$. That $B_A$ is symmetric means that $B_A(\tilde{a}, \tilde{a}) = 1$ for all $a \in A$; this is evident here. Let $(a, b) = aba^{-1}b^{-1}$. Since $B_A$ maps into a central subgroup of $A^*$, the commutator identities $[a, bc] = [a, b]b[a, c]b^{-1}$ and $[ab, c] = a[b, c]a^{-1}[a, c]$ show that $B_A$ is bimultiplicative. Let $e = \exp(A/F^*)$. The bimultiplicativity of $B_A$ shows that $\exp(A/F^*)$ is a finite group $G$.

A radical Abelian extension of a field $F$ is an Abelian Galois field extension $K$ of $F$ such that $K = F(U)$, where $U$ is a subgroup of $K^*$ with $U \supseteq F^*$ and $U/F^*$ finite.

A projective Schur algebra $A = F(G)$ is said to be reduced if for every subgroup $H$ of $G$ with $H \supseteq G'$ (the derived group of $G$) and every subfield $L$, $F \subseteq L \subseteq A$ such that $G$ acts on $L$ by conjugation, the subalgebra $L$ is simple. Recall from [2, Theorem 1.4] that every projective Schur algebra is Brauer equivalent to a reduced projective Schur algebra. We now collect basic properties of subalgebras $F(H)$ of a reduced projective Schur algebra $F(G)$, where $G' \subseteq H \subseteq G$.

Let $A = F(G)$ be a projective Schur algebra with $G \supseteq G'$. Assume $A$ is reduced. Let $H$ be any subgroup of $G$ such that $H \supseteq G'$. Let

$B = F(H)$, a simple algebra, as $A$ is reduced;
$L = Z(B)$, a field;
$\tilde{H} = C_B(L)$, a normal subgroup of $G$;
$E = F(\tilde{H})$;
\[ T = C_{\hat{H}B^*}(B); \]
\[ T = L(T). \]

**Proposition 2.2.** In the setting just described,

(a) \( L \) is Abelian Galois over \( F \) with Galois group \( \mathcal{G}(L/F) \cong \mathcal{G}/\hat{H} \), and \( L \) lies in a radical Abelian extension of \( F \).

(b) \( E \) is a simple algebra with \( Z(E) = L \) and \( E = C_A(L) = B \otimes_L T \).

(c) \( L^* \subseteq T \) and \( T/L^* \cong \hat{H}/(\hat{H} \cap B^*) \), a finite Abelian group.

(d) \( T \) is a projective Schur algebra of Abelian type over \( L \).

(e) Let \( G = \hat{G}/\hat{H} \), a finite Abelian group. Then \( A = E \ast G \), a ring-theoretic crossed product.

(f) If \( e = \exp(T) \), then \( \mu_e \subseteq F^* \).

**Proof.** Since \( \mathcal{G}' \subseteq \mathcal{H} \), we have \( \mathcal{H} \lhd \mathcal{G} \), so \( \mathcal{G} \) acts by conjugation on \( B \). Hence \( B \) is simple as \( A \) is reduced. So \( L \) is a field with \( [L : F] < \infty \), and \( \mathcal{G} \) acts by conjugation on \( L \). Clearly, \( F \subseteq L^G \subseteq A^G = F \), so \( L \) is Galois over \( F \) and \( \mathcal{G} \) maps onto \( \mathcal{G}(L/F) \) with kernel \( \mathcal{H} \). Therefore, \( \mathcal{G}(L/F) \cong \mathcal{G}/\hat{H} \), which is Abelian as \( \hat{H} \subseteq \mathcal{H} \). Since \( F(\mathcal{H}) \) is a simple \( F \)-algebra with \( \mathcal{H}/F^* \) finite and \( L = Z(F(\mathcal{H})) \) is Galois over \( F \), by [3, Theorem 1.3] \( L \) lies in a radical extension of \( F \). Because \( L \) is also Abelian Galois over \( F \), [3, Proposition 2.1] shows that \( L \) lies in a radical Abelian extension of \( F \), proving (a).

As for (e), note first that since \( \hat{H} \supseteq \mathcal{G}' \) and \( A \) is reduced, \( E = F(\hat{H}) \) is simple. Let \( n = [\mathcal{G}/\hat{H}] = [L : F] \), and let \( g_1, \ldots, g_n \) be a set of coset representatives of \( \hat{H} \) in \( \mathcal{G} \). Then \( \mathcal{G} \subseteq \sum_{i=1}^n E_{g_i} \), so \( \sum_{i=1}^n E_{g_i} = A \). Hence \( \dim_F(E) \geq \dim_F(A)/n \). Clearly \( E \subseteq C_A(L) \). So using the Double Centralizer Theorem,

\[ \dim_F(E) \leq \dim(C_A(L)) = \dim(A)/[L : F] \leq \dim(A). \]

So, equality holds throughout, which shows that \( E = C_A(L) \); hence \( Z(E) = L \). The dimension equality shows that the sum \( A = \sum_{i=1}^n E_{g_i} = A \) is direct. Hence \( A = E \ast (\mathcal{G}/\hat{H}) \), a ring-theoretic crossed product, proving (e). We have also proved the first three assertions of (b).

Now consider \( T \). Since \( \mathcal{H} \) normalizes \( \mathcal{H} \) (as \( \mathcal{H} \lhd \mathcal{G} \)), \( \hat{H} \) also normalizes \( B^* \). So \( \hat{H}B^* \) is a subgroup of \( A^* \), and the definition of \( T \) as \( C_{\hat{H}B^*}(B) \) makes sense. Observe also that \( B = F(\mathcal{H}) \subseteq F(\hat{H}) = E \); so \( T \subseteq E \). Since \( L^* \subseteq T \cap B^* \subseteq Z(B^*) = L^* \), we have \( T \cap B^* = L^* \). Also, for any \( \hat{h} \in \hat{H} \), conjugation by \( \hat{h} \) gives an \( L \)-linear automorphism of \( B \). By Skolem–Noether, there is a \( b \in B^* \) conjugation by which is produced the same automorphism of \( B \). Then \( \hat{h}b^{-1} \in T \). This shows that \( \hat{H} \subseteq T B^* \); since \( T \subseteq \hat{H}B^* \) by definition, we have \( \hat{H}B^* = T B^* \). Then,

\[ T/L^* = T/(T \cap B^*) \cong T B^*/B^* \cong \hat{H}B^*/B^* \cong \hat{H}/(\hat{H} \cap B^*). \]

Because \( F^* \subseteq \hat{H} \subseteq \hat{H}/B^* \), \( \hat{H}/(\hat{H} \cap B^*) \) is finite (as \( \mathcal{H}/F^* \) is finite) and Abelian (as \( \hat{H}/\hat{H} \) is Abelian), proving (c).

Next, consider \( T = L(T) \). We have \( E = F(\mathcal{H}) \subseteq BT \subseteq E \), so \( E = B \otimes_L C_E(B) \) by the Double Centralizer Theorem and \( T \subseteq C_E(B) = E \), we have \( E = BT = B \otimes_L T \subseteq B \otimes_L C_E(B) = E \). So \( T \subseteq C_E(B) \). This completes the proof of (b) and shows also that \( L = Z(T) \). Since \( T = L(T) \) is a central simple \( L \)-algebra, part (c) shows that \( T \) is a projective Schur algebra of Abelian type, proving (d).

Because \( T/L^* \) is Abelian, we have the pairing \( B_T : T/L^* \times T/L^* \rightarrow L^* \) described in Proposition 2.1. The key to proving (f) is to see that \( \text{im}(B_T) \subseteq F^* \). For this, observe that \( \mathcal{G} \) acts by conjugation on \( \mathcal{H}, B, L, \) and \( \hat{H} \), so on \( \mathcal{H}B^* \), so on \( T = C_{\hat{H}B^*}(B) \) and on \( T/L^* \). Take any \( g \in \mathcal{G} \) and \( t \in T \); write \( t = \hat{h}b \) with \( \hat{h} \in \hat{H} \) and \( b \in B^* \). Let \( c = gtg^{-1}t^{-1} \). Then \( c \in T \), as \( gtg^{-1} \in T \); likewise, \( gb^{-1}b^{-1} \in B^* \). So

\[
c = \{g^tg^{-1}\hat{h}^{-1}g^tg^{-1}b^{-1}h^{-1}\} \in G^\mathcal{H}B^* = \mathcal{H}B^* \subseteq B^*.\]

Hence \( c \in T \cap B^* = L^* \). Since \( gtg^{-1} = ct \), this shows that \( \mathcal{G} \) acts trivially on \( T \). Because the pairing \( B_T \) is clearly compatible with the \( \mathcal{G} \)-action, \( \mathcal{G} \) must also act trivially on \( \text{im}(B_T) \), i.e. \( \text{im}(B_T) \subseteq L^g = F^* \), as claimed. We have \( \text{im}(B_T) = \mu_\ell \), where \( \ell = \mid \text{im}(B_T) \mid \) (see Proposition 2.1(b)); so \( \mu_\ell \subseteq F^* \). Let \( e = \exp(T) \). Then by Proposition 2.1(d) and (c), \( e \exp(T/L^*) = \ell \). Thus \( \mu_e \subseteq \mu_\ell \subseteq F^* \), proving (f). \qed

We digress to show that the preceding results yield a considerably simplified proof of the exponent reduction theorem that was the main result of [5]. For this, let \( F_{\text{cyc}} \) be the maximal cyclotomic extension of \( F \).

**Proposition 2.3 ([5, Theorem 1.3]).** Let \( A \) be a projective Schur algebra over \( F \), and let \( e = \exp(A \otimes_F F_{\text{cyc}}) \). Then \( \mu_e \subseteq F^* \).
Proof. Let $A = F(G)$ with $G \supseteq F^*$. We may assume that $A$ is reduced. We apply Proposition 2.2 with $\mathcal{H} = G^*$. Recall (see [30, p. 443]) that $G$ has a central subgroup $F^*$ with $G/F^*$ finite, $|G'| < \infty$. Hence $B = F(G^*)$ is a Schur algebra. By the Brauer splitting theorem [13, pp. 385, 418], $B$ is split by $F_{cycl}$ and $L = Z(B) \subseteq F_{cycl}$. Now $E = C_A(L) \sim A \otimes_F L$ in $Br(L)$. Since $E = B \otimes_T L$ as in Proposition 2.2, we have $A \otimes_F F_{cycl} \sim E \otimes_L F_{cycl} \sim T \otimes L F_{cycl}$. Let $d = \exp(T)$. Then
\[
eq \exp(A \otimes_F F_{cycl}) = \exp(T \otimes_L F_{cycl})|\exp(T) = d.
\]
Since $\mu_d \subseteq F^*$ by Proposition 2.2(f), this shows $\mu_e \subseteq F^*$. $\Box$

We now return to the setup of Proposition 2.2, with $A = F(G)$, a reduced projective Schur algebra ($F^* \subseteq G$), and $\mathcal{H}$ a subgroup of $G$ with $G' \subseteq \mathcal{H}$, and the objects associated to $\mathcal{H}$ described there, including the projective Schur algebra of Abelian type $T = L(T)$. We will show how to use subgroups $L/L^*$ of $T/L^*$ to build new subgroups $\mathcal{H}_1$ of $G$ containing $\mathcal{H}$ so that the objects associated to $\mathcal{H}_1$ by Proposition 2.2 have a nice description in terms of $\mathcal{H}$ and the corresponding objects for $\mathcal{H}$. This will be needed in Section 4. It also provides a unified approach to constructions that were given in [2,3,5,7]. In each of those papers an $\mathcal{H}_1$ is chosen after starting with some $\mathcal{H}$, so that $\mathcal{H}_1 \supseteq \mathcal{H}$ and $F(\mathcal{H}_1)$ is maximal with respect to some property. In examining these constructions, one sees that what was needed was primarily an $\mathcal{H}_1$ so that its associated $T$ as in Proposition 2.2 is trivial and $F(\mathcal{H}_1) = F(\mathcal{H}) \otimes_L Z(F(\mathcal{H}_1))$. We will see that this occurs whenever $L/L^*$ is a Lagrangian of $T/L^*$.

For the projective Schur algebra of Abelian type $T = L(T)$ of Proposition 2.2 (with $L \subseteq T$), we have the nondegenerate symmetric pairing $B_T : T/L^* \times T/L^* \rightarrow L^*$ described in Proposition 2.1. Take any subgroup $A$ of $T/L^*$, and let $L$ be the inverse image of $A$ in $T$. Then $L(L)$ is an $L$-subalgebra of $T$ with $\dim_T(L(L)) = |A|$ (see [32, Example 2.4(c)]). Let $A^L = \{\gamma \in T/L^*|B_T(\gamma, \lambda) = 1$ for all $\lambda \in A\}$, a subgroup of $T/L^*$. Because $B_T$ is nondegenerate, we have $|A||A^L| = |T/L^*|$ (cf. [32, (2.2)]), so $A^L = A$. Abusing notation slightly, let $L^L$ denote the inverse image of $L^L$ in $T$. It is easy to check that $L(L^L) = C_T(L(L))$ (see [32, lemma 2.5(i)]). So $Z(L(L)) = L(L) \cap L(L^L) = L(L \cap L^L)$. In particular, $L(L)$ is commutative iff $L \subseteq A^L$, iff $B_T$ is trivial on $A \times A$. The subgroup $A$ is called a Lagrangian of $T/L^*$ with respect to $B_T$ if $A^L = A$. It is easy to see that if $A$ is any subgroup of $T/L^*$ with $A \subseteq A^L$, then $L$ lies in some Lagrangian of $T/L^*$. Thus the algebras $L(L)$ for $L/L^*$ a Lagrangian are the maximal commutative subalgebras of $B_T$ of the form $L(L')$ for $L' \subseteq T$.

Now fix a subgroup $A$ of $T/L^*$, let $L$ be its inverse image in $T$. Let $A = \{a_1 L^*, \ldots , a_m L^*\}$, with $a_i \in L \subseteq T \subseteq \hat{H}B^*$. Write each $a_i = \hat{h}_i b_i$ with $\hat{h}_i \in \hat{H}$ and $b_i \in B^*$. Let $\mathcal{H}_1 = (\hat{h}_1, \hat{h}_1, \ldots , \hat{h}_m)$, a subgroup of $G$ with $G' \subseteq \mathcal{H} \subseteq \mathcal{H}_1 \subseteq \hat{H}$. Associated to $\mathcal{H}_1$ we have the objects of Proposition 2.2: $B_1 = F(\mathcal{H}_1)$, $L_1 = Z(B_1)$, $\hat{H}_1 = C_{\hat{H}_1}(L_1)$, $E_1 = F(\hat{H}_1) = B_1 \otimes_L T$, where $T_1 = L_1(T_1)$ with $T_1 = C_{\hat{H}_1}b_1^*(B_1)$.

Proposition 2.4. In the situation described in the preceding paragraph,

(a) $B_1 = B \otimes_L L(L)$; $L_1 = L(L \cap L^L)$, which is a Kummer extension field of $L$, and lies in a radical Abelian extension of $F$; $H \subseteq \mathcal{H} \subseteq \mathcal{H}_1 \subseteq \hat{H}$; $E_1 = C_A(L_1) = B \otimes_L L(L \cap L^L) \subseteq E$; and $T_1 = L(L^L) \subseteq T$.
(b) $[T_1 : L_1] = |L^L/(L \cap L^L)|$
(c) If $L$ is a Lagrangian of $T/L^*$, then $L_1 = L(L) = T_1$ and $B_1 = E_1$.

Proof. For each generator $a_i$ of $L$, we have $a_i = \hat{h}_i b_i$, with each $b_i \in B = F(\mathcal{H}) \subseteq F(\mathcal{H}_1) = B_1$, and $\hat{h}_i \in F(\mathcal{H}_1) = B_1$; so $a_i \in B_1$. Also, $L \subseteq B \subseteq B_1$. Thus $B \cdot L(L) \subseteq B_1$. On the other hand, each $\hat{h}_i a_i = a_i b_i^{-1} = b_i^{-1} a_i \in B \cdot L(L)$, and $\mathcal{H} \subseteq B \subseteq B_1$. Hence $B_1 = B \cdot L(L)$. But, $B \cdot L(L) = B \otimes L(L)$ because $L(L) \subseteq T$ and $E = B \otimes T$ by Proposition 2.2(b); so $B_1 = B \otimes L(L)$. Since $C_{\hat{H}_1}(L(L)) = L(L^L)$, as noted above, we have $Z(L(L)) = L(L) \cap L(L^L) = L(L \cap L^L)$. Hence $L_1 = Z(B_1) = Z(B \otimes_L L(L)) = Z(B) \otimes Z(L(L)) = L \otimes L(L \cap L^L) = L(L \cap L^L)$. This $L_1$ is a field as is reduced and $B_1 = F(\mathcal{H}_1)$ with $\mathcal{H} \subseteq \mathcal{H}_1 \subseteq \hat{H}$. Moreover, $L_1$ is a Kummer extension of $L$ since $\exp((L \cap L^L)/L^L) \mid \exp(T/L^*) = |\text{im}(B_T)|$ and $|\text{im}(B_T)| \subseteq \mu(L)$ by Proposition 2.1(b), (c). Proposition 2.2(a), applied with $\mathcal{H}_1$ in place of $\mathcal{H}$, says that $L_1$ lies in a radical Abelian extension of $F$. Because $L_1 \supseteq L$, we have $\mathcal{H}_1 = C_{\hat{H}_1}(L_1) \subseteq C_{\hat{H}_1}(L_1) = \mathcal{H}_1$. The other inclusions in $\mathcal{H}_1 \subseteq \mathcal{H}_1 \subseteq \hat{H}_1 \subseteq \hat{H}$ are clear. Now by Proposition 2.2(b), $E_1 = C_A(L_1) \subseteq C_A(L_1) = E$, as $L \subseteq L_1$. Therefore, $T_1 = C_{E_1}(L_1) \subseteq C_{E_1}(L_1) = C_B \otimes_T (B \otimes_L L(L)) = C_T(L(L)) = L(L^L)$. But also, $L(L^L) = C_T(L(L)) \subseteq C_A(L(L \cap L^L)) = E_1$. Since $L(L^L)$ centralizes $B$ and $L(L)$, it centralizes $B \otimes_L L(L) = B_1$.\]
Thus, $L(L^\perp) \subseteq T_1$; so equality holds. Finally, $E_1 = B_1 \otimes_L^e T_1 = B \otimes_L L(L^\perp) \otimes_{L(L\cap L^\perp)} L(L^\perp) = B \otimes_L L(L^\perp)$, completing the proof of (a).

For (b), note that

$$[T_1 : L_1] = [L(L^\perp) : L(L \cap L^\perp)]/[L(L \cap L^\perp) : L] = |L^\perp|/[L \cap L^\perp] = |L^\perp/L \cap L^\perp|.$$

Part (c) follows immediately from (a) and (b), since $\Delta$ is a Lagrangian just when $\Delta = A^\perp$, so $L = L^\perp$. □

3. Splitting maps for the Brauer group of a Henselian field

In this section we give the properties of division algebras over Henselian fields that will be needed for the analysis of the projective Schur groups of such fields. We first recall some known results, then give analogues for an arbitrary Henselian valuation to Witt's direct sum decomposition theorem for the Brauer group of a complete discretely valued field.

Let $F$ be a field with a valuation $v : F^* \rightarrow \Gamma_F$, where $\Gamma_F$ is the value group of $v$, a totally ordered Abelian group, written additively. Let $V_F$ be the valuation ring of $v$ and $M_F$ the unique maximal ideal of $V_F$; let $\overline{F} = V_F/M_F$, the residue field of $v$; and let $U_F = V_F - M_F$, the group of valuation units. Let $\Delta$ be the divisible hull of $\Gamma_F$; so $\Delta \supseteq \Gamma_F$, and the ordering on $\Gamma_F$ extends uniquely to $\Delta$ making $\Delta$ an ordered Abelian group. Note that $\Delta \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_F$. If $L$ is a field algebraic over $F$ and $w$ is any extension of $v$ to $L$, then there are canonical injections which we view as inclusions $\overline{F} \hookrightarrow \overline{L}, \Gamma_F \hookrightarrow \Gamma_L$, and $\Gamma_L \hookrightarrow \Delta$. Recall the Fundamental Inequality [16, (13.10)], which says that whenever $[L : F] < \infty$, we have

$$[\overline{L} : \overline{F}]|\Gamma_L : \Gamma_F| \leq [L : F]. \quad (3.1)$$

Assume now and throughout the rest of this section that the valuation $v$ on $F$ is Henselian. This means that Hensel’s Lemma holds for $v$, or equivalently (see, e.g., [16, Corollary 16.6]), that $v$ has a unique extension to each field $L \supseteq F$ with $L$ algebraic over $F$. Thus the extension of $v$ to any such $L$, which we again denote by $v$, is also Henselian.

Examples 3.1. (a) If $v$ is a valuation on a field $F$ with $\Gamma_F$ embeddable in $\mathbb{R}$ (equivalently, if $V_F$ has Krull dimension 1) and $F$ is complete with respect to $v$, then $v$ is Henselian [16, (16.7)].

(b) Suppose $F_0$ is a field with Henselian valuation $v_0$. Let $F$ be the Laurent series field $F = F_0((x))$. Then the canonical extension $v$ of $v_0$ to $F$ (given by $v(\sum_{k=0}^{\infty} c_k x^k) = (v_0(c_k), k) \in \Gamma_F_0 \times \mathbb{Z}$ if $c_k \neq 0$) is Henselian, with $V_F = V_0 + x F_0[[x]], \overline{F} = \overline{F}_0$, and $\Gamma_F = \Gamma_F_0 \times \mathbb{Z}$. The ordering on $\Gamma_F$ is the right-to-left lexicographical ordering, in which $(\gamma, i) \leq (\delta, j)$ just when $i < j$ or both $i = j$ and $\gamma \leq \delta$. That $v$ is Henselian is a special case of the fact that composites of Henselian valuations are Henselian [29, p. 211, Proposition 10].

Let $L$ be an algebraic extension of the Henselian field $F$. If $[L : F] < \infty$, we say that $L$ is unramified over $F$ if $[\overline{L} : \overline{F}] = [L : F]$ and $\overline{L}$ is separable over $\overline{F}$. When this occurs, the Fundamental Inequality shows that $\Gamma_L = \Gamma_F$.

At the other extreme, we say that $L$ is totally ramified over $F$ if $|\Gamma_L : \Gamma_F| = [L : F]$. Also, $L$ is said to be tamely ramified over $F$ if $\text{char}(\overline{F})  \nmid |\Gamma_L : \Gamma_F|$, $\overline{L}$ is separable over $\overline{F}$, and $[\overline{L} : \overline{F}]|\Gamma_L : \Gamma_F| = [L : F]$. If $L$ is algebraic over $F$ but $[L : F] = \infty$, we say that $L$ is unramified (resp. totally ramified, tamely ramified) over $F$ if $L$ is a union of finite degree extensions of $F$ each of which is unramified (resp. totally ramified, tamely ramified) over $F$.

Let $F_s$ be a fixed separable closure of our Henselian valued field $F$. Let $F_{ns}$ denote the maximal unramified extension of $F$ in $F_s$. That is, $F_{ns}$ is the inertia field for the extension of $v$ from $F$ to $F_s$. It is known (see [16, (19.10), (19.11)]) that for fields $L$ with $F \subseteq L \subseteq F_s$, $L$ is unramified over $F$ iff $L \subseteq F_{ns}$. Furthermore (see [16, (19.12), (19.8), (19.6)]), $\overline{F_{ns}} \cong \overline{F}_s$, and $F_{ns}$ is Galois over $F$, with Galois group $G(F_{ns}/F) \cong G(\overline{F}_s/\overline{F})$, which is the absolute Galois group of $\overline{F}$, also denoted $G_F$. From the isomorphism of Galois groups, one can see that there is a one-to-one correspondence $L \mapsto \overline{L}$ between unramified field extensions $L$ of $F$ (with $L \subseteq F_s$) and separable algebraic field extensions of $\overline{F}$. If $E$ is a separable extension of $\overline{F}$, we call the field $L \supseteq F$ with $\overline{L} = E$ the inertial lift of $E$ over $\overline{F}$; we will often write $F(E)$ for this inertial lift of $E$.

When the valuation on $F$ is complete and discrete, Witt gave a description of the Brauer group $Br(F)$. We now give generalizations of Witt’s theorem, which are valid for any Henselian valued field. The basic exact sequence (3.3) below was derived in [20, pp. 154–156], but the splitting maps in Theorem 3.2 and Proposition 3.3 were not given
For any Henselian valued field $F$, there is a homomorphism $f$ which is called the \textit{inertially split part} of $Br(F)$.

The Abelian torsion group becomes the short exact sequence

$$1 \longrightarrow U_{F_{nr}} \longrightarrow F_{nr}^* \longrightarrow \Gamma \longrightarrow 0$$

induces an exact sequence in cohomology

\[
H^1(G, \Gamma) \longrightarrow H^2(G, U_{F_{nr}}) \longrightarrow H^2(G, F_{nr}^*) \longrightarrow H^2(G, \Gamma).
\]

We interpret the terms in the sequence. We have $H^1(G, \Gamma) = \text{Hom}_{\mathbb{Z}}(G, \Gamma) = 0$ (continuous homomorphisms; there are none, as $G$ is profinite and $\Gamma$ is torsion-free). It is known (see \cite{20, Theorem 5.6(a)}) that the residue map $U_{F_{nr}} \longrightarrow \overline{F}_{nr}^* \cong \overline{F}_s^*$ induces an isomorphism $H^2(G, U_{F_{nr}}) \cong H^2(G, \overline{F}_s^*) \cong Br(\overline{F})$. The next term in (3.2) is $SBr(F)$. Since $\Delta$ is uniquely divisible, $H^1(G, \Delta) = H^2(G, \Delta) = 0$, so $H^2(G, \Gamma) \cong H^1(G, \Delta/\Gamma) = \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma)$. We shall give a splitting map which shows that the last map in (3.2) is onto, so (3.2) becomes the short exact sequence

$$0 \longrightarrow Br(\overline{F}) \longrightarrow SBr(F) \longrightarrow \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma) \longrightarrow 0.$$  \hfill (3.3)

It is easy to see that (3.3) is functorial with respect to algebraic field extensions, i.e., for any field $L \supseteq F$ with $L$ algebraic over $F$, and for the unique Henselian extension of $v$ to $L$, the following diagram is commutative with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Br(\overline{F}) & \longrightarrow & SBr(F) & \longrightarrow \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Br(\overline{L}) & \longrightarrow & SBr(L) & \longrightarrow \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma_L) & \longrightarrow 0
\end{array}
\]  \hfill (3.4)

Here the right vertical map arises from the canonical inclusion $G_T \hookrightarrow G_{\overline{T}}$ and the surjection $\Delta/\Gamma_T \twoheadrightarrow \Delta/\Gamma_L$.

\textbf{Theorem 3.2.} For any Henselian valued field $F$, there is a homomorphism $f : \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma) \to SBr(F)$ splitting the $\beta$ of (3.3). Hence

$$SBr(F) \cong Br(\overline{F}) \oplus \text{Hom}_{\mathbb{Z}}(G, \Delta/\Gamma_T).$$

\textbf{Proof.} The Abelian torsion group $\Delta/\Gamma$ has its canonical primary decomposition $\Delta/\Gamma = \bigoplus_{\gamma \in \Gamma} p^{-n_{\gamma}} \Gamma/\gamma$. Fix a prime number $p$. Choose $\{\gamma_i\}_{i \in I_p} \subseteq \Gamma$ so that the $\gamma_i$ map to a $\mathbb{Z}/p\mathbb{Z}$-vector space base of $\Gamma/p\Gamma$. Then, for each $n \in \mathbb{N}$, the $\gamma_i$ also map to a base of $\Gamma/p^n \Gamma$ as a free $\mathbb{Z}/p^n\mathbb{Z}$-module. So, $\{p^{-n}\gamma_i\}_{i \in I_p}$ maps to a base of the free $\mathbb{Z}/p^n\mathbb{Z}$-module $p^n \Gamma/\Gamma$. Let $p^{-n_{\gamma}}\gamma_i$ denote $\bigcup_{j=1}^{\infty}(p^{-j}\gamma_i + \Gamma) \subseteq \Delta/\Gamma$. Then, $(\Delta/\Gamma)(p) = \bigoplus_{i \in I_p} p^{-n_{\gamma_i}}\gamma_i$.

Let $\gamma$ be any of the $\gamma_i$, and take any $t \in F^*$ with $v(t) = \gamma$. We use $t$ to define a homomorphism from $\text{Hom}_{\mathbb{Z}}(G, \gamma/\Gamma)$ to the Brauer group. For this, note that since $\mathbb{Z}[1/p]_{\gamma} \cong \mathbb{Z}[1/p]$ and $\mathbb{Z}[1/p]_{\gamma} \cap \Gamma = \mathbb{Z}_{\gamma}$ (as $\gamma \notin p\Gamma$), we have $p^{-n_{\gamma}}\gamma/\Gamma \cong \mathbb{Z}[1/p]_{\gamma}/\mathbb{Z}_{\gamma} \cong \mathbb{Z}[1/p]/\mathbb{Z}$. Thus there is a homomorphism

$$\theta : H^1(G, p^{-n_{\gamma}}\gamma/\Gamma) \longrightarrow H^1(G, \mathbb{Z}[1/p]/\mathbb{Z}) \overset{\delta}{\longrightarrow} H^2(G, \mathbb{Z}),$$

where $\delta$ is the connecting homomorphism arising from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}[1/p]/\mathbb{Z} \to 0$ of trivial $G$-modules. We use the cup product pairing $H^0(G, F_{nr}^*) \cup H^2(G, \mathbb{Z}) \to H^2(G, F_{nr}^*) \cong SBr(F)$ to define our map

$$f_\gamma : \text{Hom}_{\mathbb{Z}}(G, p^{-n_{\gamma}}\gamma/\Gamma) \longrightarrow SBr(F) \quad \text{by} \quad \chi \mapsto (t) \cup \theta(\chi).$$

(3.6)
Then \( f_i \) is a group homomorphism, as the cup product is \( \mathbb{Z} \)-bilinear. To describe \( f_i(\chi) \) as an algebra, let \( N = \ker(\chi) \), an open normal subgroup of \( G \), and let \( K \) be the fixed field of \( N \). So \( F \subseteq K \subseteq F_{nr} \) and \( \mathcal{G}(K/F) \cong \text{im}(\chi) \), which is a finite subgroup of \( p^{-\infty} \gamma/\Gamma \); thus \( \text{im}(\chi) \) is the cyclic group \( p^{-r} \gamma \Gamma \), where \([K : F] = p^r\). Choose \( \sigma \in \mathcal{G}(F_{nr}/F) = G \) with \( \chi(\sigma) = p^{-r} \gamma \Gamma \); then \( \sigma|_K \) is a generator of \( \mathcal{G}(K/F) \). It is easy to check that \( \theta(\chi) \) is represented by the cocycle \( z \in Z^2(G,\mathbb{Z}) \) given by: for \( \tau, \rho \in G \), if \( \tau|_K = (\sigma|_K)^j \) and \( \rho|_K = (\sigma|_K)^j \) with \( 0 \leq i, j \leq p^r - 1 \),
\[
  z(\tau, \rho) = \begin{cases} 
    0 & \text{if } i + j < p^r, \\
    1 & \text{if } i + j \geq p^r.
  \end{cases}
\]
Then \( f_i \) is represented by \( z' \in Z^2(G, F_{nr}^*) \) given by
\[
  z'(\tau, \rho) = \begin{cases} 
    1 & \text{if } i + j < p^r, \\
    \tau & \text{if } i + j \geq p^r.
  \end{cases}
\]

Note that the splitting map \( f \) of Theorem 3.2 depends on the choice of the \( t_i \). Similarly, in Witt’s theorem for complete discrete valuations, the splitting map depends on the choice of a uniformizing parameter. The reason for using the primary decomposition of \( \Delta/\Gamma \) in the proof of Theorem 3.2 is that the index sets \( I_p \) could have different sizes for different primes \( p \). (By definition, \( |I_p| = \dim_{\mathbb{Z}/p\mathbb{Z}}(\Gamma/p\Gamma) \).) Of course if \( \Gamma \) were a free \( \mathbb{Z} \)-module, we could choose a family \( \{t_i\} \subseteq F^* \) mapping to a \( \mathbb{Z} \)-base of \( \Gamma \). We could then use the same \( t_i \) for each prime \( p \), which would simplify the description of the splitting map.

We now show that the splitting of \( SBr(F) \) is compatible with suitably chosen scalar extensions. In the proof of Theorem 3.2, we chose \( \{\gamma_i\}_{i \in I_p} \subseteq \Gamma \) and \( \{t_i\}_{i \in I_p} \subseteq F^* \) with \( v(t_i) = \gamma_i \), for any fixed prime \( p \). To allow for consideration of all primes at once, we write \( \gamma_{p,i} \) for the earlier \( \gamma_i \) and \( t_{p,i} \) for \( t_i \).

**Proposition 3.3.** Let \( L \) be an algebraic extension of the Henselian valued field \( F \).

(a) Suppose \( L \) is unramified over \( F \). Then \( \Gamma_L = \Gamma_F \), and we can use the same \( t_{p,i} \) for the splitting map of \( SBr(L) \) as for \( SBr(F) \). Then we have a commutative diagram
\[
\begin{array}{ccc}
  Br(F) & \oplus & \text{Hom}_c(G_{\Gamma_F}, \Delta/\Gamma_F) \\
  \text{res} \downarrow & & \text{res} \downarrow \\
  Br(L) & \oplus & \text{Hom}_c(G_{\Gamma_L}, \Delta/\Gamma_L) \\
\end{array}
\]
\[
\cong \quad \cong \quad \cong
\]
\[
\text{SBr}(F) \\
\text{SBr}(L)
\]

(b) With the \( t_{p,i} \) chosen for the splitting map \( f \) of Theorem 3.2, take any subset \( \{j_{p,i}\}_{p \text{ prime}, i \in I_p} \subseteq \mathbb{N} \). For each such \( p \) and \( i \), let \( s_{p,i} \) be some \( p^{j_{p,i}} \)-th root of \( t_{p,i} \), and suppose \( L = F(s_{p,i}) \). Then \( L \) is totally ramified over \( F \), \( \{v(s_{p,i})\}_{i \in I_p} \) maps to a \( \mathbb{Z}/p\mathbb{Z} \)-base of \( \Delta/\Gamma_L \), and if we use the \( t_{p,i} \) for the splitting of \( SBr(F) \) and the \( s_{p,i} \) for the splitting of \( SBr(L) \), then there is a commutative diagram
\[
\begin{array}{ccc}
  Br(F) & \oplus & \text{Hom}_c(G_{\Gamma_F}, \Delta/\Gamma_F) \\
  \text{res} @ \text{can} \downarrow & & \text{res} \downarrow \\
  Br(L) & \oplus & \text{Hom}_c(G_{\Gamma_L}, \Delta/\Gamma_L) \\
\end{array}
\]
\[
\cong \quad \cong \quad \cong
\]
\[
\text{SBr}(F) \\
\text{SBr}(L)
\]
where \( \text{can} \) is induced by the canonical surjection \( \Delta/\Gamma_F \to \Delta/\Gamma_L \).
Proof. (a) Assume $L$ is unramified over $F$, so $\Gamma_L = \Gamma_F$. Take any prime $p$, any $\gamma \in \Gamma_F$, $\not\equiv p\Gamma_F$, and any $t \in F^*$ with $v(t) = \gamma$. We have an evidently commutative diagram

$$
\begin{array}{ccc}
H^1(G_T, p^{-\infty}\gamma/\Gamma_F) & \xrightarrow{\theta} & H^2(G_T, \mathbb{Z}) \\
\downarrow{\text{res}} & & \downarrow{\text{res}}
\end{array} \xrightarrow{(t)\cup} SBr(F)
$$

and

$$
\begin{array}{ccc}
H^1(G_T, p^{-\infty}\gamma/\Gamma_L) & \xrightarrow{\theta} & H^2(G_T, \mathbb{Z}) \\
\downarrow{\text{res}} & & \downarrow{\text{res}}
\end{array} \xrightarrow{(t)\cup} SBr(L)
$$

where the horizontal maps $\theta$ are as in (3.5), the right horizontal maps are cup product with $(t)$, and the first two vertical restriction maps arise from the canonical inclusion $G_T \hookrightarrow G_T$. The composition in the top row is the map $f_t$ of (3.6) for $F$, and in the bottom row is the $f_t$ for $L$. Since the splitting map $f : \text{Hom}_c(G_T, \Delta/\Gamma_F) \to SBr(F)$ is a direct sum of such $f_t$ and likewise for $L$ replacing $F$ (using the same family of $t$’s) the commutativity of (3.7) for each $t$ yields the commutativity of the diagram in (a).

(b) For each $p, i$ let $\delta_{p,i} = v(s_{p,i}) = p^{1/j_{p,i}}\gamma_{p,i}$. The Fundamental Inequality (3.1) shows that the field $L_{p,i} = F(s_{p,i})$ is totally ramified over $F$ with $\Gamma_{L_{p,i}} = \langle \delta_{p,i} \rangle + \Gamma_F$. Since the relative value groups $\Gamma_{L_{p,i}}/\Gamma_F$ are all independent in $\Delta/\Gamma_F$, our field $L$, which is the compositum of all the $L_{p,i}$, is totally ramified over $F$ with $\Gamma_L = \sum p_i\langle \delta_{p,i} \rangle + \Gamma_F$; so $L = \overline{F}$. We have $\Gamma_L/\Gamma_F = \bigoplus P_i \langle \delta_{p,i} \rangle + \Gamma_F/\Gamma_F$ with each summand $\langle \delta_{p,i} \rangle + \Gamma_F/\Gamma_F \leq p^{-\infty}\gamma_{p,i}/\Gamma_F$. Since $\Delta/\Gamma_F = \bigoplus P_i p^{-\infty}\gamma_{p,i}/\Gamma_F$, we have a compatible decomposition $\Delta/\Gamma_L \cong \bigoplus P_i (p^{-\infty}\gamma_{p,i}/\Gamma_F)/(\langle \delta_{p,i} \rangle + \Gamma_F) \cong \bigoplus P_i p^{-\infty}\delta_{p,i}/\Gamma_L$. Hence $\{\delta_{p,i} \mid i \in I_p\}$ maps to a $\mathbb{Z}/p\mathbb{Z}$-base of $\Gamma_L/\Gamma_F$, so the $s_{p,i}$, with $v(s_{p,i}) = \delta_{p,i}$, are a valid set to use for the splitting map of $SBr(L)$. Because the canonical map $\Delta/\Gamma_F \to \Delta/\Gamma_L$ sends $p^{-\infty}\gamma_{p,i}/\Gamma_F$ onto $p^{-\infty}\delta_{p,i}/\Gamma_L$, it suffices for (b) to verify the commutativity of the diagrams

$$
\begin{array}{ccc}
\text{Hom}_c(G_T, p^{-\infty}\gamma_{p,i}/\Gamma_F) & \xrightarrow{f_{p,i}} & SBr(F) \\
\downarrow{\text{res}} & & \\
\text{Hom}_c(G_T, p^{-\infty}\delta_{p,i}/\Gamma_L) & \xrightarrow{f'_{p,i}} & SBr(L)
\end{array}
$$

where the maps $f_{p,i}$ and $f'_{p,i}$ are as in (3.6). Of course $G_T = \overline{F}$ here, as $\overline{L} = \overline{F}$. To analyze (3.8), take any $\chi \in \text{Hom}_c(G_T, p^{-\infty}\gamma_{p,i}/\Gamma_F)$, and let $\psi$ be the image of $\chi$ in $\text{Hom}_c(G_T, p^{-\infty}\delta_{p,i}/\Gamma_L)$. We have the diagram

$$
\begin{array}{ccc}
H^1(G_T, p^{-\infty}\gamma_{p,i}/\Gamma_F) & \xrightarrow{\theta} & H^2(G_T, \mathbb{Z}\gamma_{p,i}) \\
\downarrow & & \downarrow{p^{1/j_{p,i}}}
\end{array} \xrightarrow{(t)\cup} H^2(G_T, \mathbb{Z})
$$

and

$$
\begin{array}{ccc}
H^1(G_T, p^{-\infty}\delta_{p,i}/\Gamma_L) & \xrightarrow{\theta} & H^2(G_T, \mathbb{Z}\delta_{p,i}) \\
\downarrow & & \downarrow{p^{1/j_{p,i}}}
\end{array} \xrightarrow{(t)\cup} H^2(G_T, \mathbb{Z})
$$

where the middle vertical map arises from the inclusion $\mathbb{Z}\gamma_{p,i} \hookrightarrow \mathbb{Z}\delta_{p,i}$, and the right vertical map is multiplication by $p^{1/j_{p,i}}$. In (3.9), the left rectangle is commutative since the horizontal maps are connecting homomorphisms arising from compatible short exact sequences of $G_T$-modules. The right rectangle in (3.9) is evidently commutative. The composition of the top (resp. bottom) maps in (3.9) is the $\theta$ of (3.5) for $\gamma_{p,i}$ (resp. $\delta_{p,i}$). So the commutativity of (3.9) shows that $p^{1/j_{p,i}}\theta(\chi) = \theta(\psi)$. Hence $\text{res}_{L/F}(f_{p,i}(\chi)) = \text{res}_{L/F}(\langle (t)\cup \theta(\chi) \rangle) = (s_{p,i}^{p^{1/j_{p,i}}})\cup \theta(\chi) = (s_{p,i})\cup \theta(\psi) = f'_{p,i}(\psi)$, proving the commutativity of (3.8), as desired. □

There is a further well-described part of $Br(F)$ for $F$ Henselian, which is what we get when we add in the tame totally ramified division algebras. A central division algebra $T$ over our Henselian field $F$ is said to be tame and totally ramified (TTR) if (with respect to the unique extension of the valuation $v$ on $F$ to $T$) $|\Gamma_T : \Gamma_F| = |T : F|$ and $\text{char}(\overline{F}) \nmid |T : F|$. The theory of such division algebras is described in [32]. In particular, it is known (see [15, Theorem 1]) that every such $T$ is isomorphic to a tensor product of symbol algebras (so lies in $PS(F)$), and that $\exp(T) = \exp(T\gamma/\Gamma_F)$ by [32, Example 4.4(iii)]). The possible TTR algebras are thus constrained by the roots of unity in $F$. Here is a typical example of a TTR symbol algebra:
Example 3.4. Suppose $\mu_n \subseteq F^*$, char($\overline{F}$) $\nmid n$, and $s, t \in F^*$ such that $v(s)$ and $v(t)$ generate a group of order $n^2$ in $\Gamma_F/n\Gamma_F$. Then the symbol algebra $T = (s, t; F)_n$ is TTR, with $\Gamma_T = (\frac{1}{n}v(s) + \frac{1}{n}v(t)) + \Gamma_F$ (see [32, Proposition 3.5]).

A central division algebra $D$ over a Henselian field $F$ is said to be tame if $D$ is split by the maximal tamely ramified extension $F_{tr}$ of $F$. This field $F_{tr}$ is the ramification field of the extension of $v$ from $F$ to $F_s$. It has the property that for any field $L$ with $F \subseteq L \subseteq F_s$, $L$ is tamely ramified over $F$ if $L \subseteq F_{tr}$ (see [16, (20.7), (20.16)]). It is known [20, Lemma 6.2] that if $D$ is tame, then $D \sim S \otimes_T F$ in $Br(F)$, where $S$ is inertially split and $T$ is TTR. (But the $S$ and $T$ are not uniquely determined, and it is not in general possible to express $D \cong S \otimes_T F$ with $S$ inertially split and $T$ TTR.) Let the tame part of $Br(F)$ be denoted by

$$TBr(F) = \{[D] \in Br(F) | D \text{ is tame} \} = Br(F_{tr}/F).$$

For the primary components of $TBr(F)$ we have (see [19, Proposition 4.3]) for every prime $p$,

$$TBr(F)(p) = \left\{ \begin{array}{ll}
Br(F)(p) & \text{if } p \neq \text{char}($\overline{F}$), \\
SBr(F)(p) & \text{if } p = \text{char}($\overline{F}$).
\end{array} \right.$$

(3.10)

There is a noncanonical splitting for the inclusion of $SBr(F)$ in $TBr(F)$, expressed in terms of the primary components as follows. Take any prime $p \neq \text{char}($\overline{F}$), and as above take $\{t_i | i \in I_p\} \subseteq F^*$ with $v(t_i)$ mapping to a $\mathbb{Z}/p\mathbb{Z}$-basis of $\Gamma_F/p\Gamma_F$. Fix some total ordering on the index set $I_p$. For any $n \in \mathbb{N}$ with $\mu_{p^n} \subseteq F^*$, let $T_{p^n}$ be the subgroup of $TBr(F)$ generated by the symbol algebras $\{(t_i, t_j; F)_{p^n} | i, j \in I_p, i < j \}$. As the proof of Proposition 3.5 below shows, $T_{p^n}$ is a free $\mathbb{Z}/p^n\mathbb{Z}$-module, and these symbol algebras are a base. If $\mu_{p^n} \subseteq F^*$ for all $n \in \mathbb{N}$, then let $T_{p^n} = \bigcup_{n=1}^{\infty} T_{p^n}$.

Proposition 3.5. Fix any Henselian valued field $F$ and any prime $p \neq \text{char}($\overline{F}$). If $r \in \mathbb{N}$ is maximal such that $\mu_{p^r} \subseteq F^*$, then $Br(F)(p) = SBr(F)(p) \oplus T_{p^r}$. If $\mu_{p^n} \subseteq F^*$ for every $n$, then $Br(F)(p) = SBr(F)(p) \oplus T_{p^n}$.

Proof. We have the exact sequence

$$0 \longrightarrow SBr(F)(p) \longrightarrow Br(F)(p) \longrightarrow Br(F_{nr})(p),$$

(3.11)

where $\rho$ is the restriction map. Assume first that $r$ is maximal such that $\mu_{p^r} \subseteq F^*$. For $D \in Br(F)(p)$, since $p \neq \text{char}($\overline{F}$), $D$ is tame (see (3.10)). So $D \sim S \otimes_T F$ in $Br(F)$, where $S \in SBr(F)$ and $T$ is TTR, with $S$ and $T$ both $p$-primary. It follows that the division algebra $T$ has degree a power of $p$. In a tensor decomposition of $T$ into symbol algebras, each symbol algebra factor has degree $p^m$ for some $m$ with $\mu_{p^m} \subseteq F^*$, so $m \leq r$. Hence,

$$\exp(D \otimes_T F_{nr}) = \exp(T \otimes_T F_{nr})|\exp(T)|p^r.$$

That is, $\text{im}(\rho) \subseteq \rho^* Br(F_{nr})$.

We claim that $\rho^* Br(F_{nr})$ is a free $\mathbb{Z}/p^r\mathbb{Z}$-module with base $B = \{(t_i, t_j; F_{nr})_{p^r} | i, j \in I_p, i < j \}$. For this, note first that every division algebra $D \subset \rho^* Br(F_{nr})$ is tame (see (3.10)), so TTR, since $SBr(F_{nr}) = 0$; so $D$ is a tensor product of TTR symbol algebras. Since $F_{nr}^* / F_{nr}^{p^r} \cong \Gamma^* / p^r \Gamma$, $\{t_i | i \in I_p\}$ maps to a base of the free $\mathbb{Z}/p^r\mathbb{Z}$-module $F_{nr}^* / F_{nr}^{p^r}$. Hence, $B$ is a generating set for $\rho^* Br(F_{nr})$. To see $\mathbb{Z}/p^r\mathbb{Z}$-independence of $B$ it suffices to check independence for $\{(t_i, t_j; F_{nr})_{p^r} | i, j \in J, i < j \}$ for any finite subset $J \subseteq I_p$. Say $J = \{i_1, \ldots, i_\ell\}$ with $i_1 < i_2 < \cdots < i_\ell$. Take any $u = u_{i_1}^{m_1} \cdots u_{i_\ell}^{m_\ell}$ with not all $m_j \in p^r\mathbb{Z}$, and let $A = (u, t_{i_1}; F_{nr})_{p^r}$. By induction on $\ell$, it suffices to check that $[A]$ is not in the subgroup $S$ of $\rho^* Br(F)$ generated by $\{(t_i, t_j; F_{nr})_{p^r} | i, j \in \{i_1, \ldots, i_{\ell-1}\}, i < j \}$. But, if $s$ is maximal such that $p^s$ divides each of the $m_j$, then $s < r$ and $u = u_0^{n_s} \rho^{|s|r}$. Then in $Br(F_{nr})$, $A \sim A_0 = (u_0, t_{i_1}; F_{nr})_{p^{|s|r}}$. Example 3.4 shows that $A_0$ is a TTR division algebra with $\Gamma_{A_0} = \frac{1}{p^{|s|r}}(v(u_0), v(t_{i_1})) + \Gamma_F$. On the other hand, by [32, Example 4.4(i)], every division algebra in $S$ has a value group lying in $\sum_{j=1}^{\ell-1}(\frac{1}{p^r}v(t_{i_j})) + \Gamma_F$, and this group does not contain $\frac{1}{p^{|s|r}}v(t_{i_1})$. Thus, $[A] \not\in S$, and the claim is proved.

Because $\rho^* Br(F_{nr})$ is a free $\mathbb{Z}/p^r\mathbb{Z}$-module with base $B$, there is a well-defined group homomorphism $h_r : \rho^* Br(F_{nr}) \rightarrow \rho^* Br(F)$ given by $(t_i, t_j; F_{nr})_{p^r} \mapsto (t_i, t_j; F)_{p^r}$. Clearly $\text{im}(h_r) = T_{p^r}$ and $\rho \circ h_r = \text{id}$ on $\rho^* Br(F_{nr})$; combined with what we have proved above, this shows that $\text{im}(\rho) = \rho^* Br(F_{nr})$ and that $h_r$ is a splitting map for (3.11). Thus $Br(F)(p) = \ker(\rho) \oplus \text{im}(h_r) = SBr(F)(p) \oplus T_{p^r}$. 


Now assume instead that $\mu_{p^n} \subseteq F^*$ for every $n \in \mathbb{N}$. For each $n$, we have a homomorphism $h_n : \rho_n \Br(F_{nr}) \to \rho_n \Br(F)$ defined as above. Since $h_n|_{\rho_n \Br(F_{nr})} = h_m$ for $m < n$, these maps are compatible, and we can take their union, obtaining $h : \Br(F_{nr})(p) \to \Br(F)(p)$ with $\rho \circ h = \text{id}$. Thus $h$ is a splitting map for (3.11), and $\Br(F)(p) = \ker(\rho) \oplus \text{im}(h) = SBr(F)(p) \oplus T_{p^\infty}$, as desired. \hfill \Box

**Remark 3.6.** F. Chang tells us that for $p$ odd, all the division algebras in $T_{p^\infty}$ are TTR, whenever $\mu_{p^n} \subseteq F^*$. A proof of this is given in [12, Theorem 2.3.2]. But this is not true in general for $p = 2$. For example if $|f_{2}| \geq 3$, let $T$ be the underlying division algebra of $(t_1, t_2; F) = (t_1, t_3; F) \otimes (t_2, t_5; F)$. Then $T \sim (-1, t_2; F) \otimes (t_1t_2, t_2t_3; F) \in \Br(F)$. If $\mu_4 \not\subseteq F^*$, then this equivalence is an isomorphism, and $T$ is not TTR. But if $\mu_4 \subseteq F^*$, then $T \cong (t_1t_2, t_2t_3; F) \in \Br(T)$. Therefore

4. **Projective Schur groups of Henselian valued fields**

Let $F$ be an equicharacteristic Henselian valued field with residue field $k$. (Equicharacteristic means $\text{char}(k) = \text{char}(F)$.) In this section we show how $PS(F)$ is related to $PS(k)$. We show that if every projective Schur algebra over $k$ is a radical (resp. radical Abelian) algebra, then the same is true for projective Schur algebras over $F$. The results here generalize those in [7], which treated the case where $F = k((t_1)) \cdots ((t_n))$.

Let $F$ be any field, and let $K$ be a finite radical field extension of $F$, i.e., $K = F(U)$ where $U$ is a subgroup of $K^*$ with $U \supseteq F^*$ and $U/F^*$ is finite. If $L$ is an intermediate field, $F \subseteq L \subseteq K$, and $L$ is Abelian Galois over $F$, then it is shown in [3, Proposition 2.1] that $L \subseteq M$ where $M$ is a compositum of a finite cyclotomic extension of $F$ and a finite Kummer extension of $F$. In particular, if $L$ is a radical Abelian extension of $F$, then $L$ lies in such an $M$. Let $F_{ra}$ denote the maximal radical Abelian extension of $F$, i.e., the compositum $F_{ra} = F_{cyc} \cdot F_{kum}$, where $F_{cyc}$ is the maximal cyclotomic extension of $F$ and $F_{kum}$ is the maximal Kummer extension of $F$. (The notation $F_{radab}$ was used for $F_{ra}$ in [7].)

Now, suppose $F$ has a Henselian valuation, with residue field $k$. As in Section 3 we write $F_{nr}$ for the maximal unramified extension of $F$; $SBr(F)$ for $\Br(F_{nr}/F)$; and $F(k_{ra})$ for the unramified extension of $F$ with residue field $k_{ra}$. We have,

$$F(k_{ra}) = F_{nr} \cap F_{ra}. \tag{4.1}$$

To see this, note first that the maximal unramified extension of $F$ in $F_{kum}$ is $F(k_{kum})$. Therefore, $F_{nr} \cap F_{ra} = F_{nr} \cap (F_{cyc} \cdot F_{kum}) = F_{cyc} \cdot (F_{nr} \cap F_{kum}) = F(k_{cyc}) \cdot F(k_{kum}) = F(k_{ra})$, where the second equality is immediate from the corresponding equality of associated subgroups of the absolute Galois group of $F$, as $F_{nr}$ is Galois over $F$.

**Proposition 4.1** (cf. [7, Theorem 2.3]). Let $F$ be a Henselian valued field with residue field $k$, and assume $\text{char}(F) = \text{char}(k)$. Then $PS(F) \cap SBr(F) \subseteq Br(F(k_{ra})/F)$.

**Proof.** Let $A \in PS(F) \cap SBr(F)$, say $A = F(\hat{G})$, where $\hat{G}$ is a subgroup of $A^*$ spanning $A$ as an $F$-vector space, with $F^* \subseteq \hat{G}$ and $|\hat{G}/F^*| < \infty$. Assume $A$ is reduced. Let $\hat{\mathcal{H}} \subseteq \hat{\mathcal{G}}$ (the derived group of $\hat{G}$, a finite group), and as in **Proposition 2.2** let $B = F(\hat{H})$, $L = Z(B)$, $\hat{\mathcal{H}} = C_{\hat{G}}(L)$, and $E = F(\hat{H}) \cdot C_{\hat{L}}(L) = B \otimes L T$, where $T = C_F(B) = L(T)$, where $T = C_F(B)$. So, $L$ is a field, and as $B$ is a Schur algebra, by the Brauer splitting theorem ([13, pp. 385, 418]) $L \subseteq F_{cyc}$ and $F_{cyc}$ splits $B$. Note that $F_{cyc} = F(k_{cyc}) \subseteq F_{nr}$, since $F$ is Henselian and char($k$) = char($F$), so that $F$ and $k$ contain “the same” roots of unity. Because $A \in SBr(F)$ and $E = C_\lambda(L) \sim L \otimes F A$ in $Br(L)$, we have $E \in SBr(L)$. Since $E = B \otimes L T$ and $B \in SBr(L)$, we must also have $T \in SBr(L)$.

By **Proposition 2.2**(d), $T = L(T)$ is a projective Schur algebra of Abelian type over $L$. Therefore, if we let $A = T/L^*$, **Proposition 2.1** shows that we have the nondegenerate symplectic pairing $B_T : \Lambda \times \Lambda \to \mu(L)$ induced by commutators of elements of $T$. Let $n = \exp(A)$. Since $\mu_n = \text{im}(B_T) \subseteq F^*$, we have char($F$) $\nmid n$. Therefore, char($k$) = char($F$) $\not\mid |\Lambda| = \dim_F(T)$, using **Proposition 2.1**(c).

Let $v : F^* \to \Gamma$ be our Henselian valuation on $F$, where $\Gamma$ is the value group of $v$, and let $\Delta = \mathbb{Q} \otimes \mathbb{Z} \Gamma$, the divisible hull of $\Gamma$. Because commutators of elements of $T$ are roots of unity, the valuation $v$ induces a well-defined group homomorphism $w : \Lambda \to \Delta/\Gamma$ given by $w(tL^*) = \frac{1}{p} v(t^{p^n}) + \Gamma$, where $n = \exp(A)$. Let $A_0 = \ker(w) \subseteq \Lambda$. Because $T = L(T)$ is a projective Schur algebra of Abelian type, the same is true for $L_{nr} \otimes L T = L_{nr} \otimes L \Gamma$, where $\Gamma/L_{nr}^*$ is the image of $\Gamma/L^*$ under the canonical injection $T^*/L^* \hookrightarrow (L_{nr} \otimes L T)^*/L_{nr}^*$. Let $\overline{\Lambda} = \overline{\Gamma}/L_{nr}^*$, and let
Let $V$ be a Henselian valuation ring, $M$ its maximal ideal, $F$ its quotient field, and $k$ its residue field. Proposition 2.4 shows that for $V$ a field with $F \ni \gamma$ such that $L_1(\omega)$ splits $E_1$. Then $L_1(\omega)$ splits $A$ as $L_1 \otimes_F A \sim E_1$ in $\text{Br}(L_1)$; also $L_1(\omega) \subseteq F(k_{ra})$ since $L_1 \subseteq F(k_{ra})$. So, $A \in \text{Br}(F(k_{ra})/F)$, as desired. □

In order to relate unramified phenomena over a Henselian field to corresponding phenomena over the residue field, one often uses the valuation ring $V$ as a bridge. To employ that bridge here for projective Schur algebras, we need to know about the structure of “tame” twisted group rings over $V$. The next proposition gives what is needed. It may be of some interest in its own right.

Proposition 4.2. Let $V$ be a Henselian valuation ring, $M$ its maximal ideal, $F$ its quotient field, and $k = V/M$ its residue field. Let $V^G$ be a twisted group ring over $V$, where $z \in H^2(G, V^*)$ and $G$ is a finite group acting trivially on $V$. Suppose $\text{char}(k) \nmid |G|$. Then $V^G = \bigoplus_{i=1}^m S_i$, where each $S_i$ is an Azumaya algebra over $Z(S_i)$, and $(Z(S_i))$ is a valuation ring unramified over $V$. For the corresponding twisted group ring $k^G$ over $k$, we have $k^G \cong \bigoplus_{i=1}^m S_i/MS_i$, with each $S_i/MS_i$ a simple algebra with center the residue field of $Z(S_i)$.

Proof. Let $R = V^G$. We have $R/MR \cong k^G$, where $z$ is the image of $z$ in $H^2(G, k^*)$. Because $\text{char}(k) \mid |G|$ it is known by [20, p. 30, Theorem 4.2] that $k^G$ is a semisimple $k$-algebra. So, $Z(k^G) = \bigoplus_{i=1}^m L_i$, where each $L_i$ is a field with $[L_i : k] < \infty$. Moreover, we can see that each $L_i$ must be separable over $k$. For $L_j \otimes_k k^G \cong L_j^G$, which is semisimple; so its center $\bigoplus_{i=1}^m L_i \otimes_k L_i$ is a direct sum of fields. Since $L_j \otimes_k L_j$ is a direct sum of fields, $L_j$ must be separable over $k$. Conversely, $k^G$ is a separable algebra over $k$. Because $R/MR$ is a separable $V/M$-algebra, it follows by [14, p. 72, Theorem 7.1] that $R$ is a separable $V$-algebra. Let $Z = Z(R)$. Then $R$ is a central separable algebra (= Azumaya algebra) over $Z$ and $Z$ is separable over $V$ by [14, p. 55, Theorem 3.8]. Hence, $Z$ is a direct summand of $R$ as a $Z$-module, so as a $V$-module, by [14, p. 51, Lemma 3.1]. Therefore, as $R$ is a free $V$-module of rank $|G| < \infty$, $Z$ is a finitely generated projective $V$-module, hence a free $V$-module as $V$ is local. Now, $R/MR \cong R \otimes_Z (Z/MZ)$, which is a central separable $Z/MZ$-algebra (see [14, p. 61, lemma 5.1]); so $Z/MZ \cong Z(R/MR) \cong \bigoplus_{i=1}^m Z_i$. Suppose first that $Z/MZ$ is a field, i.e., $m = 1$. Now, $Z \otimes_V F$ is a commutative separable $F$-algebra by [14, p. 44, Corollary 1.7], so $Z \otimes_V F = \bigoplus_{j=1}^n K_j$, where each $K_j$ is a field separable over $F$. Let $W_j$ be the unique (as $V$ is Henselian) valuation ring of $K_j$ extending $V$; so $W_j$ is the integral closure of $V$ in $K_j$. Let $\mathbf{T} = \bigoplus_{j=1}^n W_j$, which is the integral closure of $V$ in $\bigoplus_{j=1}^n K_j$. When we view $Z \subseteq Z \otimes_V F = \bigoplus_{j=1}^n K_j$, we have $Z \subseteq T$, since $Z$ is integral over $V$. Because $MZ$ was assumed to be a maximal ideal of $Z$, we have $MT \cap Z = MZ$, so $Z/MZ \subseteq T/MT$. Thus,

$$\dim_k(T/MT) \geq \dim_k(Z/MZ) = \text{rk}_V(Z) = \dim_k(Z \otimes_V F) = \sum_{j=1}^n [K_j : F] = \sum_{j=1}^n \dim_k(W_j/MW_j) = \dim_k(T/MT).$$

See [11, Ch. VI, Section 8, No. 5, Cor. to Proposition 5] for the last inequality here. Therefore, equality holds throughout. Hence, $T/MT = Z/MZ \cong L_1$, which is a field separable over $k$. So, $n = 1$, $T = W_1$, and $W_1/MW_1 = T/MT \cong L_1$. So, $MW_1$ is the maximal ideal of $W_1$, and $\dim_k(W_1/MW_1) = [K_1 : F]$. Hence, $W_1$
is unramified over $V$. So, $W_1$ is a finitely generated $V$-module (see [11, Ch. VI, Section 8, No. 5, Theorem 2]); since $\dim_k(Z/MZ) = \dim_k(W_1/W_1)$, Nakayama’s Lemma shows that $Z = W_1$, as desired.

Now drop the assumption that $Z/MZ$ is a field. We still have $Z/MZ \cong \bigoplus_{i=1}^m L_i$, where each $L_i$ is a field separable over $k$. Let $\tilde{e}_1, \ldots, \tilde{e}_m$ be the primitive orthogonal idempotents of $Z/MZ$, with the $\tilde{e}_i$ numbered so that each $\tilde{e}_i(Z/MZ) \cong L_i$. Because $Z$ is a finitely generated module over the Henselian local ring $V$, the $\tilde{e}_i$ lift to pairwise orthogonal idempotents $e_1, \ldots, e_m \in Z$ with each $\tilde{e}_i$ the image of $e_i$ in $Z/MZ$, and $e_1 + \cdots + e_m = 1$, by [8, Theorem 24] or [24, p. 180, Theorem A.18]. Then, $Z = \bigoplus_{i=1}^m Z_i$, where each $Z_i = e_i Z$. Since each $Z_i$ is a direct summand of $Z$, $Z_i$ is a finitely generated projective, hence free, $V$-module. Moreover, each $Z_i$ is a separable $V$-algebra with $Z_i/MZ_i = e_i/Z/MZ \cong \tilde{e}_i(Z/MZ) \cong L_i$, which is a field separable over $k$. Therefore, the argument of the preceding paragraph, applied to $Z_i$, shows that $Z_i$ is a valuation ring unramified over $V$. The $e_i$ are orthogonal central idempotents of $R$; so $R = \bigoplus_{i=1}^m S_i$, where $S_i = e_i R$. Since $R$ is central separable over $Z$, each $S_i$ is central separable over $e_i Z = Z_i$, i.e., $S_i$ is an Azumaya algebra over $Z_i$, which is a valuation ring unramified over $V$. Also, $k^*G \cong R/\mathfrak{m}R = \bigoplus_{i=1}^m S_i/MZ_i$. Each $S_i/MS_i \cong S_i \otimes_Z(S_i/MZ_i)$, so $S_i/MS_i$ is an Azumaya algebra over the field $Z_i/MZ_i$, i.e., a simple algebra with center $Z_i/MZ_i$, which is the residue field of $Z_i$. \hfill \Box

For the rest of this section, we adopt the following standing hypotheses:

$F$ is a field with Henselian valuation $v$ with residue field $k$, with $\text{char}(k) = \text{char}(F) = p \neq 0$, while $\text{char}(k') = C$ if $\text{char}(F) = 0$.

**Proposition 4.3** (cf. [7, Proposition 2.6]). $\text{PS}(F') \cap Br(k) = PS(k')$.

**Proof.** Recall how the canonical inclusion $Br(k) \hookrightarrow Br(F)$ can be obtained (cf. [20, Theorem 5.6(a), Theorem 2.8]). The map $\theta : Br(V) \rightarrow Br(k)$ given by $[A] \mapsto [A \otimes_V k]$ is an isomorphism, as $V$ is Henselian. The map $\varphi : Br(V) \rightarrow Br(F)$ given by $[A] \mapsto [A \otimes_V F]$ is injective as $V$ is a valuation ring. The inclusion $Br(k) \hookrightarrow Br(F)$ is $\varphi \circ \theta^{-1}$. Of course if $V$ contains a coefficient field, which is a subfield $k_0 \subseteq V$, which maps isomorphically onto $k$ under the composition $k_0 \hookrightarrow V \twoheadrightarrow V/M = k$ (with $M$ the maximal ideal of $V$), then we can identify $k$ with $k_0$, and the map $Br(k) \hookrightarrow Br(F)$ is given by scalar extension $[A] \mapsto [A \otimes F]$. If $\text{char}(k) = 0$, then there always is a coefficient field, as $F$ is Henselian. This is provable in the same way as for a complete discrete valuation ring (see e.g. [35, p. 280, Corollary 2]), since only Hensel’s Lemma is used. If $\text{char}(k) = \text{char}(F) = p \neq 0$, then there may not always exist a coefficient field; if $k$ is separably generated over its prime field, then there is a coefficient field.

The inclusion $PS(k') \subseteq PS(F') \cap Br(k)$ is now clear if $\text{char}(k) = 0$, since then $V$ contains a coefficient field. On the other hand, if $\text{char}(k) = p \neq 0$, then we have the description in [5, Theorem 1.4] of algebras in $PS(k')$ as tensor products of symbol algebras together with a cyclic algebra $(\ell/k, \sigma, a)$, with $\ell \leq k_{\text{cyc}}$. Every such algebra has an obvious lift to an Azumaya algebra of the same type over $V$, which then extends by $- \otimes_V F$ to a central simple algebra of the same type over $F$. So again it is clear that $PS(k') \subseteq PS(F') \cap Br(k)$.

For the reverse inclusion, the argument is based on that in [7, Proposition 2.6], but adapted to work even if $V$ does not contain a coefficient field, and to apply for an arbitrary value group.

Let $\sigma \in Br(k')$ with image $\tilde{\sigma}$ in $Br(F')$ which lies in $PS(F')$. Let $\tilde{\sigma} = \sigma(G)$ be a projective Schur algebra over $F$ representing $\tilde{\sigma}$, where $G$ is a subgroup of $A^*$ with $F^* \subseteq G$, $G$ spans $\tilde{A}$ as an $F$-vector space, and $|G/F^*| < \infty$. Let $G = G/F^*$. By [7, lemma 2.5], we may assume that $|G|$ is prime to $p$ if $\text{char}(F) = p \neq 0$. We have the central extension

$$1 \rightarrow F^* \rightarrow G \rightarrow G/F^* \rightarrow 1.$$  

Denote by $z \in H^2(G, F^*)$ the cohomology class of this extension (with $G$ acting trivially on $G^*$). There is a corresponding surjective $F$-algebra homomorphism,

$$\eta : F^* \rightarrow G,$$

where $F^\eta G$ is the group algebra twisted by $z$.

Suppose for now that $z$ lies in the image of the map $H^2(G, V^*) \rightarrow H^2(G, F^*)$. Let $V^\eta G$ be the corresponding twisted group ring over $V$. We have $V^\eta G = \bigoplus_{i=1}^m S_i$, as in Proposition 4.2. So, $F^\eta G \cong F \otimes_V V^\eta G \cong \bigoplus_{i=1}^m S_i$. \hfill \Box
\( \bigoplus_{i=1}^{m} (F \otimes V S_i) \). Each \( F \otimes V S_i \) is an Azumaya algebra (= central simple algebra) over the field \( F \otimes V Z(S_i) \), which is the quotient field of the valuation ring \( Z(S_i) \). The surjection \( \eta \) above shows that one of the simple summands \( F \otimes V S_j \) of \( F^2 G \) is isomorphic to \( F(G) \). By comparing centers, we find \( F \otimes V Z(S_j) = F \), so \( Z(S_j) \cap F = V \). Let \( A = k \otimes V S_j \cong S_j / M S_j \). Then, \( A \) is a central simple \( k \)-algebra which by Proposition 4.2 is a direct summand of \( k^2 G \). Hence, \( A \) is a projective Schur algebra over \( k \). Since \([A]\) maps to \([A]\) under the injective map \( Br(k) \to Br(F) \), we have \([A] = \alpha \in Br(k)\), showing that \( \alpha \in PS(k)\)' as desired.

So far we have assumed that \( z \) was in the image of \( H^2(G, V^*) \to H^2(G, F^*) \). Now drop this assumption. The short exact sequence of trivial \( G \)-modules \( 1 \to V^* \to F^* \to \Gamma \to 0 \) yields the exactness of \( H^2(G, V^*) \to H^2(G, F^*) \to H^2(G, \Gamma) \). Since \( \Delta \) is a uniquely divisible group, we have \( H^2(G, \Gamma) \cong H^1(G, \Delta / \Gamma) = \text{Hom}(G, \Delta / \Gamma) \) (cf. the comments preceding (3.3) above). Hence we have an exact sequence \( H^2(G, V^*) \to H^2(G, F^*) \to \text{Hom}(G, \Delta / \Gamma) \). This is clearly functorial in \( F \), i.e., for any field \( K \) algebraic over \( F \), we have a commutative diagram with exact rows:

\[
\begin{array}{ccc}
H^2(G, V^*) & \longrightarrow & H^2(G, F^*) \\
\downarrow & & \downarrow \\
H^2(G, V_K^*) & \longrightarrow & H^2(G, K^*)
\end{array}
\]

Let \( \psi \in \text{Hom}(G, \Delta / \Gamma) \) be the image of \( \check{z} \in H^2(G, F^*) \). Then, \( \text{im}(\psi) \) is a finite subgroup of \( \Delta / \Gamma \). Let \( K \) be a totally ramified finite degree extension of \( F \) such that \( \text{im}(\psi) \subseteq \Gamma K / \Gamma \). Then \( \psi \) maps to 0 in \( \text{Hom}(G, \Delta / \Gamma K) \), so the commutative diagram shows that the image \( \check{z} \) of \( z \) in \( H^2(G, K^*) \) lies in the image of \( H^2(G, V_K^*) \). Thus the preceding argument applies over \( K \) (we work with \( K^{\times} G = K \otimes F F^2 G \), which maps to the central simple \( K \)-algebra \( K(\check{G}) = K \otimes F F(G) \), where \( \check{G} = K^* G \)). The argument shows that \( \check{\alpha}_K \) lies in the image of \( PS(\overline{K}) \) in \( Br(K) \), where \( \overline{K} \) is the residue field of \( V_K \). This \( \check{\alpha}_K \) is the image in \( Br(K) \) of the image \( \alpha_{\overline{K}} \) of \( \alpha \) in \( Br(\overline{K}) \), by the commutative diagram (3.4) with \( K \) replacing \( L \); so \( \alpha_{\overline{K}} \in PS(\overline{K}) \). But \( \overline{K} \cong \overline{F} = k \), as \( K \) is totally ramified over \( F \). Hence \( \alpha \in PS(k) \), as desired. \( \square \)

**Theorem 4.4.** Assuming the standing hypotheses stated above, let \( SBr(F) \) denote the inertially split part of \( Br(F) \) and let \( SPS(F) = PS(F) \cap SBr(F) \). Then there is a split exact sequence:

\[
0 \to PS(k) \to SPS(F) \to \text{Hom}_v(G(k_{ra}/k), \Delta / \Gamma) \to 0.
\]

If \( k \) is perfect, then the above sequence is split exact without \((-)'\).

**Proof.** Since the valuation \( v \) is Henselian, \( F(k_{ra}) \subseteq F_{ra} \), and \( v \) extends uniquely to \( F(k_{ra}) \). Moreover since \( F(k_{ra}) \) is an unramified extension of \( F \), the value groups \( \Gamma F(k_{ra}) \) and \( \Gamma F \) are equal. Applying Theorem 3.2 to \( F \) and \( F(k_{ra}) \) we obtain (using the functoriality noted in (3.4)) a commutative and exact diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & Br(k_{ra}) & \longrightarrow & SBr(F(k_{ra})) & \longrightarrow & \text{Hom}_v(G(k_{ra}), \Delta / \Gamma) \longrightarrow 0 \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & \downarrow & \\
0 & \longrightarrow & Br(k) & \longrightarrow & SBr(F) & \longrightarrow & \text{Hom}_v(G(k), \Delta / \Gamma) \longrightarrow 0 \\
\uparrow & & \downarrow & & \uparrow & & \downarrow & \downarrow & \\
0 & \longrightarrow & Br(k_{ra}/k) & \longrightarrow & SBr(F(k_{ra})/F) & \longrightarrow & \text{Hom}_v(G(k_{ra}/k), \Delta / \Gamma) \longrightarrow 0
\end{array}
\]

The third row is split exact because the first two rows are split exact with compatible splitting maps by the proof of Proposition 3.3(a). Now \( PS(k) \subseteq Br(k_{ra}/k) \) \([3, \text{Corollary 2.3}], \) and by Proposition 4.1, \( SPS(F) \subseteq Br(F(k_{ra})/F) \). Moreover by Proposition 4.3 we have \( PS(k) \subseteq PS(F) \cap Br(k) \), hence \( PS(k) \subseteq PS(F) \cap SBr(F) = SPS(F) \).
We therefore obtain a commutative diagram

\[
\begin{array}{c}
0 \longrightarrow Br(k_{ra}/k) \quad \text{res}(i) \quad \text{res}(\pi) \quad \text{Hom}_c(G(k_{ra}/k), \Delta/\Gamma) \quad 0 \\
0 \longrightarrow PS(k) \quad j \quad SPS(F) \quad \eta \quad \text{Hom}_c(G(k_{ra}/k), \Delta/\Gamma) \quad 0
\end{array}
\]

where the top row is split exact and the bottom row is exact at \(PS(k)\). Again by Proposition 4.3 we have \(PS(k) \supseteq PS(F) \cap Br(k_{ra}/k) = PS(F) \cap Br(k_{ra}/k)\) so the bottom row is exact at \(SPS(F)\). To complete the proof of the first assertion of the theorem, we prove that the splitting map \(f : \text{Hom}(G(k_{ra}/k), \Delta/\Gamma) \rightarrow SBr(F(k_{ra}))/F')\) takes values in \(PS(F)\) hence in \(SPS(F)\). Indeed, for \(\chi \in \text{Hom}(G(k_{ra}/k), \Delta/\Gamma)\), \(f(\chi)\) is a tensor product of cyclic algebras of the form \((F(E)/F, \sigma, t)\), where \(E\) is a cyclic extension of \(k\) lying in \(k_{ra}\) with \([F(E):F] = [E:k]\) dividing the order of \(\chi\). Since \(F(E) \subseteq F_{ira}\) each such cyclic algebra is a radical Abelian algebra of degree not a multiple of \(\text{char}(F)\), so lies in \(SPS(F)\).

For the second assertion of the theorem, we may assume \(\text{char}(k) = p \neq 0\), and we need only prove the assertion for the \(p\)-primary components. By (3.3) we have an exact sequence

\[
0 \longrightarrow Br(k) \quad \longrightarrow SBr(F) \quad \longrightarrow \text{Hom}_c(G_k, \Delta/\Gamma) \quad \longrightarrow 0.
\]

Since the \(p\)-power map is an automorphism of \(k\) (\(k\) is perfect of characteristic \(p\)), \(Br(k)(p) = 0\). Hence \(SBr(F)(p) = \text{Hom}_c(G_k, \Delta/\Gamma)(p)\). Similarly, since \(k\) has no nontrivial Kummer \(p\)-extensions, \(SBr(F(k_{ra}))(p) = SBr(F(k_{cyc}))(p) = \text{Hom}_c(G_{k_{cyc}}, \Delta/\Gamma)(p)\), and we get a commutative diagram

\[
\begin{array}{c}
SBr(F)(p) \quad \cong \quad \text{Hom}_c(G_k, \Delta/\Gamma)(p) \\
\downarrow \quad \text{res} \quad \downarrow \text{res} \\
SBr(F(k_{cyc}))(p) \quad \cong \quad \text{Hom}_c(G_{k_{cyc}}, \Delta/\Gamma)(p)
\end{array}
\]

It follows that the corresponding kernels of the restriction maps are isomorphic:

\[
SBr(F(k_{cyc}))/F(p) \cong \text{Hom}_c(G_{k_{cyc}}, \Delta/\Gamma)(p).
\]

But by Proposition 4.1, \(SPS(F)(p) \subseteq SBr(F(k_{cyc}))/F(p)\). Furthermore, using the splitting map, we see that the map \(SPS(F)(p) \rightarrow \text{Hom}(G_{k_{cyc}}, \Delta/\Gamma)(p)\) is surjective, and the result follows. \(\square\)

For an equicharacteristic Henselian field \(F\), for any prime \(p \neq \text{char}(F)\), Theorem 3.2 and Proposition 3.5 yield a direct sum decomposition

\[
Br(F)(p) = Br(k)(p) \oplus f(\text{Hom}_c(G_k, \Delta/\Gamma)(p)) \oplus T,
\]

where \(f : \text{Hom}_c(G_k, \Delta/\Gamma) \rightarrow SBr(F)\) is the (injective) splitting map of Theorem 3.2 above and \(T = T_p^o\) if \(r\) is maximal such that \(\mu_{p^r} \subseteq k\), and \(T = T_p^o\) if there is no such \(r\). We can now see that there is a compatible direct sum decomposition of \(PS(F)(p)\). For this, let \(k_{ra,p}\) be the maximal \(p\)-extension of \(k\) in the Abelian Galois extension \(k_{ra}\). Then,

**Proposition 4.5.** For any prime \(p \neq \text{char}(k)\),

\[
PS(F)(p) = PS(k)(p) \oplus f(\text{Hom}_c(G(k_{ra,p}/k), \Delta/\Gamma)) \oplus T,
\]

where \(f\) and \(T\) are as in (4.2) above.

**Proof.** Since \(T\) is generated by symbol algebras, \(T \subseteq PS(F)(p)\). Therefore, it suffices to see that \(SPS(F)(p) = PS(k)(p) \oplus f(\text{Hom}_c(G(k_{ra,p}/k), \Delta/\Gamma))\). But this is clear from Theorem 4.4 since \(G(k_{ra,p}/k)\) is the \(p\)-part of the Abelian profinite group \(G(k_{ra}/k)\). \(\square\)

**Corollary 4.6.** Suppose \(\text{char}(k) = \text{char}(F) = 0\). If every element of \(PS(k)\) is represented by a radical (resp. radical Abelian) algebra, then the same holds for \(PS(F)\).
Proof. We recall first [4, lemma 2.4] that the tensor product of radical algebras is Brauer equivalent to a radical algebra, and the tensor product of radical Abelian algebras is Brauer equivalent to a radical Abelian algebra. Now as char($k$) = 0 every element of Br($F$) is tame, so is represented by a tensor product $S \otimes_F T$, where $[S] \in SBr(F)$ and $T$ is a tensor product of symbol algebras (which are clearly radical Abelian algebras). We therefore need to prove the assertion for $PS(F) \cap SBr(F) = SPS(F)$. But this follows easily from Theorem 4.4 and the fact that every element of $\text{Hom}_c(\mathcal{G}(k_{ra}/k), \Delta/\Gamma)$ is represented by a tensor product of cyclic radical Abelian algebras. □

Corollary 4.7. Suppose the residue field $k$ is a local or global field. Then every projective Schur algebra over $F$ is Brauer equivalent to a radical Abelian algebra.

Proof. If char($F$) = 0, the result follows from the preceding corollary since it holds for $k$, by [25, (4.3)]. (To complete the argument in [25] for the case $p = 2$ one can observe that for any number field $k$, the field $k(\mu_{\infty})$ obtained by adjoining to $k$ all $2^n$-th roots of unity for all $n$ contains nonreal cyclic extensions of arbitrary $2$-power degree.) If char($F$) ≠ 0, the result holds because it holds for any field of characteristic not zero [5, Corollary 1.5]. □

5. Projective Schur groups of Henselian fields as algebraic relative Brauer groups

In this section we prove

Theorem 5.1. Let $F$ be a field with Henselian valuation $v$, value group $\Gamma = \Gamma_F$, and residue field $k$. Assume $k$ is a local or global field and that char($k$) = char($F$). Then $PS(F)$ is an algebraic relative Brauer group $Br(M/F)$ with $M/F$ a radical Abelian extension.

Proof. We consider the cases $k = \mathbb{R}$ and $k = \mathbb{C}$ separately. In the case $k = \mathbb{C}$, we have from Proposition 3.5 that for any prime $p$,

$$Br(F)(p) = SBr(F)(p) \oplus T_p^\infty,$$

and since $SBr(F)(p) = 0, Br(F)(p) = T_p^\infty \subseteq PS(F)$, so $Br(F) = PS(F) = Br(F_{kum}/F)$ and we are done.

We turn next to the case $k = \mathbb{R}$. Then the maximal unramified extension of $F$ has residue field $\mathbb{C}$, so $F_{nr} = F(\sqrt{-1})$. Hence, $SBr(F) = Br(F(\sqrt{-1})/F)$, a $2$-torsion group. By Proposition 3.5, for $p \neq 2, Br(F)(p) = 0$, so $Br(F) = Br(F(2)) \cong SBr(F)(2) \oplus T_2$. Each summand is generated by quaternion algebras, so $PS(F) = Br(F) = Br(F_{kum}/F)$ and we are done.

We now assume that $k$ is either a nonarchimedean local field or a global field.

Let $p$ be a prime number. We will prove the theorem for $p$-primary components. More precisely, we will prove that $PS(F)(p) = Br(M_p/F)$ with $M_p/F$ a radical Abelian $p$-extension. It then follows that $PS(F) = Br(M/F)$ with $M$ equal to the composite of all the $M_p$.

For a given Galois extension $E/k$, we set $X(E/k) := \text{Hom}_c(\mathcal{G}(E/k), \mathbb{Q}/\mathbb{Z})$, the character group of $E/k$, written additively. When $\mathcal{G}(E/k)$ is Abelian there is an isomorphism between the lattice of intermediate fields $K, k \subseteq K \subseteq E$ and the lattice of subgroups of $X(E/k)$ given by $K \leftrightarrow X(K/k)$. If $m \in \mathbb{N}$, let $E^{(m)}$ denote the subfield of $E$ corresponding to the subgroup $mX(E/k)$. So $E^{(m)}$ is the smallest subfield of $E$ containing $k$ such that $\mathcal{G}(E/E^{(m)})$ is $m$-torsion. Since $k_{ra} = k_{cyc}k_{kum}$ (see Section 4), we have $X(k_{ra}/k) = X(k_{cyc}/k) + X(k_{kum}/k)$. It follows that $mX(k_{ra}/k) = mX(k_{cyc}/k) + mX(k_{kum}/k)$. Note that $mX(k_{cyc}/k) = X(k_{cyc}/k)$ when char($k$) ≠ 0 since $X(k_{cyc}/k)$ is divisible in this case.

Let $m = p^r$ be the number of $p$-power roots of unity in $k$ (or equivalently in $F$). Then $m$ is finite. Denote by $k_{ra,p}$ the $p$-part of $k_{ra}$, i.e., the maximal $p$-extension of $k$ contained in $k_{ra}$. Similarly let $k_{cyc,p}$ and $k_{kum,p}$ denote the $p$-parts of $k_{cyc}$ and $k_{kum}$, respectively.

Proposition 5.2. Let $F$ be a field with Henselian valuation $v$, value group $\Gamma$, and residue field $k$. Let $p \neq \text{char}(k)$ be a prime number. Set $m = p^r$ with $r$ maximal such that $k$ contains the $p^r$-th roots of unity. Set $L := k_{ra,p}^{(m)}$. Assume $PS(k)(p) = Br(L/k)$. Then $PS(F)(p) = Br(M_p/F)$, where $M_p = F(L)(\sqrt[p^r]{i} | i \in I_p)$, where $v(t_i) = \gamma_i$ ($i \in I_p$), and $\{\sqrt[p^r]{i} | i \in I_p\}$ is a $\mathbb{Z}/p^r\mathbb{Z}$-base of $\Gamma/p^r\Gamma$. ($\sqrt[p^r]{i} = \gamma_i + p\Gamma$.)
As usual, let $\Delta = \mathbb{Q} \otimes \mathbb{Z} \Gamma$. For fields $E \supseteq K \supseteq k$ with $E/K$ Galois, let $Y(E/K) = \text{Hom}_c(\mathcal{G}(E/K), \Delta/\Gamma)$. Note that $Y(E/K)$ has similar functorial properties to $X(E/K)$. Indeed, since $\Delta/\Gamma(p) \cong \bigoplus_{i \in \mathbb{L}} \mathbb{Q}/\mathbb{Z}(p)$, we have

$$Y(E/K)(p) = \text{Hom}_c(\mathcal{G}(E/K), \Delta/\Gamma(p)) \cong \bigoplus_{i \in \mathbb{L}} \text{Hom}_c(\mathcal{G}(E/K), \mathbb{Q}/\mathbb{Z}(p)) = \bigoplus_{i \in \mathbb{L}} X(E/K)(p).$$

The isomorphism holds because every continuous homomorphism from a profinite group to a discrete group has finite image. Clearly, the direct sum decomposition is compatible with the canonical inclusion $Y(N/K) \hookrightarrow Y(M/K)$ for fields $K \subseteq N \subseteq M$.

We compute $Br(M_p/F)$ using the decomposition

$$Br(F)(p) = Br(k)(p) \oplus f(Y(k_s)(k))(p) \oplus T_{p'}$$

in (4.2). Clearly $T_{p'} \subseteq Br(M_p/F)$ since each generator of $T_{p'}$ has a maximal subfield in $M_p$. Now, take any $\alpha \in Br(k)(p)$ and $\chi \in Y(k_s)(k)(p)$. By Propositions 3.3(a) (for $\text{res}_{F(L)/F}$) and 3.3(b) (for $\text{res}_{M_p(F/L)}$), we have $\alpha + f(\chi) \in Br(M_p/F)$ iff $\alpha$ and $f(\chi)$ each lie in $Br(M_p/F)$, iff $\alpha \in Br(k_l(k))$ (as $M_p = L$) and $\chi$ maps to 0 in $\text{Hom}_c(G_L, \Delta/\Gamma M_p)$. Since $\Delta M_p/\Gamma = \frac{1}{m} \Delta/\Gamma$ is the $m$-torsion subgroup of $\Delta/\Gamma$, the condition on $\chi$ is equivalent to: $0 = \text{res}_{L/k}(\chi) = \text{res}_{L/k}(m\chi)$, i.e., $m\chi \in Y(L/k)$. Now, $Y(L/k) = \bigoplus_{i \in \mathbb{L}} X(L/k) = \bigoplus_{i \in \mathbb{L}} \text{Hom}_c(G(k_{ra,p}/k), \mathbb{Q}/\mathbb{Z}(p))$. Hence, $m\chi \in Y(L/k)$ iff $m\chi = m\psi$ with $\psi \in Y(k_{ra,p}/k)$. We claim that this last equality holds iff $\chi \in Y(k_{ra,p}/k)$. “If” is clear, taking $\psi = \chi$. Conversely, suppose $m\chi = m\psi$. Since $G_k/\ker(\chi - \psi)$ is isomorphic to the $m$-torsion part of the fixed field of $\ker(\chi - \psi)$ is an $m$-Kummer extension of $k$. So $\chi - \psi \in Y(k_{ra,p}/k).$ We have then that $Br(M_p/F) = Br(k_l(k)) \oplus f(Y(k_{ra,p}/k)(p)) \oplus T_{p'}$. Since $Br(M_p/F) = Br(k_l(k))$ by hypothesis, Proposition 4.5 shows that $Br(M_p/F) = Br(F)(p)$. This completes the proof of Proposition 5.2.

We now apply Proposition 5.2 to prove Theorem 5.1. We will handle separately below an exceptional case when $k$ is a number field which is not totally imaginary and $p = 2$. If $k$ is a local field or a global field in the nonexceptional case and $p \neq \text{char}(k)$, we will verify that the hypothesis $PS(k)(p) = Br(k_{ra,p}/k)$ is satisfied, proving Theorem 5.1.

If $k$ is a local or global field, then $PS(k) = Br(k)$ [23, Satz 3]. We need to check that $L = k_{ra,p}^{(m)}$ splits every element of $Br(k)(p)$. If $k$ is local, then since $X(k_{cyc,p}/k)$ contains a copy of $\mathbb{Q}/\mathbb{Z}(p)$, so does $mX(k_{cyc,p}/k)$, and $mX(k_{cyc,p}/k) \subseteq mX(k_{ra,p}/k) = X(L/k)$. Hence, $L$ contains extensions of $k$ of all $p$-power degrees, so $Br(L/k) = Br(k)(p)$. If $k$ is global, then for any finite prime $p$ of $k$ and any divisor $\mathfrak{P}$ of $p$ in $L$, we have $L_\mathfrak{P}$ contains $k_{cyc,p}$, so $Br(L_\mathfrak{P}/k_p) = Br(k_p)(p)$. Since $p$ is odd or $\mathfrak{p}$ does not divide $2$, either $k$ is totally imaginary, the local global principle for $Br(k)$ [28, p. 276, Theorem 32.11] shows that $Br(L/k) = Br(k)(p)$.

It remains to treat the exceptional case, for which we will need the following lemma:

**Lemma 5.3.** Let $k$ be a number field which is not totally imaginary. Then there exists a totally imaginary quadratic extension $\ell = k(\sqrt{b})$ of $k$ such that $\ell$ does not embed into a cyclic degree 4 extension of $k$.

**Proof.** By the approximation theorem there is $\beta \in k$ with $\beta < 0$ in each real completion of $k$. Then, $\ell = k(\sqrt{b})$ is totally imaginary. Since there is a real place of $k$, this $\beta$ cannot be a sum of two squares in $k$. Therefore, by Albert’s criterion [1, p. 207, Theorem 11, p. 208, Example 1] $\ell$ cannot embed in a cyclic degree 4 extension of $k$. 

We now return to the proof of Theorem 5.1 in the exceptional case. So, $p = 2$ and $m = 2$, and we replace the previous $L = k_{ra,2}^{(2)}$ by $L' = k_{ra,2}^{(2)}(\sqrt{b}) = L(\sqrt{b})$ with $h(\sqrt{b})$ as in Lemma 5.3, and replace $M_2$ by $M_2' = M_2(\sqrt{b})$. The earlier argument shows that $Br(L_\mathfrak{P}/k_p) = Br(k_p)(2)$ for each finite prime $p$ of $k$. Since $L'$ has no real embeddings, it follows that $Br(L'/k) = Br(k)(2) = PS(k)(2)$. We compute $Br(M_2')$ by a variant of the proof of Proposition 5.2. For $\chi \in Y(k_{ra,2}(2)$, the condition for $\text{res}_{M_2'/F}(f(\chi)) = 0$ is now that $2\chi \in Y(L'/k) = Y(L/k) + Y(k(\sqrt{b})/k) = 2Y(k_{ra,2}/k) + Y(k(\sqrt{b})/k)$, $2\chi$ is an $h$-extender of $\chi$ in $Y(k_{ra,2}/k)(2)$. Again, “if” is clear; for the converse, we suppose $2\chi = 2\psi + \varphi$ with $\psi \in Y(k_{ra,2}/k)$ and $\varphi \in Y(k(\sqrt{b})/k)$. We have

$$Y(k(\sqrt{b})/k) \cap 2Y(k_s/k)(2) \cong \bigoplus_{i \in \mathbb{L}} X(k(\sqrt{b})/k) \cap 2X(k_s/k)(2) = 0,$$

since $k(\sqrt{b})$ lies in no cyclic extension of $k$ of degree 4. But $\varphi = 2(\chi - \psi)$ lies in this intersection, so $\varphi = 0$. Then, $2\chi = 2\psi$, and the rest of the argument to see that $PS(F)(2) = Br(M_2'/F)$ is the same as for Proposition 5.2.
It remains only to prove that \( PS(F)(p) = Br(M_p/F) \) when \( p = \text{char}(k) \). For this, note first that for any field \( K \) with \( \text{char}(K) = p \), we have \( Br(K_{\text{cyc}}, K) \subseteq PS(K)(p) \subseteq Br(K_{\text{cyc}}, K) \). The first inclusion holds as every finite subextension of \( K_{\text{cyc}}, K \) is cyclic, so \( Br(K_{\text{cyc}}, K) \) consists of cyclotomic cyclic algebras, which clearly lie in \( PS(K) \). The second inclusion holds because \( PS(K)(p) \subseteq Br(K_{\text{ra}}, K) \) by \cite{3}, Corollary 2.3 (this is also deducible from Propositions 2.2(a) and 2.4(c) above) and \( K_{\text{ra}}, K \) as \( K \) contains no \( p \)-th roots of unity. Thus, \( PS(K)(p) = Br(K_{\text{cyc}}, K) \). For the fields in the proof of Theorem 5.1 with \( p = \text{char}(k) \), we have \( m = 1 \), so \( L = k_{\text{ra}} = k_{\text{cyc}}, p \) and \( M_p = F(L) = F(k_{\text{cyc}}, p) = F_{\text{cyc}}, p \). Hence, \( Br(M_p/F) = Br(F_{\text{cyc}}, p/F) = PS(K)(p) \). This completes the proof of Theorem 5.1. \( \square \)

6. The Schur group case

For \( F \) a number field, \( PS(F) = Br(F) \), so trivially \( PS(F) \) is the algebraic relative Brauer group \( Br(F_k/F) \). On the other hand, we now show that this need not be the case for the classical Schur group \( S(F) \). We are grateful to Allan Herman for suggesting that \( S(F) \) is not an algebraic relative Brauer group, because the local invariants of an element of the Schur group are uniformly distributed \cite[Theorem 1]{10}. Here is an explicit example. Let \( \zeta_n \) be a primitive \( n \)-th root of unity in \( \mathbb{Q}_{\text{cyc}} \). Let \( F = \mathbb{Q}((\zeta_{12})) \) and let \( M = F((\zeta_{13})) \). Let \( B \) be the cyclic algebra \( (M/F, \sigma, \zeta_{12}) \). This is a Schur (division) algebra, of exponent 12 over \( F \) as one can check by looking at it over the completion \( F_p \), where \( p \) is a prime of \( F \) over \( (13) \). (Indeed, tensoring \( B \) with \( F_p \), we get an algebra \( B_p \) and it suffices to show that it has order 12. This is equivalent to \( \zeta_{12} \) having order 12 mod norms from \( M_p \). But this is the case by local class field theory because \( M_p/F_p \) is totally and tamely ramified: the norm group is generated by a local parameter and the one-units.) \( (13) \) splits completely in \( F \) into a product of four primes, and the local invariant of \( B \) at each of these is of order 12 (in fact these are the only nontrivial invariants of \( B \), so they must be 1/12, 5/12, 7/12, 11/12 by \cite[Theorem 1]{10}). Any splitting field must have local degree divisible by 12 at these four places. Now take an algebra with local invariants, say 1/2, 1/2, 0, 0 at these four places and 0 everywhere else. It is not in the Schur group because these local invariants are not uniformly distributed \cite[Theorem 1]{10} (in fact, since these four invariants are not of the same order, it is enough to invoke \cite[Theorem 1]{9} and is split by any field that splits \( B \).

One can also prove that \( S(F) \) is not an algebraic relative Brauer group for \( F \) a formal power series field \( k((t)) \) over certain fields \( k \). Here is a sketch of the proof. If \( A = F(G) \) is a Schur algebra over \( F \) with \( G \) finite, then \( A_0 := k(G) \) is a Schur algebra over \( k \) and \( A = A_0 \otimes_k F \). It follows that \( S(F) = \text{res}_{F/k}(S(k)) \). Recall that \( S(k)(p) = 0 \) if \( \text{char}(k) = p \neq 0 \); so \( S(k) \subseteq Br(k) \). Discretely valued fields have no TTR algebras (as \( |l'/p l'| < p^{2} \) for \( l' = \mathbb{Z} \), see the definition of \( T_{\rho'} \) in Section 3). Thus, Proposition 3.5 and Theorem 3.2 above reduce to the classical Witt decomposition

\[
Br(k((t)))' = SBr(k((t)))' \cong Br(k)' \oplus \text{Hom}_{\mathbb{C}}(G_{k}, \mathbb{Q}/\mathbb{Z})'.
\]

Here, \( S(F) \) sits in the first component. For the sake of simplicity let \( k = \mathbb{R} \) (but the same type of argument works for any field \( k \) with \( S(k) \) nontrivial and \( G_{k} \) pronilpotent). Suppose \( S(F) \) were an algebraic relative Brauer group \( Br(L/F) \) with \( L/F \) algebraic. Let \( L_0/F \) be the maximal unramified subextension of \( L/F \), and let \( \ell = \overline{L}_0 \). Then, \( l \) is either \( \mathbb{R} \) or \( \mathbb{C} \). If \( l = \mathbb{C} \), then \( L \) will split the nontrivial quaternion algebra \((-1, -1) \). This algebra belongs to the second component in the Witt decomposition, hence is not in \( S(F) \), contradiction. It follows that \( L/F \) is totally ramified, hence its residue field is \( k \). So (see (3.4)) \( Br(k) \) injects into \( Br(L) \), hence \( L \) does not split the nontrivial quaternion Schur algebra \((-1, -1) \), another contradiction.

Acknowledgement

The research was supported by the Fund for the Promotion of Research at the Technion.

References